Vol. 39 (2019)
No. 5

数 学 杂 志
J. of Math. (PRC)

LAZY 2-COCYCLE ON RADFORD BIPRODUCT HOM-HOPF ALGEBRA

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Abstract: In this paper, we study Lazy 2-cocycle on Radford's biproduct Hom-Hopf algebra. By using twisting method, we mainly investigate the relations between the left Hom-2-cocycles σ on (B,β) and $\overline{\sigma}$ on $(B^{\#}_{\times}H,\beta\otimes\alpha)$ which generalise the corresponding results in the case of usual Hopf algebras.

Keywords: lazy 2-cocycle; Radford's biproduct Hom-Hopf algebra; Yetter-Drinfeld category

2010 MR Subject Classification: 16T05

Document code: A **Article ID:** 0255-7797(2019)05-0677-17

1 Introduction and Preliminaries

A lazy 2-cocycle of a Hopf algebra H is a 2-cocycle $\sigma: H\otimes H\longrightarrow K$, which commutes with multiplication in the Hopf algebra. The second lazy cohomology group generalizes Sweedler's second cohomology group of a cocommutative Hopf algebra and the Schur multiplier of a group. Let $B\Diamond H$ be a Radford biproduct, where H is a Hopf algebra and B is a Hopf algebra in the category of Yetter-Drinfeld modules over H. A group morphism $H^2_L(B)\longrightarrow H^2_L(B\Diamond H)$ is constructed by Cuadra and Panaite in [1]. In [2], Panaite et al. introduced the concepts of pure and neat lazy 2-cocycle and extended pure and neat lazy cocycles to the Radford biproducts.

The origins of the study of Hom-algebras can be found in [3] by Hartwig, Larsson and Silvestrov, and earlier precursors of Hom-Lie algebras can be found in Hu's paper (see [4]). Subsequently, Hom-type algebra has been studied by many researchers. Especially, in 2014, Li and Ma introduced the notions of Radford biproduct Hom-Hopf algebra $(B^{\#}_{\times}H,\beta\otimes\alpha)$ and Hom-Yetter-Drinfeld category ${}^H_H\mathbb{YD}$ (see [5]), which generalize the corresponding concepts in usual Hopf algebras. In 2017, the authors presented a more general version of $(B^{\#}_{\times}H,\beta\otimes\alpha)$ (see [6]).

Radford biproduct Hom-Hopf algebra was given below.

* **Received date:** 2018-08-9 **Accepted date:** 2018-11-05

Foundation item: Supported by China Postdoctoral Science Foundation (2017M611291) and Natural Science Foundation of Henan Province (17A110007).

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Theorem 1.1 Let (H,α) be a Hom-bialgebra, (B,β) a left (H,α)-module Hom-algebra with module structure $\triangleright : H\otimes B \longrightarrow B$ and a left (H,α)-comodule Hom-coalgebra with comodule structure $\rho : B \longrightarrow H\otimes B$. Then the following conclusions are equivalent.

(i) $(B^{\#}_{\times}H,\beta\otimes\alpha)$ is a Hom-bialgebra, where $(B\#H,\beta\otimes\alpha)$ is a smash product Hom-algebra (see [7]) and $(B\times H,\beta\otimes\alpha)$ is a smash coproduct Hom-coalgebra.

(ii) The following conditions hold ($\forall\, a,b\in B$ and $h\in H$)

(R1) (B,ρ,α) is an (H,β)-comodule Hom-algebra,

(R2) $(B,\triangleright,\alpha)$ is an (H,β)-module Hom-coalgebra,

(R3) ε_B is a Hom-algebra map and $\Delta_B(1_B)=1_B\otimes 1_B$,

(R4) $\Delta_B(ab)=a_1(\alpha^2(a_{2-1})\triangleright\beta^{-1}(b_1))\otimes\beta^{-1}(a_{20})b_2$,

(R5) $h_1\alpha(a_{-1})\otimes(\alpha^3(h_2)\triangleright a_0)=(\alpha^2(h_1)\triangleright a)_{-1}h_2\otimes(\alpha^2(h_1)\triangleright a)_0$.

Definition 1.2 Let (H,α) be a Hom-bialgebra, $(M,\triangleright_M,\alpha_M)$ a left (H,α)-module with action $\triangleright_M : H\otimes M\longrightarrow M, h\otimes m\mapsto h\triangleright_M m$ and (M,ρ^M,α_M) a left (H,α)-comodule with coaction $\rho^M : M\longrightarrow H\otimes M, m\mapsto m_{-1}\otimes m_0$. Then we call $(M,\triangleright_M,\rho^M,\alpha_M)$ a (left-left) Hom-Yetter-Drinfeld module over (H,α) if the following condition holds:

$$\text{(HYD)}\quad h_1\alpha(m_{-1})\otimes(\alpha^3(h_2)\triangleright_M m_0)=(\alpha^2(h_1)\triangleright_M m)_{-1}h_2\otimes(\alpha^2(h_1)\triangleright_M m)_0,$$

where $h\in H$ and $m\in M$.

When (H,α) is a Hom-Hopf algebra, then the condition (HYD) is equivalent to

$$\text{(HYD)}'\ (\alpha^4(h)\triangleright_M m)_{-1}\otimes(\alpha^4(h)\triangleright_M m)_0=\alpha^{-2}(h_{11}\alpha(m_{-1}))S_H(h_2)\otimes(\alpha^3(h_{12})\triangleright_M m_0).$$

So it is natural to consider the relations between the 2-cocycles σ on (B,β) and $\overline{\sigma}$ on $(B^{\#}_{\times}H,\beta\otimes\alpha)$.

In this paper, we mainly investigate the relations between the left 2-cocycles σ on (B,β) and $\overline{\sigma}$ on $(B^{\#}_{\times}H,\beta\otimes\alpha)$, and also provide two non-trivial examples.

Next we recall some definitions and results in [8] which will be used later.

Definition 1.3 A left 2-cocycle on a Hom-bialgebra (H,α) is a linear map $\sigma : H\otimes H\to K$ satisfying

$$\sigma\circ(\alpha\otimes\alpha)=\sigma, \tag{1.1}$$

$$\sigma(b_1,c_1)\sigma(\alpha^2(a),b_2c_2)=\sigma(a_1,b_1)\sigma(a_2b_2,\alpha^2(c)) \tag{1.2}$$

for all $a,b,c\in H$.

Furthermore, σ is normal if $\sigma(1,h)=\sigma(h,1)=\varepsilon(h)$ for all $h\in H$.

Remarks (1) Similarly if eq. (1.2) is replaced by

$$\sigma(\alpha^2(a),b_1c_1)\sigma(b_2,c_2)=\sigma(a_1b_1,\alpha^2(c))\sigma(a_2,b_2),$$

then σ is a right 2-cocycle.

(2) If $\sigma : H\otimes H\longrightarrow K$ is a normal and convolution invertible, then σ is a left 2-cocycle if and only if σ^{-1} is a right 2-cocycle.

Proposition 1.4 Let (H,α) be a Hom-Hopf algebra.

(1) If σ is a normal left 2-cocyle on (H,α), for all $h,g\in H$, define a new multiplication on H as follows

$$h\cdot_\sigma g=\sigma(h_1,g_1)\alpha^{-1}(h_2g_2). \tag{1.3}$$

Then $(H,\cdot_\sigma,\alpha)$ is a Hom-algebra, we denote the algebra by $({}_\sigma H,\alpha)$.

(2) If σ is a normal right 2-cocyle on (H,α) for all $h,g\in H$, define multiplication on H as follows $h_\sigma\cdot g=\alpha^{-1}(h_1g_1)\sigma(h_2,g_2)$. Then $(H,{}_\sigma\cdot,\alpha)$ is also a Hom-algebra, we denote the algebra by (H_σ,α).

Definition 1.5 A left 2-cocyle σ on (H,α) is called lazy if for all $h,g\in H$,

$$\sigma(h_1,g_1)h_2g_2=h_1g_1\sigma(h_2,g_2). \tag{1.4}$$

Remark A lazy left 2-cocyle on (H,α) is also a right 2-cocyle on (H,α).

Lemma 1.6 Let $\gamma:H\longrightarrow K$ be a normal (i.e. $\gamma(1)=1$) and convolution invertible linear map such that $\gamma\circ\alpha=\gamma$, define $D^1(\gamma):H\otimes H\longrightarrow K$ by

$$D^1(\gamma)(h,g)=\gamma(h_1)\gamma(g_1)\gamma^{-1}(h_2g_2) \tag{1.5}$$

for all $h,g\in H$. Then $D^1(\gamma)$ is a normal and convolution invertible left 2-cocycle on (H,α).

Remarks (1) The set $Reg^1(H,\alpha)$ (respectively $\mathrm{Re}g^2(H,\alpha)$) consisting of normal and convolution invertible linear maps $\gamma:H\longrightarrow K$ such that $\gamma\circ\alpha=\gamma$ (respectively $\sigma:H\otimes H\longrightarrow K$ such that $\sigma\circ(\alpha\otimes\alpha)=\sigma$), is a group with respect to the convolution product.

(2) γ is lazy if for all $h\in H$, $\gamma(h_1)h_2=h_1\gamma(h_2)$. The set of all normal and convolution invertible linear maps $\gamma:H\longrightarrow K$ satisfying $\gamma\circ\alpha=\gamma$ is denoted by $\mathrm{Re}g^1_L(H)$, which is a group under convolution.

Lemma 1.7 The set of convolution invertible lazy 2-cocycle on (H,α) denoted by $Z^2_L(H,\alpha)$ is a group.

Proposition 1.8 $D^1:\mathrm{Re}g^1_L(H,\alpha)\longrightarrow Z^2_L(H,\alpha)$ is a group homomorphism, whose image denoted by $B^2_L(H,\alpha)$ (its elements are called lazy 2-coboundary), is contained in the center of $Z^2_L(H,\alpha)$. Thus we call quotient group $H^2_L(H,\alpha):=Z^2_L(H,\alpha)/B^2_L(H,\alpha)$ the second lazy cohomology group of H.

2 Main Results and Examples

In this section, we investigate the relations between the left 2-cocycles σ on (B,β) and $\overline{\sigma}$ on $(B^{\#}_{\times}H,\beta\otimes\alpha)$, and also provide two non-trivial examples. In what follows, let (H,α) be a Hom-Hopf algebra with bijective antipode S and $(B^{\#}_{\times}H,\beta\otimes\alpha)$ Radford biproduct Hom-Hopf algebra such that $\alpha^2=id$.

First we give some useful formulas. The Hom-coalgebra structure on $(B\otimes B,\beta\otimes\beta)$ in ${}^H_H\mathbb{YD}$ is given by

$$\begin{aligned}\Delta_{B\otimes B}(b\otimes b') &= (id\otimes C_{B,B}\otimes id)\circ(\Delta_B\otimes\Delta_B)(b\otimes b')\\ &= (b_1\otimes b_{2(-1)}\cdot\beta^{-1}(b'_1))\otimes(\beta^{-1}(b_{2(0)})\otimes b'_2).\end{aligned} \tag{2.1}$$

So by (2.1), if $\sigma, \tau : B \otimes B \longrightarrow K$ are morphisms in ${}_H^H\mathbb{YD}$, their convolution in ${}_H^H\mathbb{YD}$ is given by

$$(\sigma * \tau)(b, b') = \sigma(b_1, b_{2(-1)} \cdot \beta^{-1}(b'_1))\tau(\beta^{-1}(b_{2(0)}), b'_2). \tag{2.2}$$

Let $\sigma : B \otimes B \longrightarrow K$ be a morphism in ${}_H^H\mathbb{YD}$, that is, it satisfies the conditions

$$\sigma(h_1 \cdot b, h_2 \cdot b') = \varepsilon(h)\sigma(b, b'), \tag{2.3}$$

$$\sigma(b_{(0)}, b'_{(0)})b_{(-1)}b'_{(-1)} = \sigma(b, b')1_H, \tag{2.4}$$

$$\sigma \circ (\beta \otimes \beta) = \sigma \tag{2.5}$$

for all $h \in H$ and $b, b' \in B$.

Lemma 2.1 For a morphism $\sigma : B \otimes B \longrightarrow K$ in ${}_H^H\mathbb{YD}$, we can get the following useful formula

$$\sigma(a, \alpha(h) \cdot b) = \sigma(S^{-1}(h) \cdot \beta^{-2}(a), b) \tag{2.6}$$

for all $a, b \in B$ and $h \in H$.

Proof We can check that as follows

$$\begin{aligned}
\sigma(a, \alpha(h) \cdot b) &= \sigma((h_{12}S^{-1}(h_{11})) \cdot \beta^{-1}(a), h_2 \cdot b) \\
&= \sigma(\alpha(h_{12}) \cdot (S^{-1}(h_{11}) \cdot \beta^{-2}(a)), h_2 \cdot b) \\
&= \sigma(\alpha(h_{21}) \cdot (\alpha(S^{-1}(h_1)) \cdot \beta^{-2}(a)), \alpha^{-1}(h_{22}) \cdot b) \\
&\overset{(2.3)}{=} \sigma(\alpha(S^{-1}(h_1)) \cdot \beta^{-2}(a), b)\varepsilon(h_2) \\
&= \sigma(S^{-1}(h) \cdot \beta^{-2}(a), b).
\end{aligned}$$

Definition 2.2 Let $\sigma : B \otimes B \longrightarrow K$ be a morphism in ${}_H^H\mathbb{YD}$. Then σ is lazy in ${}_H^H\mathbb{YD}$ if it satisfies the categorical laziness condition (for all $b, b' \in B$)

$$\sigma(b_1, b_{2(-1)} \cdot \beta^{-1}(b'_1))\beta^{-1}(b_{2(0)})b'_2 = \sigma(\beta^{-1}(b_{2(0)}), b'_2)b_1(b_{2(-1)} \cdot \beta^{-1}(b'_1)). \tag{2.7}$$

Definition 2.3 Let $\sigma : B \otimes B \longrightarrow K$ be a morphism in ${}_H^H\mathbb{YD}$. Then σ is a normal left 2-cocycle on (B, β) in ${}_H^H\mathbb{YD}$ if it is a normal morphism in ${}_H^H\mathbb{YD}$ and satisfies the categorical left 2-cocycle condition

$$\begin{aligned}
&\sigma(a_1, a_{2(-1)} \cdot \beta^{-1}(b_1))\sigma(\beta^{-1}(a_{2(0)})b_2, \beta^2(c)) \\
= \ &\sigma(b_1, b_{2(-1)} \cdot \beta^{-1}(c_1))\sigma(\beta^2(a), \beta^{-1}(b_{2(0)})c_2)
\end{aligned} \tag{2.8}$$

for all $a, b, c \in B$.

Proposition 2.4 If we define a Hom-multiplication $\cdot_\sigma$ on (B, β) by

$$b \cdot_\sigma b' = \sigma(b_1, b_{2(-1)} \cdot \beta^{-1}(b'_1))\beta^{-1}(\beta^{-1}(b_{2(0)})b'_2) \tag{2.9}$$

for any $b, b' \in B$, then

(1) $({}_\sigma B,\beta)$ is a Hom-algebra if and only if σ is a normal left 2-cocycle in ${}^H_H\mathbb{YD}$.

(2) $({}_\sigma B,\beta)$ is a left (H,α) Hom-module algebra with the same action as (B,β).

Proof (1) For any $b\in B$, it is easy to check that $b\cdot_\sigma 1_B=\beta(b)$ if and only if $\sigma(b,1_B)=\varepsilon(b)$ and $1_B\cdot_\sigma b=\beta(b)$ if and only if $\sigma(1_B,b)=\varepsilon(b)$. For any $a,b,c\in B$, we have

$$\begin{aligned}\beta(a)\cdot_\sigma(b\cdot_\sigma c)\overset{(2.9)}{=}&\ \beta^{-1}(a_{2(0)})\beta^{-1}((\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))_2)\sigma(\beta(a_1),\alpha(a_{2(-1)})\\&\cdot\beta^{-1}((\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))_1))\sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(c_1))\end{aligned}$$

and

$$\begin{aligned}(a\cdot_\sigma b)\cdot_\sigma\beta(c)\overset{(2.9)}{=}&\ \beta^{-1}(\beta^{-1}((\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_{2(0)})\beta(c_2))\sigma((\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_1,\\&(\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_{2(-1)}\cdot c_1)\sigma(a_1,a_{2(-1)}\cdot\beta^{-1}(b_1)).\end{aligned}$$

Hence, if $\cdot_\sigma$ is Hom-associative, we get

$$\begin{aligned}&\beta^{-1}(a_{2(0)})\beta^{-1}((\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))_2)\sigma(\beta(a_1),\alpha(a_{2(-1)})\cdot\beta^{-1}((\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))_1))\\&\sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(c_1))=\beta^{-1}(\beta^{-1}((\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_{2(0)})\beta(c_2))\sigma((\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_1,\\&(\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_{2(-1)}\cdot c_1)\sigma(a_1,a_{2(-1)}\cdot\beta^{-1}(b_1)).\end{aligned}$$

Applying ε to both sides of the above equation, we get (2.8).

Conversely, if σ is a left 2-cocycle in ${}^H_H\mathbb{YD}$, we have

$$\begin{aligned}\beta(a)\cdot_\sigma(b\cdot_\sigma c)\overset{(2.9)}{=}&\ \beta^{-1}(a_{2(0)})\beta^{-1}((\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))_2)\sigma(\beta(a_1),\alpha(a_{2(-1)})\\&\cdot\beta^{-1}((\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))_1))\sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(c_1))\\\overset{(R4)}{=}&\ \beta^{-1}(a_{2(0)})\beta^{-1}(\beta^{-1}(\beta^{-2}(b_{2(0)})_{2(0)})\beta^{-1}(c_2)_2)\sigma(\beta(a_1),\alpha(a_{2(-1)})\\&\cdot\beta^{-1}(\beta^{-2}(b_{2(0)})_1(\beta^{-2}(b_{2(0)})_{2(-1)}\cdot\beta^{-1}(\beta^{-1}(c_2)_1))))\\&\sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(c_1))\\=&\ \beta^{-1}(a_{2(0)})(\beta^{-4}(b_{2(0)2(0)})\beta^{-2}(c_{22}))\sigma(\beta(a_1),\alpha(a_{2(-1)})\\&\cdot(\beta^{-3}(b_{2(0)1})(\alpha^{-1}(b_{2(0)2(-1)})\cdot\beta^{-3}(c_{21}))))\sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(c_1))\\=&\ \beta^{-1}(a_{2(0)})(\beta^{-4}(b_{22(0)(0)})\beta^{-2}(c_{22}))\\&\sigma(\beta(a_1),\alpha(a_{2(-1)})\cdot(\beta^{-3}(b_{21(0)})(\alpha^{-1}(b_{22(0)(-1)})\cdot\beta^{-3}(c_{21}))))\\&\sigma(b_1,(b_{21(-1)}b_{22(-1)})\cdot\beta^{-1}(c_1))\\=&\ \beta^{-1}(a_{2(0)})(\beta^{-3}(b_{22(0)})\beta^{-2}(c_{22}))\\&\sigma(\beta(a_1),\alpha(a_{2(-1)})\cdot(\beta^{-3}(b_{21(0)})(\alpha^{-1}(b_{22(-1)2})\cdot\beta^{-3}(c_{21}))))\\&\sigma(b_1,(b_{21(-1)}\alpha^{-1}(b_{22(-1)1}))\cdot\beta^{-1}(c_1))\end{aligned}$$

$$
\begin{aligned}
&\overset{(2.6)}{=} && \beta^{-1}(a_{2(0)})(\beta^{-3}(b_{22(0)})\beta^{-2}(c_{22}))\\
& && \sigma(S^{-1}(a_{2(-1)})\cdot\beta^{-1}(a_1),\beta^{-3}(b_{21(0)})(\alpha^{-1}(b_{22(-1)2})\cdot\beta^{-3}(c_{21})))\\
& && \sigma(b_1,\alpha(b_{21(-1)})\cdot(\alpha^{-1}(b_{22(-1)1})\cdot\beta^{-2}(c_1)))\\
&= && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\\
& && \sigma(S^{-1}(a_{2(-1)})\cdot\beta^{-1}(a_1),\beta^{-3}(b_{12(0)})(b_{2(-1)2}\cdot\beta^{-3}(c_{12})))\\
& && \sigma(\beta^{-1}(b_{11}),\alpha(b_{12(-1)})\cdot(b_{2(-1)1}\cdot\beta^{-3}(c_{11})))\\
&= && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\\
& && \sigma(\beta^{2}(S^{-1}(a_{2(-1)})\cdot\beta^{-1}(a_1)),\beta^{-1}(b_{12(0)})(b_{2(-1)2}\cdot\beta^{-1}(c_{12})))\\
& && \sigma(b_{11},b_{12(-1)}\cdot(\alpha(b_{2(-1)1})\cdot\beta^{-2}(c_{11})))\\
&= && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\\
& && \sigma(\beta^{2}(S^{-1}(a_{2(-1)})\cdot\beta^{-1}(a_1)),\beta^{-1}(b_{12(0)})(b_{2(-1)}\cdot\beta^{-1}(c_1))_2)\\
& && \sigma(b_{11},b_{12(-1)}\cdot\beta^{-1}((b_{2(-1)}\cdot\beta^{-1}(c_1))_1))\\
&\overset{(2.8)}{=} && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma((S^{-1}(a_{2(-1)})\cdot\beta^{-1}(a_1))_1,\\
& && (S^{-1}(a_{2(-1)})\cdot\beta^{-1}(a_1))_{2(-1)}\cdot\beta^{-1}(b_{11}))\\
& && \sigma(\beta^{-1}((S^{-1}(a_{2(-1)})\cdot\beta^{-1}(a_1))_{2(0)})b_{12},b_{2(-1)}\cdot\beta(c_1))\\
&= && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(S^{-1}(a_{2(-1)})_1\cdot\beta^{-1}(a_1)_1,\\
& && (S^{-1}(a_{2(-1)})_2\cdot\beta^{-1}(a_1)_2)_{(-1)}\cdot\beta^{-1}(b_{11}))\\
& && \sigma(\beta^{-1}((S^{-1}(a_{2(-1)})_2\cdot\beta^{-1}(a_1)_2)_{(0)})b_{12},b_{2(-1)}\cdot\beta(c_1))\\
&= && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(S^{-1}(a_{2(-1)2})\cdot\beta^{-1}(a_{11}),\\
& && (S^{-1}(a_{2(-1)1})\cdot\beta^{-1}(a_{12}))_{(-1)}\cdot\beta^{-1}(b_{11}))\\
& && \sigma(\beta^{-1}((S^{-1}(a_{2(-1)1})\cdot\beta^{-1}(a_{12}))_{(0)})b_{12},b_{2(-1)}\cdot\beta(c_1))\\
&\overset{(\mathrm{HYD})'}{=} && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(S^{-1}(a_{2(-1)2})\cdot\beta^{-1}(a_{11}),\\
& && ((S^{-1}(a_{2(-1)1})_{11}a_{12(-1)})S(S^{-1}(a_{2(-1)1})_2))\cdot\beta^{-1}(b_{11}))\\
& && \sigma(\beta^{-1}(\alpha(S^{-1}(a_{2(-1)1})_{12})\cdot\beta^{-1}(a_{12(0)}))b_{12},b_{2(-1)}\cdot\beta(c_1))\\
&= && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(S^{-1}(a_{2(-1)2})\cdot\beta^{-1}(a_{11}),\\
& && ((S^{-1}(a_{2(-1)122})a_{12(-1)})a_{2(-1)11})\cdot\beta^{-1}(b_{11}))\\
& && \sigma((S^{-1}(a_{2(-1)121})\cdot\beta^{-2}(a_{12(0)}))b_{12},b_{2(-1)}\cdot\beta(c_1))\\
&= && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(S^{-1}(\alpha^{-1}(a_{2(-1)2}))\cdot\beta^{-2}(a_{11}),\\
& && ((S^{-1}(\alpha^{-1}(a_{2(-1)122}))\alpha^{-1}(a_{12(-1)}))\alpha^{-1}(a_{2(-1)11}))\cdot\beta^{-2}(b_{11}))\\
& && \sigma((S^{-1}(a_{2(-1)121})\cdot\beta^{-2}(a_{12(0)}))b_{12},b_{2(-1)}\cdot\beta(c_1))\\
&\overset{(2.6)}{=} && \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},a_{2(-1)2}\\
& && \cdot(((S^{-1}(\alpha^{-1}(a_{2(-1)122}))\alpha^{-1}(a_{12(-1)}))\alpha^{-1}(a_{2(-1)11}))\cdot\beta^{-2}(b_{11})))\\
& && \sigma((S^{-1}(a_{2(-1)121})\cdot\beta^{-2}(a_{12(0)}))b_{12},b_{2(-1)}\cdot\beta(c_1))
\end{aligned}
$$

$$
\begin{aligned}
&= \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},(\alpha^{-1}(a_{2(-1)2})\\
&\quad((S^{-1}(\alpha^{-1}(a_{2(-1)122}))\alpha^{-1}(a_{12(-1)}))\alpha^{-1}(a_{2(-1)11}))) \cdot \beta^{-1}(b_{11}))\\
&\quad\sigma((S^{-1}(a_{2(-1)121}) \cdot \beta^{-2}(a_{12(0)}))b_{12}, b_{2(-1)} \cdot \beta(c_1))\\
&= \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},(\alpha^{-1}(a_{2(-1)222})\\
&\quad((S^{-1}(\alpha^{-1}(a_{2(-1)221}))\alpha^{-1}(a_{12(-1)}))a_{2(-1)1})) \cdot \beta^{-1}(b_{11}))\\
&\quad\sigma((S^{-1}(\alpha(a_{2(-1)21})) \cdot \beta^{-2}(a_{12(0)}))b_{12}, b_{2(-1)} \cdot \beta(c_1))\\
&= \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},((a_{2(-1)222}\\
&\quad S^{-1}(a_{2(-1)221}))(a_{12(-1)}a_{2(-1)1})) \cdot \beta^{-1}(b_{11}))\\
&\quad\sigma((S^{-1}(\alpha(a_{2(-1)21})) \cdot \beta^{-2}(a_{12(0)}))b_{12}, b_{2(-1)} \cdot \beta(c_1))\\
&= \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},\alpha(a_{12(-1)}a_{2(-1)1}) \cdot \beta^{-1}(b_{11}))\\
&\quad\varepsilon(a_{2(-1)22})\sigma((S^{-1}(\alpha(a_{2(-1)21})) \cdot \beta^{-2}(a_{12(0)}))b_{12}, b_{2(-1)} \cdot \beta(c_1))\\
&\overset{(2.3)}{=} \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},\alpha(a_{12(-1)}a_{2(-1)1}) \cdot \beta^{-1}(b_{11}))\\
&\quad\sigma(a_{2(-1)221} \cdot ((S^{-1}(\alpha(a_{2(-1)21})) \cdot \beta^{-2}(a_{12(0)}))b_{12}), a_{2(-1)222}\\
&\quad\cdot(b_{2(-1)} \cdot \beta(c_1)))\\
&= \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},\alpha(a_{12(-1)}a_{2(-1)1}) \cdot \beta^{-1}(b_{11}))\\
&\quad\sigma((a_{2(-1)2211} \cdot (S^{-1}(\alpha(a_{2(-1)21})) \cdot \beta^{-2}(a_{12(0)})))(a_{2(-1)2212} \cdot b_{12}),\\
&\quad(\alpha^{-1}(a_{2(-1)222})b_{2(-1)}) \cdot \beta^2(c_1))\\
&= \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},\alpha(a_{12(-1)}a_{2(-1)1}) \cdot \beta^{-1}(b_{11}))\\
&\quad\sigma(((\alpha^{-1}(a_{2(-1)2211})S^{-1}(\alpha(a_{2(-1)21}))) \cdot \beta^{-1}(a_{12(0)}))(a_{2(-1)2212} \cdot b_{12}),\\
&\quad(\alpha^{-1}(a_{2(-1)222})b_{2(-1)}) \cdot \beta^2(c_1))\\
&= \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},\alpha(a_{12(-1)}a_{2(-1)1}) \cdot \beta^{-1}(b_{11}))\\
&\quad\sigma(((\alpha^{-1}(a_{2(-1)2112})S^{-1}(\alpha^{-1}(a_{2(-1)2111}))) \cdot \beta^{-1}(a_{12(0)}))\\
&\quad(\alpha(a_{2(-1)212}) \cdot b_{12}), (a_{2(-1)22}b_{2(-1)}) \cdot \beta^2(c_1))\\
&= \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11},\alpha(a_{12(-1)}a_{2(-1)1}) \cdot \beta^{-1}(b_{11}))\\
&\quad\sigma(a_{12(0)}(a_{2(-1)21} \cdot b_{12}), (a_{2(-1)22}b_{2(-1)}) \cdot \beta^2(c_1))
\end{aligned}
$$

and

$$
\begin{aligned}
(a \cdot_\sigma b) \cdot_\sigma \beta(c) &\overset{(2.9)}{=} \beta^{-1}(\beta^{-1}((\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_{2(0)})\beta(c_2))\sigma((\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_1,\\
&\quad(\beta^{-2}(a_{2(0)})\beta^{-1}(b_2))_{2(-1)} \cdot c_1)\sigma(a_1, a_{2(-1)} \cdot \beta^{-1}(b_1))\\
&\overset{(R4)}{=} \beta^{-2}((\beta^{-1}(\beta^{-2}(a_{2(0)})_{2(0)})\beta^{-1}(b_2)_2)_{(0)})c_2\\
&\quad\sigma(\beta^{-2}(a_{2(0)})_1(\beta^{-2}(a_{2(0)})_{2(-1)} \cdot \beta^{-1}(\beta^{-1}(b_2)_1)),\\
&\quad(\beta^{-1}(\beta^{-2}(a_{2(0)})_{2(0)})\beta^{-1}(b_2)_2)_{(-1)} \cdot c_1)\sigma(a_1, a_{2(-1)} \cdot \beta^{-1}(b_1))\\
&= (\beta^{-5}(a_{2(0)2(0)(0)})\beta^{-3}(b_{22(0)}))c_2\sigma(\beta^{-2}(a_{2(0)1})(a_{2(0)2(-1)} \cdot \beta^{-2}(b_{21})),
\end{aligned}
$$

$$
\begin{aligned}
& (\alpha^{-3}(a_{2(0)2(0)(-1)})\alpha^{-1}(b_{22(-1)}))\cdot c_1)\sigma(a_1, a_{2(-1)}\cdot\beta^{-1}(b_1))\\
= \;& (\beta^{-4}(a_{2(0)2(0)})\beta^{-3}(b_{22(0)}))c_2\sigma(\beta^{-2}(a_{2(0)1})(\alpha^{-1}(a_{2(0)2(-1)1})\cdot\beta^{-2}(b_{21})),\\
& (\alpha^{-1}(a_{2(0)2(-1)2})\alpha^{-1}(b_{22(-1)}))\cdot c_1)\sigma(a_1, a_{2(-1)}\cdot\beta^{-1}(b_1))\\
= \;& (\beta^{-4}(a_{22(0)(0)})\beta^{-3}(b_{22(0)}))c_2\sigma(\beta^{-2}(a_{21(0)})(\alpha^{-1}(a_{22(0)(-1)1})\cdot\beta^{-2}(b_{21})),\\
& (\alpha^{-1}(a_{22(0)(-1)2})\alpha^{-1}(b_{22(-1)}))\cdot c_1)\sigma(a_1, (a_{21(-1)}a_{22(-1)})\cdot\beta^{-1}(b_1))\\
= \;& (\beta^{-3}(a_{22(0)})\beta^{-3}(b_{22(0)}))c_2\sigma(\beta^{-2}(a_{21(0)})(\alpha^{-1}(a_{22(-1)21})\cdot\beta^{-2}(b_{21})),\\
& (\alpha^{-1}(a_{22(-1)22})\alpha^{-1}(b_{22(-1)}))\cdot c_1)\sigma(a_1, (a_{21(-1)}\alpha^{-1}(a_{22(-1)1}))\cdot\beta^{-1}(b_1))\\
= \;& (\beta^{-2}(a_{2(0)})\beta^{-2}(b_{2(0)}))c_2\sigma(\beta^{-2}(a_{12(0)})(a_{2(-1)21}\cdot\beta^{-2}(b_{12})),\\
& (a_{2(-1)22}b_{2(-1)})\cdot c_1)\sigma(\beta^{-1}(a_{11}), (a_{12(-1)}a_{2(-1)1})\cdot\beta^{-2}(b_{11}))\\
= \;& \beta^{-1}(a_{2(0)})(\beta^{-2}(b_{2(0)})\beta^{-1}(c_2))\sigma(a_{11}, \alpha(a_{12(-1)}a_{2(-1)1})\cdot\beta^{-1}(b_{11}))\\
& \sigma(a_{12(0)}(a_{2(-1)21}\cdot b_{12}), (a_{2(-1)22}b_{2(-1)})\cdot\beta^2(c_1)),
\end{aligned}
$$

then we can get $\beta(a)\cdot_\sigma(b\cdot_\sigma c)=(a\cdot_\sigma b)\cdot_\sigma\beta(c)$, i.e., $\cdot_\sigma$ is Hom-associative.

(2) We check that $({}_\sigma B,\beta)$ is a left (H,α)-Hom-module algebra. Clearly, $h\cdot 1_B=\varepsilon(h)1_B$ for any $h\in H$. Next we only need to check the identity $h\cdot(b\cdot_\sigma b')=(h_1\cdot b)\cdot_\sigma(h_2\cdot b')$ for any $h\in H$ and $b,b'\in B$. Indeed, we have

$$
\begin{aligned}
& (h_1\cdot b)\cdot_\sigma(h_2\cdot b')\\
\overset{(2.9)}{=}\;& \sigma(h_{11}\cdot b_1, (h_{12}\cdot b_2)_{(-1)}\cdot\beta^{-1}(h_{21}\cdot b'_1))\beta^{-1}(\beta^{-1}((h_{12}\cdot b_2)_{(0)})(h_{22}\cdot b'_2))\\
\overset{(\mathrm{HYD})'}{=}\;& \sigma(h_{11}\cdot b_1, (h_{1211}\alpha(b_{2(-1)}))S(h_{122})\cdot\beta^{-1}(h_{21}\cdot b'_1))\\
& \beta^{-1}(\beta^{-1}(\alpha(h_{1212})\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
=\;& \sigma(h_{11}\cdot b_1, (\alpha(h_{121})\alpha(b_{2(-1)}))S(\alpha^{-1}(h_{1222}))\cdot\beta^{-1}(h_{21}\cdot b'_1))\\
& \beta^{-1}(\beta^{-1}(\alpha(h_{1221})\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
=\;& \sigma(\alpha^{-1}(h_{111})\cdot b_1, ((\alpha(h_{112})\alpha(b_{2(-1)}))S(h_{122}))\cdot\beta^{-1}(h_{21}\cdot b'_1))\\
& \beta^{-1}(\beta^{-1}(h_{121}\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
=\;& \sigma(\alpha^{-1}(h_{111})\cdot b_1, h_{112}(\alpha(b_{2(-1)})S(\alpha^{-1}(h_{122})))\cdot\beta^{-1}(h_{21}\cdot b'_1))\\
& \beta^{-1}(\beta^{-1}(h_{121}\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
=\;& \sigma(\alpha^{-1}(h_{111})\cdot b_1, \alpha(h_{112})\cdot((\alpha(b_{2(-1)})S(\alpha^{-1}(h_{122})))\cdot\beta^{-2}(h_{21}\cdot b'_1)))\\
& \beta^{-1}(\beta^{-1}(h_{121}\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
=\;& \sigma(\alpha^{-1}(h_{111})\cdot b_1, \alpha^{-1}(h_{112})\cdot((\alpha(b_{2(-1)})S(\alpha^{-1}(h_{122})))\cdot\beta^{-2}(h_{21}\cdot b'_1)))\\
& \beta^{-1}(\beta^{-1}(h_{121}\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
\overset{(2.3)}{=}\;& \sigma(b_1, \alpha(b_{2(-1)})S(h_{12})\cdot\beta^{-2}(h_{21}\cdot b'_1))\beta^{-1}(\beta^{-1}(\alpha(h_{11})\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
=\;& \sigma(b_1, \alpha^{-1}(\alpha(b_{2(-1)})S(h_{12}))h_{21}\cdot\beta^{-1}(b'_1))\beta^{-1}(\beta^{-1}(\alpha(h_{11})\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
=\;& \sigma(b_1, \alpha(b_{2(-1)})(S(\alpha^{-1}(h_{12}))\alpha^{-1}(h_{21}))\cdot\beta^{-1}(b'_1))\beta^{-1}(\beta^{-1}(\alpha(h_{11})\cdot b_{2(0)})(h_{22}\cdot b'_2))\\
=\;& \sigma(b_1, \alpha(b_{2(-1)})(S(h_{211})h_{212})\cdot\beta^{-1}(b'_1))\beta^{-1}(\beta^{-1}(h_1\cdot b_{2(0)})(h_{22}\cdot b'_2))
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma(b_1, b_{2(-1)} \cdot \beta^{-1}(b'_1))(h_1 \cdot \beta^{-2}(b_{2(0)}))(h_2 \cdot \beta^{-1}(b'_2)) \\
&= h \cdot (b \cdot_\sigma b'),
\end{aligned}
$$

the proof is completed.

Let $\gamma : B \longrightarrow K$ be a morphism in ${}_H^H\mathbb{YD}$, that is

$$\gamma(h \cdot b) = \varepsilon(h)\gamma(b), \tag{2.10}$$

$$\gamma(b_{(0)})b_{(-1)} = \gamma(b)1_H, \tag{2.11}$$

$$\gamma \circ \beta = \gamma \tag{2.12}$$

for all $h \in H$ and $b \in B$.

If γ is a normal and convolution invertible linear map in ${}_H^H\mathbb{YD}$, with convolution inverse γ^{-1} in ${}_H^H\mathbb{YD}$, the analogue of the operator D^1 is given in ${}_H^H\mathbb{YD}$ by

$$
\begin{aligned}
D^1(\gamma)(b, b') &= \gamma(b_1)\gamma(b_{2(-1)} \cdot \beta^{-1}(b'_1))\gamma^{-1}(\beta^{-1}(b_{2(0)})b'_2) \\
&\overset{(2.10)}{=} \gamma(b_1)\gamma(b'_1)\gamma^{-1}(b_2 b'_2).
\end{aligned}
$$

For a morphism $\gamma : B \longrightarrow K$ in ${}_H^H\mathbb{YD}$, the laziness condition is identical to the usual one: $\gamma(b_1)b_2 = b_1\gamma(b_2)$ for all $b \in B$. $\mathrm{Reg}_L^1(B, \beta)$ is a group in ${}_H^H\mathbb{YD}$.

Theorem 2.5 (i) For a normal left 2-cocycle $\sigma : B \otimes B \longrightarrow K$ in ${}_H^H\mathbb{YD}$, define $\overline{\sigma} : B^{\#}_{\times}H \otimes B^{\#}_{\times}H \longrightarrow K$ by

$$\overline{\sigma}(b \otimes h, b' \otimes h') = \sigma(b, \alpha(h) \cdot \beta^{-1}(b'))\varepsilon(h'), \tag{2.13}$$

then $\overline{\sigma}$ is a normal left 2-cocycle on $(B^{\#}_{\times}H, \beta \otimes \alpha)$ and we have ${}_\sigma B\#H =_{\overline{\sigma}} (B^{\#}_{\times}H)$ as Hom-algebras. Moreover, $\overline{\sigma}$ is unique with this property.

(ii) If $\overline{\sigma}$ is a left 2-cocycle on $(B^{\#}_{\times}H, \beta \otimes \alpha)$, then σ is a left 2-cocycle on (B, β) in ${}_H^H\mathbb{YD}$.

(iii) If σ is convolution invertible in ${}_H^H\mathbb{YD}$, then $\overline{\sigma}$ is convolution invertible, with inverse

$$\overline{\sigma}^{\,-1}(b \otimes h, b' \otimes h') = \sigma^{-1}(b, \alpha(h) \cdot \beta^{-1}(b'))\varepsilon(h'), \tag{2.14}$$

where σ^{-1} is the convolution inverse of σ in ${}_H^H\mathbb{YD}$.

(iv) σ is lazy in ${}_H^H\mathbb{YD}$ if and only if $\overline{\sigma}$ is lazy.

(v) If $\sigma, \tau : B \otimes B \longrightarrow K$ are lazy 2-cocycles in ${}_H^H\mathbb{YD}$, then $\overline{\sigma * \tau} = \overline{\sigma} * \overline{\tau}$, hence the map $\sigma \longrightarrow \overline{\sigma}$ is a group homomorphism from $Z_L^2(B, \beta)$ to $Z_L^2(B^{\#}_{\times}H, \beta \otimes \alpha)$.

(vi) If $\gamma : B \longrightarrow K$ is a normal and convolution invertible morphism in ${}_H^H\mathbb{YD}$, define $\overline{\gamma} : B^{\#}_{\times}H \longrightarrow K$ by

$$\overline{\gamma}(b \otimes h) = \gamma(b)\varepsilon(h), \tag{2.15}$$

then $\overline{\gamma}$ is normal and convolution invertible and $\overline{D^1(\gamma)} = D^1(\overline{\gamma})$. If γ is lazy in ${}_H^H\mathbb{YD}$, then $\overline{\gamma}$ is also lazy.

Proof (i) It is easy to see that $\overline{\sigma}$ is normal. Since ${}_\sigma B\#H$ is a Hom-algebra, we have

$$((\beta\otimes\alpha)(b\#h))((b'\#h')(b''\#h''))=((b\#h)(b'\#h'))((\beta\otimes\alpha)(b''\#h'')). \tag{2.16}$$

Applying $\varepsilon_B\otimes\varepsilon_H$ to both sides of (2.16), then $\overline{\sigma}$ is a normal left 2-cocycle on $(B^{\#}_{\times}H,\beta\otimes\alpha)$.

We prove that the multiplication in ${}_\sigma B\#H$ and ${}_{\overline{\sigma}}(B^{\#}_{\times}H)$ coincide.

$$\begin{aligned}
(b\#h)(b'\#h') &= b\cdot(h_1\cdot\beta^{-1}(b'))\#\alpha^{-1}(h_2)h'\\
&\overset{(2.9)}{=} \sigma(b_1,b_{2(-1)}\cdot(\alpha^{-1}(h_{11})\cdot\beta^{-2}(b'_1)))\\
&\quad\ \beta^{-2}(b_{2(0)})(\alpha^{-1}(h_{12})\cdot\beta^{-2}(b'_2))\#\alpha^{-1}(h_2)h'\\
&= \sigma(b_1,(\alpha(b_{2(-1)})\alpha^{-1}(h_{11}))\cdot\beta^{-1}(b'_1))\\
&\quad\ \beta^{-2}(b_{2(0)})(\alpha^{-1}(h_{12})\cdot\beta^{-2}(b'_2))\#\alpha^{-1}(h_2)h'\\
&= \sigma(b_1,(\alpha(b_{2(-1)})\alpha^{-1}(h_{11}))\cdot\beta^{-1}(b'_1))\varepsilon(b'_{2(-1)})\varepsilon(h'_1)\\
&\quad\ (\beta^{-1}\otimes\alpha^{-1})(\beta^{-1}(b_{2(0)})(h_{12}\cdot\beta^{-2}(b'_{2(0)}))\#h_2h'_2\\
&= \overline{\sigma}(b_1\otimes b_{2(-1)}h_{11},b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))\\
&\quad\ (\beta^{-1}\otimes\alpha^{-1})(\beta^{-1}(b_{2(0)})(h_{12}\cdot\beta^{-2}(b'_{2(0)}))\#h_2h'_2\\
&= \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1),b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))\\
&\quad\ (\beta^{-1}\otimes\alpha^{-1})(\beta^{-1}(b_{2(0)})(h_{21}\cdot\beta^{-2}(b'_{2(0)}))\#\alpha^{-1}(h_{22})h'_2)\\
&= \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1),b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))\\
&\quad\ (\beta^{-1}\otimes\alpha^{-1})((\beta^{-1}(b_{2(0)})\otimes h_2)(\beta^{-1}(b'_{2(0)})\otimes h'_2))\\
&= \overline{\sigma}((b\otimes h)_1,(b'\otimes h')_1)(\beta^{-1}\otimes\alpha^{-1})((b\otimes h)_2(b'\otimes h')_2)\\
&\overset{(1.3)}{=} (b\otimes h)\cdot_{\overline{\sigma}}(b'\otimes h').
\end{aligned}$$

The uniqueness of $\overline{\sigma}$ follows easily by applying $\varepsilon_B\otimes\varepsilon_H$ to the multiplications in ${}_\sigma B\#H$ and ${}_{\overline{\sigma}}(B^{\#}_{\times}H)$. We check that as follows

$$\begin{aligned}
&\quad\ (\varepsilon_B\otimes\varepsilon_H)((b\#h)(b'\#h'))\\
&= \sigma(b_1,(\alpha(b_{2(-1)})\alpha^{-1}(h_{11}))\cdot\beta^{-1}(b'_1))(\varepsilon_B\otimes\varepsilon_H)(\beta^{-2}(b_{2(0)})(\alpha^{-1}(h_{12})\cdot\beta^{-2}(b'_2))\#\alpha^{-1}(h_2)h')\\
&= \sigma(b_1,(\alpha(b_{2(-1)})\alpha^{-1}(h_{11}))\cdot\beta^{-1}(b'_1))\varepsilon(b_{2(0)})\varepsilon(h_{12})\varepsilon(b'_2)\varepsilon(h_2)\varepsilon(h')\\
&= \sigma(\beta(b),h\cdot b')\varepsilon(h')\\
&= \sigma(b,\alpha(h)\cdot\beta^{-1}(b'))\varepsilon(h')
\end{aligned}$$

and

$$\begin{aligned}
&\quad\ (\varepsilon_B\otimes\varepsilon_H)((b\otimes h)\cdot_{\overline{\sigma}}(b'\otimes h'))\\
&= \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1),b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))\\
&\quad\ (\varepsilon_B\otimes\varepsilon_H)(\beta^{-1}\otimes\alpha^{-1})(\beta^{-1}(b_{2(0)})(h_{21}\cdot\beta^{-2}(b'_{2(0)}))\#\alpha^{-1}(h_{22})h'_2)\\
&= \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1),b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))\varepsilon(b_{2(0)})\varepsilon(h_{21})\varepsilon(b'_{2(0)})\varepsilon(h_{22})\varepsilon(h'_2)\\
&= \overline{\sigma}(\beta(b)\otimes\alpha(h),\beta(b')\otimes\alpha(h'))\\
&= \overline{\sigma}(b\otimes h,b'\otimes h').
\end{aligned}$$

(ii) Let $a, b, c \in B$ and $h, g, l \in H$, we have

$$\begin{aligned} & \overline{\sigma}((a\otimes h)_1, (b\otimes g)_1)\overline{\sigma}((a\otimes h)_2(b\otimes g)_2, (\beta^2\otimes\alpha^2)(c\otimes l)) \\ = \; & \overline{\sigma}((b\otimes g)_1, (c\otimes l)_1)\overline{\sigma}((\beta^2\otimes\alpha^2)(a\otimes h), (b\otimes g)_2(c\otimes l)_2). \end{aligned}$$

Then

$$\begin{aligned} LHS &= \overline{\sigma}(a_1\otimes a_{2(-1)}\alpha^{-1}(h_1), b_1\otimes b_{2(-1)}\alpha^{-1}(g_1)) \\ &\quad \overline{\sigma}((\beta^{-1}(a_{2(0)})\otimes h_2)(\beta^{-1}(b_{2(0)})\otimes g_2), \beta^2(c)\otimes l) \\ &= \overline{\sigma}(a_1\otimes a_{2(-1)}\alpha^{-1}(h_1), b_1\otimes b_{2(-1)}\alpha^{-1}(g_1)) \\ &\quad \overline{\sigma}(\beta^{-1}(a_{2(0)})(h_{21}\cdot\beta^{-2}(b_{2(0)}))\otimes\alpha^{-1}(h_{22})g_2, \beta^2(c)\otimes l) \\ &\overset{(2.13)}{=} \sigma(a_1, (\alpha(a_{2(-1)})h_1)\cdot\beta^{-1}(b_1)) \\ &\quad \sigma(\beta^{-1}(a_{2(0)})(h_{21}\cdot\beta^{-1}(b_2)), (h_{22}g)\cdot\beta(c))\varepsilon(l) \end{aligned}$$

and

$$\begin{aligned} RHS &= \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(g_1), c_1\otimes c_{2(-1)}\alpha^{-1}(l_1)) \\ &\quad \overline{\sigma}(\beta^2(a)\otimes h, (\beta^{-1}(b_{2(0)})\otimes g_2)(\beta^{-1}(c_{2(0)})\otimes l_2)) \\ &= \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(g_1), c_1\otimes c_{2(-1)}\alpha^{-1}(l_1)) \\ &\quad \overline{\sigma}(\beta^2(a)\otimes h, \beta^{-1}(b_{2(0)})(g_{21}\cdot\beta^{-2}(c_{2(0)}))\otimes\alpha^{-1}(g_{22})l_2) \\ &\overset{(2.13)}{=} \sigma(b_1, (\alpha(b_{2(-1)})g_1)\cdot\beta^{-1}(c_1)) \\ &\quad \sigma(\beta^2(a), \alpha(h)\cdot(\beta^{-2}(b_{2(0)})(g_2\cdot\beta^{-2}(c_2))))\varepsilon(l). \end{aligned}$$

Let $h = g = l = 1_H$, we can get (2.8).

(iii)

$$\begin{aligned} & (\overline{\sigma}*\overline{\sigma}^{-1})(b\otimes h, b'\otimes h') \\ = \; & \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1), b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))\overline{\sigma}^{-1}(\beta^{-1}(b_{2(0)})\otimes h_2, \beta^{-1}(b'_{2(0)})\otimes h'_2) \\ \overset{(2.13,2.14)}{=} \; & \sigma(b_1, \alpha(b_{2(-1)}\alpha^{-1}(h_1))\cdot\beta^{-1}(b'_1))\varepsilon(b'_{2(-1)})\varepsilon(h'_1) \\ & \sigma^{-1}(\beta^{-1}(b_{2(0)}), \alpha(h_2)\cdot\beta^{-2}(b'_{2(0)}))\varepsilon(h'_2) \\ = \; & \sigma(b_1, b_{2(-1)}\cdot(h_1\cdot\beta^{-2}(b'_1)))\sigma^{-1}(\beta^{-1}(b_{2(0)}), \alpha(h_2)\cdot\beta^{-1}(b'_2))\varepsilon(h') \\ \overset{(2.2)}{=} \; & (\sigma*\sigma^{-1})(b, \alpha(h)\cdot\beta^{-1}(b'))\varepsilon(h') \\ = \; & \varepsilon(b)\varepsilon(h)\varepsilon(b')\varepsilon(h'). \end{aligned}$$

(iv) Now let $b, b' \in B$ and $h, h' \in H$ and assume that σ is lazy in ${}^H_H\mathbb{YD}$, then we prove (1.4) for $\overline{\sigma}$ on $(B^{\#}_{\times}H, \beta\otimes\alpha)$ as follows:

$$\begin{aligned} RHS &= \overline{\sigma}((b\otimes h)_2, (b'\otimes h')_2)(b\otimes h)_1(b'\otimes h')_1 \\ &= \overline{\sigma}(\beta^{-1}(b_{2(0)})\otimes h_2, \beta^{-1}(b'_{2(0)})\otimes h'_2)(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1))(b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1)) \\ &\overset{(2.13)}{=} \sigma(\beta^{-1}(b_{2(0)}), \alpha(h_2)\cdot\beta^{-2}(b'_{2(0)}))(b_1((b_{2(-1)1}\alpha^{-1}(h_{11}))\cdot\beta^{-1}(b'_1)) \\ &\quad \otimes(\alpha^{-1}(b_{2(-1)2})h_{12})(b'_{2(-1)})h') \end{aligned}$$

$$\begin{aligned}
&= \sigma(\beta^{-1}(b_{2(0)}), \alpha(h_2)\cdot\beta^{-2}(b'_{2(0)}))(b_1((b_{2(-1)1}\alpha^{-1}(h_{11}))\cdot\beta^{-1}(b'_1))\\
&\quad \otimes(\alpha^{-1}(b_{2(-1)2})h_{12})(b'_{2(-1)}h'))\\
&= \sigma(\beta^{-2}(b_{2(0)(0)}), \alpha(h_2)\cdot\beta^{-2}(b'_{2(0)}))(b_1((\alpha(b_{2(-1)})\alpha^{-1}(h_{11}))\cdot\beta^{-1}(b'_1))\\
&\quad \otimes(\alpha^{-1}(b_{2(0)(-1)})h_{12})(b'_{2(-1)}h'))\\
&= \sigma(\beta^{-2}(b_{2(0)(0)}), h_{22}\cdot\beta^{-2}(b'_{2(0)}))(b_1((\alpha(b_{2(-1)})h_1)\cdot\beta^{-1}(b'_1))\\
&\quad \otimes(\alpha^{-1}(b_{2(0)(-1)})h_{21})(b'_{2(-1)}h'))\\
&= \sigma(\beta^{-2}(b_{2(0)(0)}), h_{22}\cdot\beta^{-2}(b'_{2(0)}))(b_1((\alpha(b_{2(-1)})h_1)\cdot\beta^{-1}(b'_1))\\
&\quad \otimes(\alpha^{-1}(b_{2(0)(-1)})(\alpha^{-1}(h_{21})\alpha^{-1}(b'_{2(-1)})))\alpha(h'))\\
&= \sigma(b_{2(0)(0)}, h_{22}\cdot b'_{2(0)})(b_1((\alpha(b_{2(-1)})h_1)\cdot\beta^{-1}(b'_1))\\
&\quad \otimes(\alpha^{-1}(b_{2(0)(-1)})\alpha^{-1}(h_{21}b'_{2(-1)}))\alpha(h'))\\
&\overset{(R5)}{=} \sigma(\beta(b_{2(0)(0)}), (h_{21}\cdot\beta(b'_2))_{(0)})(b_1((\alpha(b_{2(-1)})h_1)\cdot\beta^{-1}(b'_1))\\
&\quad \otimes(\alpha^{-1}(b_{2(0)(-1)})\alpha^{-1}(((h_{21}\cdot\beta(b'_2))_{(-1)})h_{22}))\alpha^{-1}(h'))\\
&= \sigma(\beta(b_{2(0)(0)}), (h_{21}\cdot\beta(b'_2))_{(0)})(b_1((\alpha(b_{2(-1)})h_1)\cdot\beta^{-1}(b'_1))\\
&\quad \otimes((\alpha^{-2}(b_{2(0)(-1)})\alpha^{-1}((h_{21}\cdot\beta(b'_2))_{(-1)}))h_{22})\alpha(h'))\\
&= \sigma(b_{2(0)(0)}, \beta^{-1}(h_{21}\cdot\beta(b'_2))_{(0)})(b_1((\alpha(b_{2(-1)})h_1)\cdot\beta^{-1}(b'_1))\\
&\quad \otimes((\alpha^{-2}(b_{2(0)(-1)})\beta^{-1}(h_{21}\cdot\beta(b'_2))_{(-1)})h_{22})\alpha(h'))\\
&\overset{(2.4)}{=} \sigma(b_{2(0)}, \beta^{-1}(h_{21}\cdot\beta(b'_2)))(b_1((\alpha(b_{2(-1)})h_1)\cdot\beta^{-1}(b'_1))\otimes(1_H h_{22})\alpha(h'))\\
&= \sigma(b_{2(0)}, \alpha(h_{12})\cdot b'_2)(b_1((\alpha(b_{2(-1)})\alpha^{-1}(h_{11}))\cdot\beta^{-1}(b'_1))\otimes h_2\alpha(h'))\\
&= \sigma(\beta^{-2}(b_{2(0)}), (\alpha^{-1}(h_1)\cdot\beta^{-2}(b'))_2)(b_1(b_{2(-1)}\cdot(\alpha^{-1}(h_1)\cdot\beta^{-2}(b'))_1)\\
&\quad \otimes h_2\alpha(h'))\\
&= \sigma(\beta^{-1}(b_{2(0)}), (h_1\cdot\beta^{-1}(b'))_2)(b_1(b_{2(-1)}\cdot\beta^{-1}((h_1\cdot\beta^{-1}(b'))_1))\otimes h_2\alpha(h'))
\end{aligned}$$

and

$$\begin{aligned}
LHS &= \overline{\sigma}((b\otimes h)_1, (b'\otimes h')_1)(b\otimes h)_2(b'\otimes h')_2\\
&= \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1), b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))(\beta^{-1}(b_{2(0)})\otimes h_2)(\beta^{-1}(b'_{2(0)})\otimes h'_2)\\
&\overset{(2.13)}{=} \sigma(b_1, \alpha(b_{2(-1)}\alpha^{-1}(h_1))\cdot\beta^{-1}(b'_1))\\
&\quad \varepsilon(b'_{2(-1)})\varepsilon(h'_1)\beta^{-1}(b_{2(0)})(h_{21}\cdot\beta^{-2}(b'_{2(0)}))\otimes\alpha^{-1}(h_{22})h'_2\\
&= \sigma(b_1, b_{2(-1)}\cdot(\alpha^{-1}(h_{11})\cdot\beta^{-2}(b'_1)))\beta^{-1}(b_{2(0)})(h_{12}\cdot\beta^{-1}(b'_2))\otimes h_2\alpha(h')\\
&= \sigma(b_1, b_{2(-1)}\cdot\beta^{-1}((h_1\cdot\beta^{-1}(b'))_1))\beta^{-1}(b_{2(0)})(h_1\cdot\beta^{-1}(b'))_2\otimes h_2\alpha(h')\\
&\overset{(2.7)}{=} \sigma(\beta^{-1}(b_{2(0)}), (h_1\cdot\beta^{-1}(b'))_2)b_1(b_{2(-1)}\cdot\beta^{-1}((h_1\cdot\beta^{-1}(b'))_1))\otimes h_2\alpha(h'),
\end{aligned}$$

which proves that $\overline{\sigma}$ is lazy.

Conversely, if $\overline{\sigma}$ is lazy, we have

$$\overline{\sigma}((b\otimes h)_2, (b'\otimes h')_2)(b\otimes h)_1(b'\otimes h')_1 = \overline{\sigma}((b\otimes h)_1, (b'\otimes h')_1)(b\otimes h)_2(b'\otimes h')_2,$$

then we can get

$$\begin{aligned} & \sigma(\beta^{-1}(b_{2(0)}),(h_1\cdot\beta^{-1}(b'))_2)(b_1(b_{2(-1)}\cdot\beta^{-1}((h_1\cdot\beta^{-1}(b'))_1))\otimes h_2\alpha(h')) \\ = \; & \sigma(b_1,b_{2(-1)}\cdot\beta^{-1}((h_1\cdot\beta^{-1}(b'))_1))\beta^{-1}(b_{2(0)})(h_1\cdot\beta^{-1}(b'))_2\otimes h_2\alpha(h'). \end{aligned}$$

Applying $id\otimes\varepsilon$ to both sides of the above equation and let $h=1_H$, we get σ is lazy in ${}^H_H\mathbb{YD}$.

(v) Using (2.2) for the convolution in ${}^H_H\mathbb{YD}$, we compute

$$\begin{aligned} & \overline{(\sigma*\tau)}(b\otimes h,b'\otimes h') \\ \overset{(2.13)}{=} \; & (\sigma*\tau)(b\otimes\alpha(h)\cdot\beta^{-1}(b'))\varepsilon(h') \\ \overset{(2.2)}{=} \; & \sigma(b_1,b_{2(-1)}\cdot(h_1\cdot\beta^{-2}(b'_1)))\tau(\beta^{-1}(b_{2(0)}),\alpha(h_2)\cdot\beta^{-1}(b'_2))\varepsilon(h') \\ = \; & \sigma(b_1,(\alpha(b_{2(-1)})h_1)\cdot\beta^{-1}(b'_1))\varepsilon(b'_{2(-1)})\varepsilon(h'_1)\tau(\beta^{-1}(b_{2(0)}),\alpha(h_2)\cdot\beta^{-2}(b'_{2(0)}))\varepsilon(h'_2) \\ \overset{(2.13)}{=} \; & \overline{\sigma}(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1),b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))\overline{\tau}(\beta^{-1}(b_{2(0)})\otimes h_2,\beta^{-1}(b'_{2(0)})\otimes h'_2) \\ = \; & \overline{\sigma}((b\otimes h)_1,(b'\otimes h')_1)\overline{\tau}((b\otimes h)_2,(b'\otimes h')_2) \\ = \; & (\overline{\sigma}*\overline{\tau})(b\otimes h,b'\otimes h'). \end{aligned}$$

(vi) Obviously $\overline{\gamma}$ is normalized, and it is easy to see that its convolution inverse is given by $\overline{\gamma}^{\,-1}(b\times h)=\gamma^{-1}(b)\varepsilon(h)$, where $\gamma^{\,-1}$ is the convolution inverse of γ in ${}^H_H\mathbb{YD}$. Now we compute

$$\begin{aligned} & \overline{D^1(\gamma)}(b\otimes h,b'\otimes h') \\ \overset{(2.13)}{=} \; & D^1(\gamma)(b,\alpha(h)\cdot\beta^{-1}(b'))\varepsilon(h') \\ = \; & \gamma(b_1)\gamma(\alpha(h_1)\cdot\beta^{-1}(b'_1))\gamma^{-1}(b_2(\alpha(h_2)\cdot\beta^{-1}(b'_2)))\varepsilon(h') \\ \overset{(2.10)}{=} \; & \gamma(b_1)\gamma(b'_1)\gamma^{-1}(b_2(h\cdot\beta^{-1}(b'_2)))\varepsilon(h') \\ = \; & \gamma(b_1)\varepsilon(b_{2(-1)})\varepsilon(h_1)\gamma(b'_1)\varepsilon(b'_{2(-1)})\varepsilon(h'_1)\gamma^{-1}(\beta^{-1}(b_{2(0)}(h_{21}\cdot\beta^{-2}(b'_{2(0)}))))\varepsilon(h_{22})\varepsilon(h'_2) \\ = \; & \gamma(b_1)\varepsilon(b_{2(-1)})\varepsilon(h_1)\gamma(b'_1)\varepsilon(b'_{2(-1)})\varepsilon(h'_1)\overline{\gamma}^{-1}(\beta^{-1}(b_{2(0)}(h_{21}\cdot\beta^{-2}(b'_{2(0)})) \\ & \otimes\alpha^{-1}(h_{22})(h'_2))) \\ = \; & \overline{\gamma}(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1))\overline{\gamma}(b'_1\otimes b'_{2(-1)}\alpha^{-1}(h'_1))\overline{\gamma}^{-1}(\beta^{-1}(b_{2(0)}\otimes h_2)(\beta^{-1}(b'_{2(0)})\otimes h'_2)) \\ = \; & \overline{\gamma}((b\otimes h)_1)\overline{\gamma}((b'\otimes h')_1)\overline{\gamma}^{-1}((b\otimes h)_2(b'\otimes h')_2) \\ \overset{(1.5)}{=} \; & D^1(\overline{\gamma})(b\otimes h,b'\otimes h'). \end{aligned}$$

Hence we have indeed $\overline{D^1(\gamma)}=D^1(\overline{\gamma})$. If γ is lazy in ${}^H_H\mathbb{YD}$, then we have

$$\begin{aligned} \overline{\gamma}((b\otimes h)_1)(b\otimes h)_2 \; & \overset{(2.15)}{=} \; \gamma(b_1)(b_2\otimes\alpha(h)) \\ & = \; \gamma(b_2)(b_1\otimes\alpha(h)) \\ & \overset{(2.11)}{=} \; \gamma(b_{2(0)})(b_1\otimes b_{2(-1)}h) \\ & = \; \gamma(b_{2(0)}))\varepsilon(h_2)(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1)) \\ & \overset{(2.15)}{=} \; \overline{\gamma}(\beta^{-1}(b_{2(0)})\otimes h_2)(b_1\otimes b_{2(-1)}\alpha^{-1}(h_1)) \\ & = \; \overline{\gamma}((b\otimes h)_2)(b\otimes h)_1, \end{aligned}$$

so $\overline{\gamma}$ is lazy.

Remarks (1) $Z_L^2(B^{\#}_{\times}H,\beta\otimes\alpha)$ is a group by Lemma 1.7.

(2) If σ,τ are left lazy 2-cocycles on (B,β) in ${}^H_H\mathbb{YD}$, we can get $\overline{\sigma},\overline{\tau}$ are left lazy 2-cocycles on $(B^{\#}_{\times}H,\beta\otimes\alpha)$ by (i) and (iv). Then $\overline{\sigma}*\overline{\tau}$ is a left lazy 2-cocycle on $(B^{\#}_{\times}H,\beta\otimes\alpha)$. Combining (ii) with (v), we have $\sigma*\tau$ is a left lazy 2-cocycle on (B,β) in ${}^H_H\mathbb{YD}$.

By (2.13) and (2.14), we have $\overline{\sigma}^{-1}=\overline{\sigma^{-1}}$. If $\overline{\sigma}\in Z_L^2(B^{\#}_{\times}H,\beta\otimes\alpha)$, then $\overline{\sigma}^{-1}=\overline{\sigma^{-1}}\in Z_L^2(B^{\#}_{\times}H,\beta\otimes\alpha)$. Combining (ii) with (iv), then σ^{-1} is a lazy 2-cocycle on (B,β) in ${}^H_H\mathbb{YD}$.

In a word, the set of convolution invertible lazy 2-cocycle on (B,β) in ${}^H_H\mathbb{YD}$ denoted by $Z_L^2(B,\beta)$ is a group.

Proposition 2.6 $D^1:\mathrm{Reg}_L^1(B,\beta)\longrightarrow Z_L^2(B,\beta)$ is a group homomorphism in ${}^H_H\mathbb{YD}$, whose image denoted by $B_L^2(B,\beta)$ (its elements are called lazy 2-coboundary in ${}^H_H\mathbb{YD}$), is contained in the center of $Z_L^2(B,\beta)$. Thus we call quotient group $H_L^2(B,\beta):=Z_L^2(B,\beta)/B_L^2(B,\beta)$ the second lazy cohomology group of (B,β) in ${}^H_H\mathbb{YD}$.

Proof It is easy to check that $D^1:\mathrm{Reg}_L^1(B,\beta)\longrightarrow Z_L^2(B,\beta)$ is a group homomorphism in ${}^H_H\mathbb{YD}$. Now we prove $B_L^2(B,\beta)$ is contained in the center of $Z_L^2(B,\beta)$.

For all $\gamma\in\mathrm{Reg}_L^1(B,\beta)$ and $\sigma\in Z_L^2(B,\beta)$,

$$\begin{aligned}
(\sigma*D^1(\gamma))(b,b') &\overset{(2.2)}{=} \sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(b'_1))D^1(\gamma)(\beta^{-1}(b_{2(0)}),b'_2)\\
&= \sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(b'_1))\gamma(\beta^{-1}(b_{2(0)1}))\gamma(b'_{21})\gamma^{-1}(\beta^{-1}(b_{2(0)2})b'_{22})\\
&= \sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(b'_1))\gamma(\beta^{-1}(b_{2(0)2}))\gamma(b'_{22})\gamma^{-1}(\beta^{-1}(b_{2(0)1})b'_{21})\\
&= \sigma(b_1,(b_{21(-1)}b_{22(-1)})\cdot\beta^{-1}(b'_1))\\
&\quad\ \gamma(\beta^{-1}(b_{22(0)}))\gamma(b'_{22})\gamma^{-1}(\beta^{-1}(b_{21(0)})b'_{21})\\
&\overset{(2.11)}{=} \sigma(b_1,\alpha(b_{21(-1)})\cdot\beta^{-1}(b'_1))\gamma(b_{22})\gamma(b'_{22})\gamma^{-1}(\beta^{-1}(b_{21(0)})b'_{21})\\
&= \sigma(\beta^{-1}(b_{11}),\alpha(b_{12(-1)})\cdot\beta^{-2}(b'_{11}))\gamma(b_2)\gamma(b'_2)\gamma^{-1}(\beta^{-1}(b_{12(0)})b'_{12})\\
&= \sigma(b_{11},b_{12(-1)}\cdot\beta^{-1}(b'_{11}))\gamma(b_2)\gamma(b'_2)\gamma^{-1}(\beta^{-1}(b_{12(0)})b'_{12})\\
&\overset{(2.7)}{=} \sigma(\beta^{-1}(b_{12(0)}),b'_{12})\gamma^{-1}(b_{11}(b_{12(-1)}\cdot\beta^{-1}(b'_{11})))\gamma(b_2)\gamma(b'_2)\\
&= \sigma(\beta^{-1}(b_{21(0)}),b'_{21})\gamma^{-1}(\beta(b_1)(b_{21(-1)}\cdot b'_1))\gamma(b_{22})\gamma(b'_{22})\\
&= \sigma(\beta^{-1}(b_{22(0)}),b'_{22})\gamma^{-1}(\beta(b_1)(b_{22(-1)}\cdot b'_1))\gamma(b_{21})\gamma(b'_{21})\\
&= \sigma(b_{2(0)},\beta(b'_2))\gamma^{-1}(b_{11}(\alpha(b_{2(-1)})\cdot\beta^{-1}(b'_{11})))\gamma(b_{12})\gamma(b'_{12}),\\
(D^1(\gamma)*\sigma)(b,b') &\overset{(2.2)}{=} D^1(\gamma)(b_1,b_{2(-1)}\cdot\beta^{-1}(b'_1))\sigma(\beta^{-1}(b_{2(0)}),b'_2)\\
&= \gamma(b_{11})\gamma(\alpha^2(b_{2(-1)1})\cdot\beta^{-1}(b'_{11}))\gamma^{-1}(b_{12}(b_{2(-1)2}\cdot\beta^{-1}(b'_{12})))\\
&\quad\ \sigma(\beta^{-1}(b_{2(0)}),b'_2)\\
&\overset{(2.10)}{=} \gamma(b_{11})\varepsilon(b_{2(-1)1})\gamma(b'_{11})\gamma^{-1}(b_{12}(b_{2(-1)2}\cdot\beta^{-1}(b'_{12})))\sigma(b_{2(0)},\beta(b'_2))\\
&= \gamma(b_{11})\gamma(b'_{11})\gamma^{-1}(b_{12}(\alpha(b_{2(-1)})\cdot\beta^{-1}(b'_{12})))\sigma(\beta(b_{2(0)}),\beta(b'_2)).
\end{aligned}$$

Then $\sigma*D^1(\gamma)=D^1(\gamma)*\sigma$. The proof is completed.

Proposition 2.7 If σ is a lazy 2-coboundary for (B,β) in ${}^H_H\mathbb{YD}$, then $\overline{\sigma}$ is a lazy 2-coboundary for $(B^{\#}_{\times}H,\beta\otimes\alpha)$, so the group homomorphism $Z_L^2(B,\beta)\longrightarrow Z_L^2(B^{\#}_{\times}H,\beta\otimes$

$\alpha), \sigma \mapsto \overline{\sigma}$, factorizes to a group homomorphism $H_L^2(B,\beta) \longrightarrow H_L^2(B^{\#}_{\times}H, \beta\otimes\alpha)$.

Proof It follows immediately from (vi) in Theorem 2.5.

Example 2.8 Let $A = sp\{1_A, z\}$ and the automorphism $\beta : A \longrightarrow A$, $\beta(1_A) = 1_A, \beta(z) = -z$. Then (A,β) is a Hom-algebra with multiplication: $1_A 1_A = 1_A, 1_A z = z 1_A = -z, z^2 = 0$, and (A,β) is a Hom-coalgebra with comultiplication and counit

$$\Delta(1_A) = 1_A \otimes 1_A, \quad \varepsilon(1_A) = 1_k,$$
$$\Delta(z) = (-z)\otimes 1_A + 1_A \otimes (-z), \quad \varepsilon(z) = 0.$$

Let $H = sp\{1_H, g\}$ be the group Hopf algebra with $g^2 = 1_H$ and $\Delta(g) = g\otimes g, S_H(g) = g = g^{-1}$. Then (H, id_H) is a Hom-Hopf algebra.

Define $\cdot : H\otimes A \longrightarrow A$ such that $1_H \cdot 1_A = 1_A, 1_H \cdot z = -z, g\cdot 1_A = 1_A$, and $g\cdot z = z$. It is easy to check (A,β) is a left (H, id_H)-module Hom-algebra and module Hom-coalgebra.

Define $\rho : A \longrightarrow H\otimes A$ such that $\rho(1_A) = 1_H \otimes 1_A$ and $\rho(x) = g\otimes(-z)$. We get (A,β) is a left (H, id_H)-comodule Hom-algebra and comodule Hom-coalgebra. Then we can get a Radford biproduct Hom-bialgebra $(A^{\#}_{\times}H, \beta\otimes id)$ (see [5]).

Define $\sigma : A\otimes A \longrightarrow K$ by

σ	1_A	z
1_A	1	0
z	0	s

where $\forall\ s \in K$.

Then we can check that σ is normal left 2-cocycle on (A,β) in ${}_H^H\mathbb{YD}$, and σ is lazy in ${}_H^H\mathbb{YD}$. By Theorem 2.5, $\overline{\sigma}$ is defined as follows

$\overline{\sigma}$	$1_A\otimes 1_H$	$1_A\otimes g$	$z\otimes 1_H$	$z\otimes g$
$1_A\otimes 1_H$	1	1	0	0
$1_A\otimes g$	1	1	0	0
$z\otimes 1_H$	0	0	s	s
$z\otimes g$	0	0	-s	-s

and $\overline{\sigma}$ is a normal lazy left 2-cocycle.

Let $\gamma(1_A) = 1, \gamma(z) = 0$. Then γ is normal and lazy in ${}_H^H\mathbb{YD}$. By Theorem 2.5, $\overline{\gamma}$ is defined as follows

$\overline{\gamma}$	1_H	g
1_A	1	1
z	0	0

and $\overline{\gamma}$ is normal and lazy.

Example 2.9 Let $KZ_2 = K\{1,a\}$ be Hopf group algebra. Then (KZ_2, id) is a Hom-Hopf algebra. Let $T_{2,-1} = K\{1,g,x,y|g^2 = 1, x^2 = 0, y = gx, gy = -yg = x\}$ be Taft's Hopf algebra, its coalgebra structure and antipode are given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + g \otimes y;$$
$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad \varepsilon(y) = 0$$

and $S(g) = g, S(x) = y, S(y) = -x$. Define a linear map $\alpha : T_{2,-1} \longrightarrow T_{2,-1}$ by $\alpha(1) = 1, \alpha(g) = g, \alpha(x) = -x, \alpha(y) = -y$. Then α is an automorphism of Hopf algebra.

So we can get a Hom-Hopf algebra $H_\alpha = (T_{2,-1}, \alpha \circ \mu_{T_{2,-1}}, 1_{T_{2,-1}}, \Delta_{T_{2,-1}} \circ \alpha, \varepsilon_{T_{2,-1}}, \alpha)$. Define module action $\triangleright : KZ_2 \otimes H_\alpha \longrightarrow H_\alpha$ by

$$1_{KZ_2} \triangleright 1_{H_\alpha} = 1_{H_\alpha},\ 1_{KZ_2} \triangleright g = g,\ 1_{KZ_2} \triangleright x = -x,\ 1_{KZ_2} \triangleright y = -y,$$
$$a \triangleright 1_{H_\alpha} = 1_{H_\alpha},\ a \triangleright g = g,\ a \triangleright x = -x,\ a \triangleright y = -y.$$

Then by a routine computation we can get $(H_\alpha, \triangleright, \alpha)$ is a (KZ_2, id)-module Hom-algebra. Therefore, $(H_\alpha \# KZ_2, \alpha \otimes id)$ is a smash product Hom-algebra.

Define comodule action $\rho : H_\alpha \longrightarrow KZ_2 \otimes H_\alpha$ by

$$\rho : H_\alpha \longrightarrow KZ_2 \otimes H_\alpha,\ 1_{H_\alpha} \mapsto 1_{KZ_2} \otimes 1_{H_\alpha},$$
$$g \mapsto 1_{KZ_2} \otimes g,\ x \mapsto -a \otimes x,\ y \mapsto -a \otimes y.$$

Then we can get (H_α, ρ, α) is a left (KZ_2, id)-comodule Hom-coalgebra. Therefore $(H_\alpha \times KZ_2, \alpha \otimes id)$ is a smash coproduct Hom-coalgebra.

Then we can get a Radford biproduct Hom-bialgebra $(H_\alpha {}^{\#}_{\times} KZ_2, \alpha \otimes id)$ (see [5]).

Define $\sigma : H_\alpha \otimes H_\alpha \longrightarrow K$ by

σ	1	g	x	y
1	1	1	0	0
g	1	1	0	0
x	0	0	-s	s
y	0	0	-s	s

where $\forall\ s \in K$.

Then we can check that σ is a normal left 2-cocycle on (H_α, α) in ${}^H_H\mathbb{YD}$, and σ is lazy in ${}^H_H\mathbb{YD}$. By Theorem 2.5, $\overline{\sigma}$ is defined as follows

$\overline{\sigma}$	$1 \otimes 1$	$1 \otimes a$	$g \otimes 1$	$g \otimes a$	$x \otimes 1$	$x \otimes a$	$y \otimes 1$	$y \otimes a$
$1 \otimes 1$	1	1	1	1	0	0	0	0
$1 \otimes a$	1	1	1	1	0	0	0	0
$g \otimes 1$	1	1	1	1	0	0	0	0
$g \otimes a$	1	1	1	1	0	0	0	0
$x \otimes 1$	0	0	0	0	-s	-s	s	s
$x \otimes a$	0	0	0	0	-s	-s	s	s
$y \otimes 1$	0	0	0	0	-s	-s	s	s
$y \otimes a$	0	0	0	0	-s	-s	s	s

and $\overline{\sigma}$ is a normal lazy left 2-cocycle on $(H_{\alpha}{}^{\#}_{\times}KZ_2, \alpha\otimes id)$.

Let $\gamma(1)=1, \gamma(g)=1, \gamma(x)=0, \gamma(y)=0$. Then γ is normal and lazy in ${}^H_H\mathbb{YD}$. By Theorem 2.5, $\overline{\gamma}$ is defined as follows

$\overline{\gamma}$	1	a
1	1	1
g	1	1
x	0	0
y	0	0

and $\overline{\gamma}$ is normal and lazy.

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Radford双积Hom-Hopf代数上的Lazy 2-余循环

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摘要: 本文研究了Radford双积Hom-Hopf代数上的lazy 2-余循环. 利用扭曲方法得到了(B,β)上的左Hom-2-余循环σ和$(B^{\#}_{\times}H, \beta\otimes\alpha)$ 上的左Hom-2-余循环$\overline{\sigma}$之间的关系, 推广了通常Hopf代数情形下的相应结论.

关键词: lazy 2-余循环; Radford双积Hom-Hopf代数; Yetter-Drinfeld范畴

MR(2010)主题分类号: 16T05 中图分类号: O153.3