Vol. 39 (2019)
No. 5

数 学 杂 志
J. of Math. (PRC)

AN ESTIMATE FOR MAXIMAL BOCHNER-RIESZ MEANS ON MUSIELAK-ORLICZ HARDY SPACES

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Abstract: In this paper, we study the boundedness of maximal Bochner-Riesz means. By using the pointwise of maximal Bochner-Riesz means and the atomic decomposition of weak Musielak-Orlicz Hardy space, we establish the boundedness of maximal Bochner-Riesz means from weak Musielak-Orlicz Hardy space to weak Musielak-Orlicz space. This result is new even when $\varphi(x, t) := \Phi(t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, where Φ is an Orlicz function, and it is an extension to Musielak-Orlicz spaces from the setting of the weighted spaces of Wang [1].

Keywords: Bochner-Riesz means; Muckenhoupt weight; Orlicz function; Hardy space

2010 MR Subject Classification: 42B30; 46E30

Document code: A **Article ID:** 0255-7797(2019)05-0694-11

1 Introduction

The Bochner-Riesz means of order $\delta \in (0, \infty)$ are defined initially for Schwartz functions f on $\mathbb{R}^n$ by, for any $x \in \mathbb{R}^n$,

$$T_R^\delta(f)(x) := \int_{\mathbb{R}^n} \hat{f}(\xi)\left(1-\frac{|\xi|^2}{R^2}\right)_+^\delta e^{2\pi i x\cdot\xi}\, d\xi,\ R \in (0, \infty),$$

where $\hat{f}$ denotes the Fourier transform of f. The Bochner-Riesz means can be also expressed as convolution operator $T_R^\delta(f)(x) = (f * \phi_{1/R})(x)$, where, for any $x \in \mathbb{R}^n$ and $\varepsilon \in (0, \infty)$, $\phi(x) := \{(1-|\cdot|^2)_+^\delta\}\hat{}\,(x)$ and $\phi_\varepsilon(x) := \varepsilon^{-n}\phi(x/\varepsilon)$. The corresponding maximal Bochner-Riesz means are defined by, for any $x \in \mathbb{R}^n$,

$$T_*^\delta(f)(x) := \sup_{R\in(0,\infty)} T_R^\delta(f)(x).$$

The Bochner-Riesz means were first introduced by Bochner [2] in connection with summation of multiple Fourier series. Questions concerning the convergence of multiple Fourier series led to the study of their $L^p(\mathbb{R}^n)$ boundedness.

* **Received date:** 2018-05-11 **Accepted date:** 2018-07-06

Foundation item: Supported by National Natural Science Foundation of China (11461065; 11661075).

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In 2013, Wang [1] considered the values of δ greater than the critical index $n/p-(n+1)/2$ and proved the following weighted weak type estimate of T_*^δ, which is bounded from weighted weak Hardy space $WH^p_\omega(\mathbb{R}^n)$ to weighted weak Lebesgue space $WL^p_\omega(\mathbb{R}^n)$.

Theorem A Let $p\in(0,\,1]$ and $\delta\in(n/p-(n+1)/2,\,\infty)$. If $\omega\in A_1$ (the Muckenhoupt weight class), then there exists a positive constant C independent of f such that

$$\left\|T_*^\delta(f)\right\|_{WL^p_\omega(\mathbb{R}^n)}\le C\|f\|_{WH^p_\omega(\mathbb{R}^n)}.$$

Recently, Liang et al. [3] introduced weak Musielak-Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$, which generalizes both the weak Orlicz-Hardy space and the weak weighted Hardy space, and hence has a wide generality. In light of Wang [1] and Liang et al. [3], it is a natural and interesting problem to ask whether T_*^δ is bounded from $WH^\varphi(\mathbb{R}^n)$ to $WL^\varphi(\mathbb{R}^n)$. In this paper, we shall answer this problem affirmatively.

Precisely, this paper is organized as follows.

In Section 2, we recall some notions concerning Muckenhoupt weights, growth functions and weak Musielak-Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$. Then we present the boundedness of maximal Bochner-Riesz means T_*^δ from $WH^\varphi(\mathbb{R}^n)$ to $WL^\varphi(\mathbb{R}^n)$ (see Theorem 2.6 below), the proof of which are given in Sections 3. This result is also new even it comes back to Orlicz Hardy space.

In the process of the proof of main result, it is worth pointing out that a more subtle pointwise estimate of T_*^δ on atom (see Lemma 3.5 below) plays a crucial role for the desired estimate of T_*^δ. Moreover, towards the boundedness of maximal Bochner-Riesz means T_*^δ from $WH^\varphi(\mathbb{R}^n)$ to $WL^\varphi(\mathbb{R}^n)$, the range of δ (see Theorem 2.6 below) coincides with that of the known best conclusion of Theorem A even $\varphi(x,\,t):=\omega(x)t^p$ for all $(x,\,t)\in\mathbb{R}^n\times[0,\,\infty)$ with $p\in(0,\,1]$.

Finally, we make some conventions on notation. Let $\mathbb{Z}_+:=\{1,\,2,\,\cdots\}$ and $\mathbb{N}:=\{0\}\cup\mathbb{Z}_+$. For any $\alpha:=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n$, let $|\alpha|:=\alpha_1+\cdots+\alpha_n$. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $D\lesssim F$ means that $D\le CF$. If $D\lesssim F$ and $F\lesssim D$, we then write $D\sim F$. For any sets $E,\,F\subset\mathbb{R}^n$, we use $(E)^\complement$ to denote the set $\mathbb{R}^n\setminus E$, $|E|$ its n-dimensional Lebesgue measure and χ_E its characteristic function. For any $a\in\mathbb{R}$, $\lfloor a\rfloor$ denotes the maximal integer not larger than a. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$. For example, $L^p(\mathbb{R}^n)$ is simply denoted by L^p. For any index $q\in[1,\,\infty]$, we denote by q' its conjugate index, namely, $1/q+1/q'=1$. For any set E of $\mathbb{R}^n$, $t\in[0,\,\infty)$ and measurable function φ, let $\varphi(E,\,t):=\int_E\varphi(x,\,t)\,dx$ and $\{|f|>t\}:=\{x\in\mathbb{R}^n:|f|>t\}$. As usual we use B_r to denote the ball $\{x\in\mathbb{R}^n:\ |x|<r\}$ with $r\in(0,\,\infty)$.

2 Notion and Main Results

In this section, we first recall the notion concerning the weak Musielak-Orlicz Hardy space WH^{φ} via the grand maximal function, and then present the boundedness of maximal Bochner-Riesz means T_*^{δ} from WH^{φ} to WL^{φ}.

Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an Orlicz function, if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0, \infty)$, and $\lim\limits_{t\to\infty} \Phi(t) = \infty$. An Orlicz function Φ is said to be of lower (resp. upper) type p with $p \in (0, +\infty)$, if there exists a positive constant $C := C_p$ such that for any $t \in [0, \infty)$ and $s \in (0, 1]$ (resp. $s \in [1, \infty)$),

$$\Phi(st) \le Cs^p\Phi(t).$$

Given a function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ such that for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is an Orlicz function, φ is said to be of uniformly lower (resp. upper) type p with $p \in (0, +\infty)$, if there exists a positive constant $C := C_p$ such that, for any $x \in \mathbb{R}^n$, $t \in [0, \infty)$ and $s \in (0, 1]$ (resp. $s \in [1, \infty)$), $\varphi(x, st) \le Cs^p\varphi(x, t)$. The critical uniformly lower type index of φ is defined by

$$i(\varphi) := \sup\{p \in (0, +\infty) : \varphi \text{ is of uniformly lower type } p\}. \tag{2.1}$$

Observe that $i(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$ (see [4, p. 415] for more details).

Definition 2.1 [5, p. 120] Let $q \in [1, \infty)$. A function $\varphi(\cdot, t) : \mathbb{R}^n \to [0, \infty)$ is said to satisfy the uniform Muckenhoupt condition, denoted by $\varphi \in \mathbb{A}_q$, if there exists a positive constant C such that for any ball $B \subset \mathbb{R}^n$ and $t \in (0, \infty)$, when $q \in (1, \infty)$,

$$\frac{1}{|B|}\int_B \varphi(y, t)\,dy \left\{\frac{1}{|B|}\int_B [\varphi(y, t)]^{-\frac{1}{q-1}}\,dy\right\}^{q-1} \le C$$

and, when $q = 1$,

$$\frac{1}{|B|}\int_B \varphi(y, t)\,dy \left\{\operatorname*{ess\,sup}_{y\in B}\,[\varphi(y, t)]^{-1}\right\} \le C.$$

Let $\mathbb{A}_\infty := \bigcup_{q\in[1,\,\infty)} \mathbb{A}_q$. The critical weight index of $\varphi \in \mathbb{A}_\infty$ is defined as follows

$$q(\varphi) := \inf\{q \in [1, \infty) :\ \varphi \in \mathbb{A}_q\}. \tag{2.2}$$

Observe that, if $q(\varphi) \in (1, \infty)$, then $\varphi \notin \mathbb{A}_{q(\varphi)}$, and there exists $\varphi \notin \mathbb{A}_1$ such that $q(\varphi) = 1$ (see, for example, [6]).

Definition 2.2 [5, Definition 2.1] A function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is called a growth function if the following conditions are satisfied

(i) φ is a Musielak-Orlicz function, namely,

(a) the function $\varphi(x, \cdot) : [0, \infty) \to [0, \infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$,

(b) the function $\varphi(\cdot, t)$ is a Lebesgue measurable function on $\mathbb{R}^n$ for all $t \in [0, \infty)$;

(ii) $\varphi \in \mathbb{A}_\infty$;

(iii) φ is of uniformly lower type p for some $p \in (0, 1]$ and of uniformly upper type 1.

Clearly, $\varphi(x, t) := \omega(x)\Phi(t)$ is a growth function if $\omega \in A_\infty$ (the Musielak weight class) and Φ is an Orlicz function of lower type p for some $p \in (0, 1]$ and of upper type 1. It is well known that, for $p \in (0, 1]$, if $\Phi(t) := t^p$ for all $t \in [0, \infty)$, then Φ is an Orlicz function of lower type p and of upper p; for $p \in [1/2, 1]$, if $\Phi(t) := t^p / \ln(e+t)$ for all $t \in [0, \infty)$, then Φ is an Orlicz function of lower type q for $q \in (0, p)$ and of upper type p; for $p \in (0, 1/2]$, if $\Phi(t) := t^p \ln(e+t)$ for all $t \in [0, \infty)$, then Φ is an Orlicz function of lower type p and of upper type q for $q \in (p, 1]$. Recall that if an Orlicz function is of upper type $p \in (0, 1)$, then it is also of upper type 1. Another typical and useful growth function is

$$\varphi(x, t) := \frac{t^\alpha}{[\ln(e+|x|)]^\beta + [\ln(e+t)]^\gamma}$$

for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ with any $\alpha \in (0, 1]$, $\beta \in [0, \infty)$ and $\gamma \in [0, 2\alpha(1+\ln 2)]$; more precisely, $\varphi \in \mathbb{A}_1$, φ is of uniformly upper type α and $i(\varphi) = \alpha$ which is not attainable (see [5]).

Recall that the Musielak-Orlicz space L^φ is defined to be the set of all measurable functions f such that for some $\lambda \in (0, \infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx < \infty$$

equipped with the (quasi-) norm

$$\|f\|_{L^\varphi} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

Similarly, the weak Musielak-Orlicz space WL^φ is defined to be the set of all measurable functions f such that for some $\lambda \in (0, \infty)$,

$$\sup_{t \in (0, \infty)} \varphi(\{|f| > t\}, t/\lambda) < \infty$$

equipped with the quasi-norm

$$\|f\|_{WL^\varphi} := \inf\left\{\lambda \in (0, \infty) : \sup_{t \in (0, \infty)} \varphi\left(\{|f| > t\}, \frac{t}{\lambda}\right) \le 1\right\}.$$

Remark 2.3 Let ω be a classical Muckenhoupt weight and Φ an Orlicz function.

(i) If $\varphi(x, t) := \omega(x) t^p$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ with $p \in (0, \infty)$, then L^φ (resp. WL^φ) is reduced to weighted Lebesgue space L^p_ω (resp. weighted weak Lebesgue space WL^p_ω), and particularly, when $\omega \equiv 1$, the corresponding unweighted spaces are also obtained.

(ii) If $\varphi(x, t) := \omega(x)\Phi(t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, then L^φ (resp. WL^φ) is reduced to weighted Orlicz space L^Φ_ω (resp. weighted weak Orlicz space WL^Φ_ω), and particularly, when $\omega \equiv 1$, the corresponding unweighted spaces are also obtained.

In what follows, we denote by $\mathcal{S}$ the space of all Schwartz functions and by $\mathcal{S}'$ its dual space (namely, the space of all tempered distributions). For any $m \in \mathbb{N}$, let

$$\mathcal{S}_m := \left\{\psi \in \mathcal{S} : \sup_{\alpha \in \mathbb{N}^n, |\alpha| \le m+1} \sup_{x \in \mathbb{R}^n} (1+|x|)^{(m+2)(n+1)} |\partial^\alpha \psi(x)| \le 1\right\}.$$

Then, for any $m \in \mathbb{N}$ and $f \in \mathcal{S}'$, the non-tangential grand maximal function f_m^* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\psi \in \mathcal{S}_m} \sup_{|y-x|<t,\, t\in(0,\,\infty)} |f * \psi_t(y)|, \tag{2.3}$$

where, for any $t \in (0,\,\infty)$, $\psi_t(\cdot) := t^{-n}\psi(\frac{\cdot}{t})$. When

$$m = m(\varphi) := \left\lfloor n\left(\frac{q(\varphi)}{i(\varphi)} - 1\right)\right\rfloor, \tag{2.4}$$

we denote f_m^* simply by f^*, where $q(\varphi)$ and $i(\varphi)$ are as in (2.2) and (2.1), respectively.

Definition 2.4 [5, Definition 2.2] Let φ be a growth function as in Definition 2.2. The weak Musielak-Orlicz Hardy space WH^φ is defined as the set of all $f \in \mathcal{S}'$ such that $f^* \in WL^\varphi$ equipped with the quasi-norm $\|f\|_{WH^\varphi} := \|f^*\|_{WL^\varphi}$.

Remark 2.5 Let ω be a classical Muckenhoupt weight and Φ an Orlicz function.

(i) If $\varphi(x,\,t) := \omega(x)t^p$ for all $(x,\,t) \in \mathbb{R}^n \times [0,\,\infty)$ with $p \in (0,\,1]$, then WH^φ is reduced to weighted weak Hardy space WH^p_ω, and particularly, when $\omega \equiv 1$, the corresponding unweighted spaces are also obtained.

(ii) If $\varphi(x,\,t) := \omega(x)\Phi(t)$ for all $(x,\,t) \in \mathbb{R}^n \times [0,\,\infty)$, then WH^φ is reduced to weighted weak Orlicz Hardy space WH^Φ_ω, and particularly, when $\omega \equiv 1$, the corresponding unweighted spaces are also obtained.

The main results of this paper are as follows, the proof of which are given in Section 3.

Theorem 2.6 Let $p \in (0,\,1]$, $\delta \in (n/p - (n+1)/2,\,\infty)$ and φ be a growth function as in Definition 2.2, which is of uniformly lower type p and of uniformly upper type 1. If $\varphi \in \mathbb{A}_1$, then there exists a positive constant C independent of f such that

$$\left\|T_*^\delta(f)\right\|_{WL^\varphi} \le C\|f\|_{WH^\varphi}.$$

Corollary 2.7 Let $p \in (0,\,1]$, $\delta \in (n/p-(n+1)/2,\,\infty)$, ω be a classical Muckenhoupt weight and Φ an Orlicz function, which is of uniformly lower type p and of uniformly upper type 1. If $\omega \in \mathbb{A}_1$, then there exists a positive constant C independent of f such that

$$\left\|T_*^\delta(f)\right\|_{WL^\Phi_\omega} \le C\|f\|_{WH^\Phi_\omega}.$$

Remark 2.8 Let ω be a classical Muckenhoupt A_1 weight and Φ an Orlicz function.

(i) When $\varphi(x,\,t) := \omega(x)t^p$ for all $(x,\,t) \in \mathbb{R}^n \times [0,\,\infty)$ with $p \in (0,\,1]$, we have $WH^\varphi = WH^p_\omega$, Theorem 2.6 is reduced to Theorem A.

(ii) When $\varphi(x,\,t) := \omega(x)\Phi(t)$ for all $(x,\,t) \in \mathbb{R}^n \times [0,\,\infty)$, we have $WH^\varphi = WH^\Phi_\omega$, and particularly, when $\omega \equiv 1$, the corresponding result on unweighted space is also obtained.

3 Proof of Theorem 2.6

To prove Theorem 2.6, we need some auxiliary lemmas. Let us begin with some notions.

For any measurable set E of $\mathbb{R}^n$, the space $L^q_\varphi(E)$ for $q \in [1, \infty]$ is defined as the set of all measurable functions f on E such that

$$\|f\|_{L^q_\varphi(E)} := \begin{cases} \sup\limits_{t\in(0,\infty)} \left[\dfrac{1}{\varphi(E, t)} \displaystyle\int_E |f(x)|^q \varphi(x, t)\, dx\right]^{1/q} < \infty, & q \in [1, \infty), \\ \\ \|f\|_{L^\infty(E)} < \infty, & q = \infty. \end{cases}$$

Definition 3.1 [3, Definition 3.1] Let φ be a growth function as in Definition 2.2.

(i) A triplet (φ, q, s) is said to be admissible, if $q \in (q(\varphi), \infty]$ and $s \in [m(\varphi), \infty) \cap \mathbb{N}$, where $q(\varphi)$ and $m(\varphi)$ are as in (2.2) and (2.4), respectively.

(ii) For an admissible triplet (φ, q, s), a measurable function a is called a (φ, q, s)-atom associated with some ball $B \subset \mathbb{R}^n$ if it satisfies the following three conditions

(a) supp $a \subset B$;

(b) $\|a\|_{L^q_\varphi(B)} \le \|\chi_B\|^{-1}_{L^\varphi}$;

(c) $\displaystyle\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \le s$.

(iii) For an admissible triplet (φ, q, s), the weak Musielak-Orlicz atomic Hardy space $WH^{\varphi, q, s}_{\rm at}$ is defined as the set of all $f \in \mathcal{S}'$ satisfying that there exist a sequence of (φ, q, s)-atoms, $\{a_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$, and a positive constant C such that $\sum\limits_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \le C$ for all $x \in \mathbb{R}^n$, and $i \in \mathbb{Z}$, and $f = \sum\limits_{i\in\mathbb{Z}}\sum\limits_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j}$ in $\mathcal{S}'$, where $\lambda_{i,j} := \widetilde{C} 2^i \|\chi_{B_{i,j}}\|_{L^\varphi}$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\widetilde{C}$ is a positive constant independent of f.

Moreover, define

$$\|f\|_{WH^{\varphi, q, s}_{\rm at}} := \inf\left\{\inf\left\{\lambda \in (0, \infty) : \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{N}} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\} \le 1\right\}\right\},$$

where the first infimum is taken over all admissible decompositions of f as above.

Lemma 3.2 [7, Lemma 6] Let $p_1 \in (0, 1)$, $\delta := n/p_1 - (n+1)/2$ and $\alpha \in \mathbb{N}^n$. Then there exists a positive constant $C := C_{n, p_1, \alpha}$ such that the kernel ϕ of Bochner-Riesz means of order δ satisfies the inequality

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^{n/p_1} |\partial^\alpha \phi(x)| \le C.$$

Lemma 3.3 [3, Theorem 3.5] Let (φ, q, s) be an admissible triplet as in Definition 3.1. Then $WH^\varphi = WH^{\varphi, q, s}_{\rm at}$ with equivalent quasi-norms.

Lemma 3.4 [5, Lemma 4.5 (i)] Let $\varphi \in \mathbb{A}_q$ with $q \in [1, \infty)$. Then there exists a positive constant C such that for any ball $B \subset \mathbb{R}^n$, $\lambda \in (1, \infty)$ and $t \in (0, \infty)$,

$$\varphi(\lambda B, t) \le C\lambda^{nq} \varphi(B, t).$$

Lemma 3.5 Let $p \in (0, 1)$, $\delta := n/p - (n+1)/2$ and φ be a growth function as in Definition 2.2, which is of uniformly lower type p and of uniformly upper type 1. Suppose b

is a multiple of a $(\varphi, \infty, \lfloor n(q(\varphi)/p-1)\rfloor)$-atom associated with some ball $B(x_0, r)$, where $q(\varphi)$ is as in (2.2). Then there exists a positive constant C independent of b such that, for any $x \in \mathbb{R}^n$,

$$T_*^\delta(b)(x) \le C\|b\|_{L^\infty} \left(\frac{r}{r+|x-x_0|}\right)^{\frac{n}{p}}. \tag{3.1}$$

Proof We show this lemma by borrowing some ideas from the proof of [8, Lemma 2]. It suffices to show (3.1) holds for $x_0 = \mathbf{0}$ and $r = 1$. Indeed, for any multiple of a $(\varphi, \infty, \lfloor n(q(\varphi)/p-1)\rfloor)$-atom b associated with some ball $B(x_0, r)$, it is easy to see that

$$b_1(\cdot) := \|\chi_{B_1}\|_{L^\varphi}^{-1}\|b\|_{L^\infty}^{-1} b(x_0 + r\cdot)$$

is a $(\varphi, \infty, \lfloor n(q(\varphi)/p-1)\rfloor)$-atom associated with the ball $B(\mathbf{0}, 1)$. For any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}(b*\phi_\varepsilon)(x) &= \varepsilon^{-n}\int_{\mathbb{R}^n} b(x-y)\phi\left(\frac{y}{\varepsilon}\right)dy\\ &= \|b\|_{L^\infty}\|\chi_{B_1}\|_{L^\varphi}\varepsilon^{-n}\int_{\mathbb{R}^n} b_1\left(\frac{x-x_0}{r}-\frac{y}{r}\right)\phi\left(\frac{y}{\varepsilon}\right)dy\\ &= \|b\|_{L^\infty}\|\chi_{B_1}\|_{L^\varphi}(b_1*\phi_{\varepsilon/r})\left(\frac{x-x_0}{r}\right),\end{aligned}$$

which implies that

$$T_*^\delta(b)(x) = \|b\|_{L^\infty}\|\chi_{B_1}\|_{L^\varphi}T_*^\delta(b_1)\left(\frac{x-x_0}{r}\right).$$

If we assume (3.1) holds for $x_0 = \mathbf{0}$ and $r = 1$, then, for any $x \in \mathbb{R}^n$, we obtain

$$\begin{aligned}T_*^\delta(b)(x) &\lesssim \|b\|_{L^\infty}\|\chi_{B_1}\|_{L^\varphi}\|b_1\|_{L^\infty}\left(\frac{1}{1+\left|\frac{x-x_0}{r}\right|}\right)^{n/p}\\ &\lesssim \|b\|_{L^\infty}\left(\frac{r}{r+|x-x_0|}\right)^{n/p}.\end{aligned}$$

It remains to prove (3.1) holds for $x_0 = \mathbf{0}$ and $r = 1$. Let b be a multiple of a $(\varphi, \infty, \lfloor n(q(\varphi)/p-1)\rfloor)$-atom associated with the ball $B(\mathbf{0}, 1)$. From Lemma 3.2 and $p \in (0, 1)$, we deduce that, for any $x \in B(\mathbf{0}, 2)$,

$$\begin{aligned}T_*^\delta(b)(x) &= \sup_{1/\varepsilon\in(0,\infty)}|(b*\phi_\varepsilon)(x)| \le \|b\|_{L^\infty}\int_{\mathbb{R}^n}|\phi(y)|\,dy\\ &\le \|b\|_{L^\infty}\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n/p}}\,dy \sim \|b\|_{L^\infty}\left(\frac{1}{1+2}\right)^{n/p} \lesssim \|b\|_{L^\infty}\left(\frac{1}{1+|x|}\right)^{n/p},\end{aligned}$$

which is wished.

By repeating the estimate of (2) in the proof of [8, Lemma 2], we know that, for any $x \in [B(\mathbf{0}, 2)]^{\complement}$ and $\varepsilon \in (0, \infty)$,

$$|(b*\phi_\varepsilon)(x)| \lesssim \|b\|_{L^\infty}|x|^{-n/p}.$$

From this and the inequality $|x| \sim |x|+1$ with $x \in [B(\mathbf{0},2)]^{\complement}$, it follows that, for any $x \in [B(\mathbf{0},2)]^{\complement}$,

$$T_*^\delta(b)(x) = \sup_{1/\varepsilon \in (0,\infty)} |(b * \phi_\varepsilon)(x)| \lesssim \|b\|_{L^\infty} \left(\frac{1}{1+|x|}\right)^{n/p},$$

which is also wished. This finishes the proof of Lemma 3.5.

Lemma 3.6 [5, Lemma 4.5 (ii)] Let $\varphi \in \mathbb{A}_q$ with $q \in (1, \infty)$. Then there exists a positive constant C such that, for any ball $B := x_0 + B_r$ and $t \in (0, \infty)$,

$$\int_{B^{\complement}} \frac{\varphi(x,\, t)}{|x-x_0|^{nq}}\, dx \le C \frac{\varphi(B,\, t)}{r^{nq}}.$$

Proof of Theorem 2.6 By Lemma 3.3, we know that, for any $f \in WH^\varphi = WH_{at}^{\varphi,\, q,\, s}$ with $q \in (q(\varphi), \infty)$, where $q(\varphi)$ and $m(\varphi)$ are, respectively, as in (2.2) and (2.4), let $x_{i,\, j}$ denote the center of $B_{i,\, j}$ and $r_{i,\, j}$ its radius, then there exists a sequence of multiple of $(\varphi,\, q,\, s)$-atoms, $\{b_{i,\, j}\}_{i\in\mathbb{Z},\, j\in\mathbb{N}}$, associated with balls $\{B_{i,\, j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$ such that

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} b_{i,\, j} \text{ in } \mathcal{S}',$$

$\sum_{j\in\mathbb{N}} \chi_{B_{i,\, j}}(x) \lesssim 1$ for all $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\|b_{i,\, j}\|_{L^\infty_\varphi(B_{i,\, j})} \lesssim 2^i$ for $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, and

$$\|f\|_{WH^\varphi} \sim \inf\left\{\lambda \in (0,\, \infty) : \sup_{i\in\mathbb{Z}} \left\{\sum_{j\in\mathbb{N}} \varphi\left(B_{i,\, j}, \frac{2^i}{\lambda}\right)\right\} \le 1\right\}.$$

Thus, to prove $T_*^\delta(f) \in WL^\varphi$, it suffices to prove that, for all $\alpha, \lambda \in (0,\infty)$ and $f \in WH^\varphi$,

$$\varphi\left(\left\{x \in \mathbb{R}^n : T_*^\delta(f)(x) > \alpha\right\}, \frac{\alpha}{\lambda}\right) \lesssim \sup_{i\in\mathbb{Z}} \left\{\sum_{j\in\mathbb{N}} \varphi\left(B_{i,\, j}, \frac{2^i}{\lambda}\right)\right\}. \tag{3.2}$$

To prove (3.2), we may assume that there exists $i_0 \in \mathbb{Z}$ such that $\alpha = 2^{i_0}$, without loss of generality. Write

$$f = \sum_{i=-\infty}^{i_0-1} \sum_{j\in\mathbb{N}} + \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{N}} =: F_1 + F_2.$$

Let $a \in (0,\, 1-1/q)$ be a positive constant, by the well-known weighted L^q boundedness of T_*^δ with $\varphi \in \mathbb{A}_1 \subset \mathbb{A}_q$, Hölder's inequality, $\sum_{j\in\mathbb{N}} \chi_{B_{i,\, j}}(x) \lesssim 1$ for all $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\|b_{i,\, j}\|_{L^\infty_\varphi(B_{i,\, j})} \lesssim 2^i$ for $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, and the uniformly upper type 1 property of φ, we see that

$$
\begin{aligned}
&\varphi\left(\left\{x\in\mathbb{R}^n: T_*^\delta(F_1)(x)>2^{i_0}\right\},\, \frac{2^{i_0}}{\lambda}\right)\\
\le&\int_{\mathbb{R}^n}\left|\frac{T_*^\delta(F_1)(x)}{2^{i_0}}\right|^q\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)\,dx\lesssim 2^{-i_0q}\int_{\mathbb{R}^n}|F_1(x)|^q\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)\,dx\\
\sim&2^{-i_0q}\int_{\mathbb{R}^n}\left|\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}b_{i,j}(x)\right|^q\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)\,dx\\
\lesssim&2^{-i_0q}\int_{\mathbb{R}^n}\left(\sum_{i=-\infty}^{i_0-1}2^{iaq'}\right)^{q/q'}\left\{\sum_{i=-\infty}^{i_0-1}2^{-iaq}\left[\sum_{j\in\mathbb{N}}|b_{i,j}(x)|\right]^q\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)\right\}\,dx\\
\lesssim&2^{-i_0q(1-a)}\sum_{i=-\infty}^{i_0-1}2^{-iaq}\int_{\mathbb{R}^n}\sum_{j\in\mathbb{N}}|b_{i,j}(x)|^q\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)\,dx\\
\lesssim&2^{-i_0q(1-a)}\sum_{i=-\infty}^{i_0-1}2^{-iaq}\sum_{j\in\mathbb{N}}2^{iq}\varphi\left(B_{i,j},\,\frac{2^{i_0}}{\lambda}\right)\,dx\\
\lesssim&2^{i_0[1-q(1-a)]}\sum_{i=-\infty}^{i_0-1}2^{i[q(1-a)-1]}\sup_i\left[\sum_{j\in\mathbb{N}}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right]\sim\sup_i\left\{\sum_{j\in\mathbb{N}}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right\},
\end{aligned}
$$

which is wished.

For F_2, let $x_{i,j}$ denote the center of $B_{i,j}$ and $r_{i,j}$ its radius, and

$$A_{i_0}:=\bigcup_{i=i_0}^{\infty}\bigcup_{j\in\mathbb{N}}\widetilde{B_{i,j}},\widetilde{B_{i,j}}:=B\left(x_{i,j},\,\left(\frac{3}{2}\right)^{p(i-i_0)/n}2r_{i,j}\right).$$

To prove that

$$\varphi\left(\left\{x\in\mathbb{R}^n: T_*^\delta(F_2)(x)>2^{i_0}\right\},\, \frac{2^{i_0}}{\lambda}\right)\lesssim\sup_i\left\{\sum_{j\in\mathbb{N}}\left[\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right]\right\},$$

we cut $\{x\in\mathbb{R}^n: T_*^\delta(F_2)(x)>2^{i_0}\}$ into A_{i_0} and $\{x\in(A_{i_0})^{\complement}: T_*^\delta(F_2)(x)>2^{i_0}\}$. Since φ is of uniformly lower type p, $\varphi\in\mathbb{A}_1$, and by Lemma 3.4, it follows that, for any $\lambda\in(0,\infty)$,

$$
\begin{aligned}
&\varphi\left(\left\{x\in A_{i_0}: T_*^\delta(F_2)(x)>2^{i_0}\right\},\, \frac{2^{i_0}}{\lambda}\right)\le\varphi\left(A_{i_0},\,\frac{2^{i_0}}{\lambda}\right)\\
\le&\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}}\varphi\left(\widetilde{B_{i,j}},\,\frac{2^{i_0}}{\lambda}\right)\lesssim\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}}\left[\left(\frac{3}{2}\right)^{p(i-i_0)/n}\right]^n2^{(i_0-i)p}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\\
\sim&\sum_{i=i_0}^{\infty}\left(\frac{3}{4}\right)^{p(i-i_0)}\sum_{j\in\mathbb{N}}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\lesssim\sum_{i=i_0}^{\infty}\left(\frac{3}{4}\right)^{p(i-i_0)}\sup_i\left\{\sum_{j\in\mathbb{N}}\left[\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right]\right\}\\
\sim&\sup_i\left\{\sum_{j\in\mathbb{N}}\left[\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right]\right\},
\end{aligned}
$$

which is also wished.

Let $p_1 := 2n/(n+1+2\delta)$, since $\delta > n/p - (n+1)/2$, we have $p_1 < p$. For any $(x, t) \in \mathbb{R}^n \times (0, \infty)$, set $\varphi_1(x, t) := \varphi(x, t)t^{p_1-p}$, then φ_1 is a Musielak-Orlicz function of uniformly lower type p_1 and of uniformly upper type $1+p_1-p$. It is easy to see that

$$b^1_{i,j} := \|\chi_{B_{i,j}}\|^{-1}_{L^{\varphi_1}}\|b_{i,j}\|^{-1}_{L^\infty} b_{i,j}$$

is a $(\varphi_1, \infty, \lfloor n(q(\varphi)/p_1 - 1)\rfloor)$-atom associated with the ball $B_{i,j}$. By this and Lemma 3.5, we know that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
T^\delta_*(b_{i,j})(x) &= T^\delta_*\left(\|b_{i,j}\|_{L^\infty}\|\chi_{B_{i,j}}\|_{L^{\varphi_1}} b^1_{i,j}\right)(x)\\
&= \|b_{i,j}\|_{L^\infty}\|\chi_{B_{i,j}}\|_{L^{\varphi_1}} T^\delta_*\left(b^1_{i,j}\right)(x)\\
&\lesssim \|b_{i,j}\|_{L^\infty}\|\chi_{B_{i,j}}\|_{L^{\varphi_1}}\left(\frac{r_{i,j}}{r_{i,j}+|x-x_{i,j}|}\right)^{n/p_1}\|b^1_{i,j}\|_{L^\infty}\\
&\lesssim \|b_{i,j}\|_{L^\infty}\left(\frac{r_{i,j}}{r_{i,j}+|x-x_{i,j}|}\right)^{n/p_1}\\
&\lesssim 2^i\left(\frac{r_{i,j}}{|x-x_{i,j}|}\right)^{n/p_1},
\end{aligned}$$

from this, by Lemma 3.6, Lemma 3.4 with $\varphi \in \mathbb{A}_1$, and the uniformly lower type p property of φ, it follows that

$$\begin{aligned}
&\varphi\left(\left\{x \in (A_{i_0})^\complement : T^\delta_*(F_2)(x) > 2^{i_0}\right\}, \frac{2^{i_0}}{\lambda}\right)\\
\le& 2^{-i_0 p}\sum_{i=i_0}^\infty\sum_{j\in\mathbb{N}}\int_{(\widetilde{B_{i,j}})^\complement}|T^\delta_*(b_{i,j})(x)|^p\varphi\left(x, \frac{2^{i_0}}{\lambda}\right)dx\\
\lesssim& 2^{-i_0 p}\sum_{i=i_0}^\infty\sum_{j\in\mathbb{N}}\int_{(\widetilde{B_{i,j}})^\complement}2^{ip}\left(\frac{r_{i,j}}{|x-x_{i,j}|}\right)^{pn/p_1}\varphi\left(x, \frac{2^{i_0}}{\lambda}\right)dx\\
\lesssim& 2^{-i_0 p}\sum_{i=i_0}^\infty\sum_{j\in\mathbb{N}}(r_{i,j})^{pn/p_1}2^{ip}\int_{(\widetilde{B_{i,j}})^\complement}\left(\frac{1}{|x-x_{i,j}|}\right)^{pn/p_1}\varphi\left(x, \frac{2^{i_0}}{\lambda}\right)dx\\
\lesssim& 2^{-i_0 p}\sum_{i=i_0}^\infty\sum_{j\in\mathbb{N}}(r_{i,j})^{pn/p_1}2^{ip}\left[\left(\frac{3}{2}\right)^{p(i-i_0)/n}2r_{i,j}\right]^{-pn/p_1}\varphi\left(B\left(x_{i,j}, \left(\frac{3}{2}\right)^{p(i-i_0)/n}2r_{i,j}\right), \frac{2^{i_0}}{\lambda}\right)\\
\lesssim& 2^{-i_0 p}\sum_{i=i_0}^\infty\sum_{j\in\mathbb{N}}2^{ip}\left(\frac{3}{2}\right)^{p^2(i_0-i)/p_1}\left(\frac{3}{2}\right)^{p(i-i_0)}2^{(i_0-i)p}\varphi\left(B(x_{i,j}, r_{i,j}), \frac{2^i}{\lambda}\right)\\
\lesssim& \sum_{i=i_0}^\infty\left(\frac{3}{2}\right)^{p(i-i_0)(1-p/p_1)}\sup_i\left\{\sum_{j\in\mathbb{N}}\left[\varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right]\right\}\\
\sim& \sup_i\left\{\sum_{j\in\mathbb{N}}\left[\varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right]\right\}.
\end{aligned}$$

This finishes the proof of Theorem 2.6.

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极大Bochner-Riesz平均在弱Musielak-Orlicz Hardy 空间上的估计

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摘要: 本文研究了极大 Bochner-Riesz 平均的有界性. 利用极大 Bochner-Riesz 平均的点态估计及弱 Musielak-Orlicz Hardy 空间的原子分解, 得到了极大 Bochner-Riesz 平均从弱 Musielak-Orlicz Hardy 空间到弱 Musielak-Orlicz 空间是有界的. 即使对任意的 $(x, t) \in \mathbb{R}^n \times [0, \infty)$, 当 Musielak-Orlicz 函数 $\varphi(x, t)$ 取为特殊的 Orlicz 函数 $\Phi(t)$ 时, 上述结果也是新的. 这个结果是王华加权空间上的结果 (见文献[1]) 在 Musielak-Orlicz 空间情形下的推广.

关键词: Bochner-Riesz 平均; Muckenhoupt 权; Orlicz 函数; Hardy 空间

MR(2010)主题分类号: 42B30; 46E30 **中图分类号**: O174.2