Vol. 39 (2019)
No. 5

数 学 杂 志
J. of Math. (PRC)

MULTIPLICITY OF POSITIVE SOLUTIONS FOR QUASI-LINEAR ELLIPTIC EQUATIONS INVOLVING CONCAVE-CONVEX NONLINEARITY AND SOBOLEV-HARDY TERM

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Abstract: In this paper, we investigate the quasi-linear elliptic equations involving concave-convex nonlinearity and Sobolev-Hardy term. By using the theory of the Lusternik-Schnirelmann category and the relationship between the Nehari manifold and fibering maps, we get some improvement on existence and multiplicity of positive solution.

Keywords: subcritical Sobolev-Hardy exponent; Nehari manifold; sign-changing weight; concave-convex nonlinearity

2010 MR Subject Classification: 35J30; 35J75

Document code: A **Article ID:** 0255-7797(2019)05-0633-23

1 Introduction

In this paper, we consider the following equation

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \lambda \dfrac{|u|^{p-2}u}{|x|^{p(a+1)}} = f_\mu(x)|u|^{q-2}u + g(x)|u|^{r-2}u, \\ u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geqslant 3$, $1 < p < N$, $0 \leqslant a < \frac{N-p}{p}$, $1 \leqslant q < p < r < p^*[a] = \frac{Np}{N-p(a+1)}$, and $p^*[a]$ is the critical Sobolev-Hardy exponent. The parameter λ satisfies $0 \leqslant \lambda < \overline{\lambda} = (\frac{N-p}{p} - a)^p$, $\mu \geqslant 0$, and $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$ will be explained later. The weight functions $f_\mu(x) = \mu f_+(x) + f_-(x)$ and $g(x) = g_1(x) + g_2(x)$ satisfy the following conditions

(A_1) $f \in L^{q^*}(\mathbb{R}^N)(q^* = \frac{r}{r-q})$ with $f_\pm(x) = \pm\max\{\pm f(x), 0\} \not\equiv 0$ and there exists a positive constant r_f such that

$$f_-(x) \geqslant -c_f|x|^{-r_f} \quad \text{for some } c_f > 0 \text{ and for all } x \in \mathbb{R}^N;$$

* **Received date:** 2017-03-11 **Accepted date:** 2017-08-11

Foundation item: Supported by National Natural Science Foundation of China (11371282; 11571259).

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(A_2) $g \in C(\mathbb{R}^N)$ with $g_0 = \max\limits_{x\in\mathbb{R}^N} g(x)$ and there exists constants r_{g_1}, r_{g_2} with $0 < r_{g_2} < \min\{r_f - N, r_{g_1} - N\}$ such that

$$1 \geqslant g_1(x) \geqslant 1 - c_{g_1}|x|^{-r_{g_1}} \quad \text{for some } c_{g_1} < 1 \text{ and for all } x \in \mathbb{R}^N$$

and

$$g_2(x) \geqslant c_{g_2}|x|^{-r_{g_2}} \quad \text{for some } c_{g_2} > 0 \text{ and for all } x \in \mathbb{R}^N.$$

Such kind of problem arised from various fields of geometry and physics and was widely used in the applied sciences. We refer to [1–3] for details on the description about the background.

Elliptic problems on bounded domains involving concave-convex nonlinearity were studied extensively since Ambrosetti, Brezis and Cerami [4] considered the following equation

$$\begin{cases} -\Delta u = \mu u^{q-1} + u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

where $1 < q < 2 < p \leqslant 2^*$, $\mu > 0$. They found that there exists $\mu_0 > 0$ such that (1.2) admits at least two positive solutions for $\mu \in (0, \mu_0)$, a positive solution for $\mu = \mu_0$ and no positive solution exists for $\mu > \mu_0$ (see also Ambrosetti, Azorero and Peral [5, 6] for more references therein). In recent years, several authors studied semilinear or quasilinear problems with the help of Nehari manifold (see [7–9]). In particular, Lin [9] studied the following critical problem

$$\begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u) - \lambda \dfrac{u}{|x|^{2(a+1)}} = \dfrac{|u|^{2^*(a,b)-2}u}{|x|^{b2^*(a,b)}} + \mu|u|^{q-2}u & \text{in } \Omega\backslash\{0\}, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a bounded domain with smooth boundary, $0 \leqslant a < \frac{N-2}{2}$, $a \leqslant b < a+1$, $2^*(a,b) = \frac{2N}{N-2(a+1-b)}$, $0 \leqslant \lambda < \overline{\lambda} = \frac{(N-2(a+1))^2}{4}$, $\mu > 0$, and $1 < q < 2$. He found that (1.3) admits at least two positive and one sign-changing solutions.

Actually, Fan and Liu [10] established multiple positive solutions of standard p-Laplacian elliptic equations without Hardy term on a bounded domain Ω in $\mathbb{R}^N$. Some other theorems for p-Laplacian elliptic equations without Hardy term can be found in [11, 12]. Hsu and Lin [13] studied the following critical problem via generalized Mountain Pass Theorem [14]

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \lambda \dfrac{|u|^{p-2}u}{|x|^{p(a+1)}} = \dfrac{|u|^{p^*(a,b)-2}u}{|x|^{bp^*(a,b)}} + \mu \dfrac{|u|^{q-2}u}{|x|^{dp^*(a,d)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $a \leqslant b, d < a+1$, $p^*(a,b) = \frac{Np}{N-p(a+1-b)}$ is the critical Sobolev-Hardy exponent. They found that (1.4) admits at least two positive solutions.

However, little is done on $\mathbb{R}^N$ for the operator $-\mathrm{div}(|x|^{-ap}|\nabla\cdot|^{p-2}\nabla\cdot)-\lambda\frac{|\cdot|^{p-2}\cdot}{|x|^{p(a+1)}}$ involving the concave-convex nonlinearity. Since the embedding is not compact on $\mathbb{R}^N$ and the weight functions f and g are sign-changing, we will discuss the concentration behavior of solutions on the corresponding Nehari manifold to overcome these difficulties. Moreover, we get some improvement on multiplicity of positive solutions via the theory of Lusternik-Schnirelmann category (see [15]).

Throughout our paper, we denote by $\mathbf{W}^{1,p}_a(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the standard norm $(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx)^{\frac{1}{p}}$. The function $u\in\mathbf{W}^{1,p}_a(\mathbb{R}^N)$ is said to be a solution of problem (1.1) if u satisfies

$$\int_{\mathbb{R}^N}(|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla v-\lambda\frac{|u|^{p-2}uv}{|x|^{p(a+1)}}-f_\mu|u|^{q-2}uv-g|u|^{r-2}uv)dx=0 \tag{1.5}$$

for all $v\in\mathbf{W}^{1,p}_a(\mathbb{R}^N)$. It is well known that the nontrivial solution of problem (1.1) is equivalent to the corresponding nonzero critical point of the energy functional

$$I_\mu(u)=\frac{1}{p}\int_{\mathbb{R}^N}(|x|^{-ap}|\nabla u|^p-\lambda\frac{|u|^p}{|x|^{p(a+1)}})dx-\frac{1}{q}\int_{\mathbb{R}^N}f_\mu|u|^q dx-\frac{1}{r}\int_{\mathbb{R}^N}g|u|^r dx. \tag{1.6}$$

Then $I_\mu(u)$ is well-defined on $\mathbf{W}^{1,p}_a(\mathbb{R}^N)$ and belongs to $C^1(\mathbf{W}^{1,p}_a(\mathbb{R}^N),\mathbb{R})$.

Problem (1.1) is related to well-known Caffarelli-Kohn-Nirenberg inequality in [16]

$$\left(\int_{\mathbb{R}^N}\frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}dx\right)^{\frac{p}{p^*(a,b)}}\leqslant C\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx,\quad \forall u\in C_0^\infty(\mathbb{R}^N). \tag{1.7}$$

If $b=a+1$, then $p^*(a,b)=p$ and the following Hardy inequality holds [17]

$$\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{p(a+1)}}dx\leqslant\frac{1}{\overline{\lambda}}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx,\quad \forall u\in C_0^\infty(\mathbb{R}^N), \tag{1.8}$$

where $\overline{\lambda}=(\frac{N-p}{p}-a)^p$ is the best Hardy constant. Consequently, for $\lambda<\overline{\lambda}$, we endow the space $\mathbf{W}^{1,p}_a(\mathbb{R}^N)$ with the following norm

$$\|u\|=\|u\|_{\mathbf{W}^{1,p}_a(\mathbb{R}^N)}=(\int_{\mathbb{R}^N}(|x|^{-ap}|\nabla u|^p-\lambda\frac{|u|^p}{|x|^{p(a+1)}})dx)^{\frac{1}{p}}, \tag{1.9}$$

which is equivalent to the usual norm $(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx)^{\frac{1}{p}}$.

We get our main result as follows.

Theorem 1.1 Suppose that the functions f and g satisfy condition (A_1) and (A_2). Let

$$L_2=\frac{q}{p}\left(\frac{r-p}{r-q}\right)\left(\frac{p-q}{g_0(r-q)}\right)^{\frac{p-q}{r-p}}\frac{S_\lambda^{\frac{r-q}{r-p}}}{\|f_+\|_{L^{q^*}}}, \tag{1.10}$$

where S_λ is the best Sobolev constant for the embedding of $\mathbf{W}^{1,p}_a(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$ and defined by

$$S_\lambda:=\inf_{u\in\mathbf{W}^{1,p}_a(\mathbb{R}^N)\backslash\{0\}}\frac{\|u\|^p}{(\int_{\mathbb{R}^N}|u|^r dx)^{\frac{p}{r}}}. \tag{1.11}$$

Then

(i) for $\mu \in (0, L_2)$, (1.1) has at least two positive solutions in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$ corresponding to negative least energy;

(ii) there exists $\mu_0 \in (0, L_2)$ such that for $\mu \in (0, \mu_0)$, (1.1) has at least three positive solutions in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$ including two with positive energy.

The paper is organized as follows: in Sections 2–4, based on some related preliminaries, we develop the description of Palais-Smale condition and the estimate of corresponding energy functional I_μ; in Section 5, we discuss the concentration behavior of solutions on Nehari manifold; in Section 6, we complete the proof of Theorem 1.1.

2 Preliminaries

Since the energy functional I_μ in (1.6) is unbounded below on $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$, we consider the functional on Nehari manifold

$$N_\mu = \{u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid \langle I'_\mu(u), u\rangle = 0\}.$$

Note that N_μ contains all nonzero solutions of (1.1) and $u \in N_\mu$ if and only if

$$\|u\|^p - \int_{\mathbb{R}^N} f_\mu |u|^q dx - \int_{\mathbb{R}^N} g|u|^r dx = 0.$$

Lemma 2.1 The energy functional I_μ is coercive and bounded below on N_μ.

Proof For $u \in N_\mu$, by the Hölder inequality and Sobolev embedding theorem, we can deduce

$$\begin{aligned} I_\mu(u) &= \frac{1}{p}\|u\|^p - \frac{1}{q}\int_{\mathbb{R}^N} f_\mu |u|^q dx - \frac{1}{r}\int_{\mathbb{R}^N} g|u|^r dx \\ &= \left(\frac{1}{p} - \frac{1}{r}\right)\|u\|^p - \left(\frac{1}{q} - \frac{1}{r}\right)\int_{\mathbb{R}^N} f_\mu |u|^q dx \\ &\geqslant \left(\frac{1}{p} - \frac{1}{r}\right)\|u\|^p - \left(\frac{1}{q} - \frac{1}{r}\right)\mu \|f_+\|_{L^{q^*}} S_\lambda^{-\frac{q}{p}} \|u\|^q \\ &\geqslant -C\mu^{\frac{p}{p-q}}, \end{aligned}$$

where C is a positive constant depending on N, q, S_λ and $\|f_+\|_{L^{q^*}}$. This completes the proof.

Define

$$\Psi(u) = \langle I'{}_\mu(u), u\rangle = \|u\|^p - \int_{\mathbb{R}^N} f_\mu |u|^q dx - \int_{\mathbb{R}^N} g|u|^r dx.$$

Then for $u \in N_\mu$, we have

$$\begin{aligned} \langle \Psi'(u), u\rangle &= p\|u\|^p - q\int_{\mathbb{R}^N} f_\mu |u|^q dx - r\int_{\mathbb{R}^N} g|u|^r dx \\ &= (p-q)\|u\|^p - (r-q)\int_{\mathbb{R}^N} g|u|^r dx \\ &= (p-r)\|u\|^p - (q-r)\int_{\mathbb{R}^N} f_\mu |u|^q dx. \end{aligned} \tag{2.1}$$

As in [18], we divide N_μ into three parts

$$
\begin{aligned}
N_\mu^+ &= \{u \in N_\mu \mid \langle \Psi'(u), u\rangle > 0\},\\
N_\mu^0 &= \{u \in N_\mu \mid \langle \Psi'(u), u\rangle = 0\},\\
N_\mu^- &= \{u \in N_\mu \mid \langle \Psi'(u), u\rangle < 0\}.
\end{aligned}
$$

Then we have the following result.

Lemma 2.2 (i) If $u \in N_\mu^+$, then $\int_{\mathbb{R}^N} f_\mu(x)|u|^q dx > 0$.

(ii) If $u \in N_\mu^0$, then $\int_{\mathbb{R}^N} f_\mu(x)|u|^q dx > 0$ and $\int_{\mathbb{R}^N} g(x)|u|^r dx > 0$.

(iii) If $u \in N_\mu^-$, then $\int_{\mathbb{R}^N} g(x)|u|^r dx > 0$.

Proof By (2.1) we can easily derive these results.

Set $L_1 = \left(\frac{r-p}{r-q}\right)\left(\frac{p-q}{g_0(r-q)}\right)^{\frac{p-q}{r-p}} \frac{S_\lambda^{\frac{r-q}{r-p}}}{\|f_+\|_{L^{q^*}}}$ and it is easy to see $L_2 = \frac{q}{p}L_1$, where L_2 is defined in (1.10). We define

$$
\alpha = \inf_{u\in N_\mu} I_\mu(u), \quad \alpha^+ = \inf_{u\in N_\mu^+} I_\mu(u) \quad \text{and} \quad \alpha^- = \inf_{u\in N_\mu^-} I_\mu(u).
$$

Then the following lemma is essential for the main result.

Lemma 2.3 (i) For all $\mu \in (0, L_1)$, we have $N_\mu^0 = \emptyset$ and $\alpha^+ < 0$.

(ii) If $\mu < L_2$, then we have $\alpha^- > c_0$ for some $c_0 > 0$. In particular, $\inf_{u\in N_\mu} I_\mu(u) = \alpha^+$ for all $\mu \in (0, L_2)$.

Proof (i) Suppose the contrary. We may assume that there exists $\mu_* \in (0, L_1)$ such that $N_{\mu_*}^0 \neq \emptyset$. Thus, for each $u \in N_{\mu_*}^0$, by the Hölder and Sobolev inequalities, we can obtain

$$
0 = \langle \Psi'(u), u\rangle = (p-r)\|u\|^p - (q-r)\int_{\mathbb{R}^N} f_{\mu_*}|u|^q dx, \tag{2.2}
$$

that is,

$$
\|u\|^p = \frac{r-q}{r-p}\int_{\mathbb{R}^N} f_{\mu_*}|u|^q dx \leqslant \frac{r-q}{r-p}\mu_*\|f_+\|_{L^{q^*}} S_\lambda^{-\frac{q}{p}}\|u\|^q \tag{2.3}
$$

and so

$$
\mu_* \geqslant \frac{r-p}{r-q}\|u\|^{p-q}\frac{S_\lambda^{\frac{q}{p}}}{\|f_+\|_{L^{q^*}}}. \tag{2.4}
$$

But (2.1) implies that

$$
(p-q)\|u\|^p = (r-q)\int_{\mathbb{R}^N} g|u|^r dx \leqslant g_0(r-q)\|u\|^r S_\lambda^{-\frac{r}{p}},
$$

which means

$$
\|u\|^{p-q} \geqslant \left(\frac{p-q}{g_0(r-q)}S_\lambda^{\frac{r}{p}}\right)^{\frac{p-q}{r-p}}. \tag{2.5}
$$

Combined (2.4) and (2.5), we have

$$
\begin{aligned}
\mu_* &\geqslant \left(\frac{r-p}{r-q}\right)\left(\frac{p-q}{g_0(r-q)}\right)^{\frac{p-q}{r-p}}\frac{S_\lambda^{\frac{r}{p}\cdot\frac{p-q}{r-p}+\frac{q}{p}}}{\|f_+\|_{L^{q^*}}}\\
&= \left(\frac{r-p}{r-q}\right)\left(\frac{p-q}{g_0(r-q)}\right)^{\frac{p-q}{r-p}}\frac{S_\lambda^{\frac{r-q}{r-p}}}{\|f_+\|_{L^{q^*}}} = L_1.
\end{aligned}
$$

This contradicts to $\mu_* \in (0, L_1)$. Therefore, $N_\mu^0 = \emptyset$ and $N_\mu = N_\mu^+ \cup N_\mu^-$ for $\mu \in (0, L_1)$. Then for $u \in N_\mu^+$, by Lemma 2.2, we get

$$
\begin{aligned}
I_\mu(u) &= \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|^p - \left(\frac{1}{q}-\frac{1}{r}\right)\int_{\mathbb{R}^N} f_\mu|u|^q dx\\
&< \left(\frac{1}{p}-\frac{1}{q}\right)\frac{r-q}{r}\int_{\mathbb{R}^N} f_\mu|u|^q dx
\end{aligned}
$$

and so

$$
\alpha^+ = \inf_{u\in N_\mu^+} I_\mu(u) < 0. \tag{2.6}
$$

(ii) Let $u \in N_\mu^-$. By (2.1) and the Sobolev inequality, we have

$$
(p-q)\|u\|^p < (r-q)\int_{\mathbb{R}^N} g|u|^r dx \leqslant g_0(r-q)\|u\|^r S_\lambda^{-\frac{r}{p}}
$$

or

$$
\|u\| > \left(\frac{p-q}{g_0(r-q)}S_\lambda^{\frac{r}{p}}\right)^{\frac{1}{r-p}}. \tag{2.7}
$$

Then for $\mu \in (0, L_2)$, we have

$$
\begin{aligned}
I_\mu(u) &= \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|^p - \left(\frac{1}{q}-\frac{1}{r}\right)\int_{\mathbb{R}^N} f_\mu|u|^q dx\\
&= \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|^p - \left(\frac{1}{q}-\frac{1}{r}\right)\mu\|f_+\|_{L^{q^*}} S_\lambda^{-\frac{q}{p}}\|u\|^q + c_0\\
&> \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|^p - \left(\frac{1}{q}-\frac{1}{r}\right)\frac{q}{p}L_1\|f_+\|_{L^{q^*}} S_\lambda^{-\frac{q}{p}}\|u\|^q + c_0\\
&> c_0,
\end{aligned}
$$

where

$$
c_0 = \left(\frac{1}{q}-\frac{1}{r}\right)\mu\|f_+\|_{L^{q^*}} S_\lambda^{-\frac{q}{p}}\|u\|^q - \left(\frac{1}{q}-\frac{1}{r}\right)\int_{\mathbb{R}^N} f_\mu|u|^q dx > 0.
$$

This implies, for $\mu \in (0, L_2)$, $\alpha^+ < 0 < c_0 < \alpha^-$. The proof is completed.

Now we introduce the following function $m_u : \mathbb{R}^+ \to \mathbb{R}$ in the form

$$
m_u(t) = t^{p-q}\|u\|^p - t^{r-q}\int_{\mathbb{R}^N} g|u|^r dx \text{ for } t > 0.
$$

Clearly, $tu \in N_\mu$ if and only if $m_u(t) = \int_{\mathbb{R}^N} f_\mu(x)|u|^q dx$, and

$$m'_u(t) = (p-q)t^{p-q-1}\|u\|^p - (r-q)t^{r-q-1}\int_{\mathbb{R}^N} g(x)|u|^r dx. \tag{2.8}$$

It is obvious that if $tu \in N_\mu$, then $t^{q+1}m'_u(t) = \langle \Psi'(tu), tu\rangle$. Hence, $tu \in N_\mu^+$(or N_μ^-) if and only if $m'_u(t) > 0$ (or < 0).

Suppose $u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\}$. Then by (2.8), m_u admits a unique critical point at $t = t_{\max}$, where

$$t_{\max} = \left(\frac{(p-q)\|u\|^p}{(r-q)\int_{\mathbb{R}^N} g|u|^r dx}\right)^{\frac{1}{r-p}} > 0,$$

and m_u strictly increases on $(0, t_{\max})$ and decreases on $(t_{\max}, \infty)$ with $\lim\limits_{t\to\infty} m_u(t) = -\infty$. Furthermore, since $\mu \in (0, L_1)$, we have

$$\begin{aligned}
m_u(t_{\max}) &= \left(\frac{(p-q)\|u\|^p}{(r-q)\int_{\mathbb{R}^N} g|u|^r dx}\right)^{\frac{p-q}{r-p}}\|u\|^p - \left(\frac{(p-q)\|u\|^p}{(r-q)\int_{\mathbb{R}^N} g|u|^r dx}\right)^{\frac{r-q}{r-p}}\int_{\mathbb{R}^N} g|u|^r dx \\
&= \left(\frac{p-q}{r-q}\right)^{\frac{p-q}{r-p}} \frac{\|u\|^{\frac{p(r-q)}{r-p}}}{(\int_{\mathbb{R}^N} g|u|^r dx)^{\frac{p-q}{r-p}}} - \left(\frac{p-q}{r-q}\right)^{\frac{r-q}{r-p}} \frac{\|u\|^{\frac{p(r-q)}{r-p}}}{(\int_{\mathbb{R}^N} g|u|^r dx)^{\frac{p-q}{r-p}}} \\
&= \|u\|^q\left(\frac{r-p}{r-q}\right)\left(\frac{p-q}{r-q}\right)^{\frac{p-q}{r-p}}\left(\frac{\|u\|^r}{\int_{\mathbb{R}^N} g|u|^r dx}\right)^{\frac{p-q}{r-p}} \\
&\geqslant \frac{1}{\mu}\|f_+\|_{L^{q^*}}^{-1} S_\lambda^{\frac{r-q}{r-p}}\left(\frac{p-q}{r-q}\right)^{\frac{p-q}{r-p}}\left(\frac{r-p}{r-q}\right)\int_{\mathbb{R}^N} f_\mu|u|^q dx \\
&> \int_{\mathbb{R}^N} f_\mu|u|^q dx.
\end{aligned}$$

Thus, we have the following lemma.

Lemma 2.4 For each $u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\}$, we have

(i) if $\int_{\mathbb{R}^N} f_\mu|u|^q dx \leqslant 0$, then there exists a unique $t^- = t^-(u) > t_{\max}$ such that $t^-u \in N_\mu^-$, and

$$I_\mu(t^-u) = \sup_{t\geqslant 0} I_\mu(tu); \tag{2.9}$$

(ii) if $\int_{\mathbb{R}^N} f_\mu|u|^q dx > 0$, then there exist unique $0 < t^+ = t^+(u) < t_{\max} < t^-$ such that $t^+u \in N_\mu^+$, $t^-u \in N_\mu^-$ and

$$I_\mu(t^+u) = \inf_{0\leqslant t\leqslant t_{\max}} I_\mu(tu),\ I_\mu(t^-u) = \sup_{t\geqslant t^+} I_\mu(tu); \tag{2.10}$$

(iii) $t^-(u) : \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^+$ is continuous;

(iv) $N_\mu^- = \{u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\} \mid \frac{1}{\|u\|} t^-(\frac{u}{\|u\|}) = 1\}$.

Proof (i) The equation $m_u(t) = \int_{\mathbb{R}^N} f_\mu |u|^q dx$ admits a unique solution $t^- > t_{\max}$ and $m_u'(t^-) < 0$. Thus $t^- u \in N_\mu^-$, and (2.9) holds by Lemma 2.3.

(ii) The equation $m_u(t) = \int_{\mathbb{R}^N} f_\mu |u|^q dx$ admits distinctive solutions $t^+ < t_{\max} < t^-$ such that $m_u'(t^+) > 0$ and $m_u'(t^-) < 0$, and then we have $t^+ u \in N_\mu^+$ and $t^- u \in N_\mu^-$. Thus (2.10) holds by Lemma 2.3 and Lemma 2.4 (i).

(iii) By the uniqueness and extremal property of $t^-(u)$, we have $t^-(u)$ is a continuous function for $u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\}$.

(iv) For $u \in N_\mu^-$, let $v = \frac{u}{\|u\|}$. By (i) and (ii), there is a unique $t^-(v) > 0$ such that $t^-(v)v \in N_\mu^-$ or

$$t^-(\frac{u}{\|u\|}) \frac{1}{\|u\|} u \in N_\mu^-.$$

Since (i) $u \in N_\mu^-$, we have $t^-(\frac{u}{\|u\|}) \frac{1}{\|u\|} = 1$, and this implies

$$N_\mu^- \subset \{u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\} | \frac{1}{\|u\|} t^-(\frac{u}{\|u\|}) = 1\}.$$

On the other hand, let $u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\frac{1}{\|u\|} t^-(\frac{u}{\|u\|}) = 1.$$

If $u \in N_\mu^+$, then $t^-(u) > t_{\max} > 1$ and this contradicts $t_{\max} < 1$ on N_μ^-. Then

$$t^-(\frac{u}{\|u\|}) \frac{u}{\|u\|} \in N_\mu^-.$$

Thus, the proof is completed.

Remark 2.5 If $\mu = 0$, by Lemma 2.4 (i), $N_0^+ = \emptyset$ and so $N_0 = N_0^-$.

3 Palais-Smale Condition

Now we consider the limiting problem

$$\begin{cases} -\operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) - \lambda \dfrac{|u|^{p-2} u}{|x|^{p(a+1)}} = |u|^{r-2} u, \\ u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \end{cases} \tag{3.1}$$

and the corresponding energy functional I^∞ in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$ is defined by

$$I^\infty(u) = \frac{1}{p} \|u\|^p - \frac{1}{r} \int_{\mathbb{R}^N} |u|^r dx. \tag{3.2}$$

Proposition 3.1 For $0 \leqslant a < \frac{N-p}{p}$, $0 \leqslant \lambda < \overline{\lambda}$, problem (3.1) has radially symmetric ground states

$$u_\epsilon(x) = \epsilon^{-(\frac{N-p}{p} - a)} v_\epsilon(\frac{x}{\epsilon}), \qquad \forall \epsilon > 0,$$

satisfying

$$\int_{\mathbb{R}^N}(|x|^{-ap}|\nabla u_\epsilon(x)|^p-\lambda\frac{|u|^p}{|x|^{p(a+1)}})dx=\int_{\mathbb{R}^N}|u_\epsilon(x)|^r dx=S_\lambda^{\frac{r}{r-p}},$$

where $v_\epsilon(x)=v_\epsilon(|x|)$ is the unique radial solution of (3.1) up to a dilation. In particular, we have

$$v_\epsilon(1)=(\frac{r(\overline{\lambda}-\lambda)}{p})^{\frac{1}{r}-p}, \tag{3.3}$$

and v_ϵ also has the following properties:

$$\begin{aligned}&\lim_{\xi\to 0}\xi^{a(\lambda)}v_\epsilon(\xi)=c_1>0, &&\lim_{\xi\to 0}\xi^{a(\lambda)+1}v_\epsilon'(\xi)=c_1a(\lambda)\geqslant 0,\\ &\lim_{\xi\to+\infty}\xi^{b(\lambda)}v_\epsilon(\xi)=c_2>0, &&\lim_{\xi\to+\infty}\xi^{b(\lambda)+1}v_\epsilon'(\xi)=c_2b(\lambda)>0,\end{aligned} \tag{3.4}$$

where c_i $(i=1,2)$ are positive constants and $a(\lambda),b(\lambda)$ are the zeros of the function

$$\phi(t)=(p-1)t^p-\frac{N}{r}t^{p-1}+\lambda,\quad t\geqslant 0,\ 0\leqslant\lambda<\overline{\lambda}$$

with $0\leqslant a(\lambda)<\frac{N}{r}<b(\lambda)<\frac{Np}{(p-1)r}$.

Furthermore, there exist the positive constants c_3,c_4 such that

$$c_3\leqslant v_\epsilon(x)(|x|^{a(\lambda)/\delta}+|x|^{b(\lambda)/\delta})^\delta\leqslant c_4,\quad \delta=\frac{N}{r}. \tag{3.5}$$

Proof As in [19], we can prove that the limiting problem (3.1) has radially symmetric ground states, by which S_λ can be achieved. Let $u(\xi)$ be a radial solution to (3.1). Then we get that

$$(\xi^{N-1-ap}|u'|^{p-2}u')'+\xi^{N-1}(\lambda\frac{u^{p-1}}{\xi^{p(a+1)}}+u^{r-1})=0.$$

Set

$$\delta=\frac{N}{r},\quad t=\ln\xi,\quad y(t)=\xi^\delta u(\xi),\quad z(t)=\xi^{(1+\delta)(p-1)}|u'(\xi)|^{p-2}u'(\xi).$$

Then we can obtain the following system

$$\begin{cases}\dfrac{dy}{dt}=\delta y+|z|^{\frac{2-p}{p-1}}z,\\ \dfrac{dz}{dt}=-\delta z-|y|^{r-2}y-\lambda|y|^{p-2}y.\end{cases}$$

The rest of the proof follows exactly the same lines as that of the limiting problem (3.1) in [19], here we omit it.

By Proposition 3.1, we can easily derive the minimizing problem

$$\inf_{u\in N^\infty}I^\infty(u)=(\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}}, \tag{3.6}$$

where

$$N^\infty=\{u\in\mathbf{W}_a^{1,p}(\mathbb{R}^N)\setminus\{0\}|\langle(I^\infty)'(u),u\rangle=0\}.$$

For our purpose, the functional I_μ is said to satisfy the $(\mathrm{P.S.})_c$ condition if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{W}_a^{1,p}(\mathbb{R}^N)$ such that as $n \to \infty$,

$$I_\mu(u_n) \to c, \quad I'_\mu(u_n) \to 0 \quad \text{strongly in } (\mathbf{W}_a^{1,p}(\mathbb{R}^N))^*$$

contains a convergent subsequence in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$. Then the following proposition develops a precise description for the $(\mathrm{P.S.})_c$-sequence of I_μ.

Proposition 3.2 (i) If $\mu \in (0, L_1)$, then I_μ has a $(\mathrm{P.S.})_\alpha$-sequence $\{u_n\}_{n\in\mathbb{N}} \subset N_\mu$.

(ii) If $\mu \in (0, L_2)$, then I_μ has a $(\mathrm{P.S.})_{\alpha^-}$-sequence $\{u_n\}_{n\in\mathbb{N}} \subset N_\mu^-$.

Proof The proof is similar to the argument of Proposition 3.3 in [20].

Now, we establish the existence of a local minimizer for I_μ on N_μ.

Proposition 3.3 For $\mu \in (0, L_1)$, the functional I_μ has a minimizer $u_\mu^+ \in N_\mu^+$ satisfying

(i) $I_\mu(u_\mu^+) = \alpha^+ = \alpha$;

(ii) u_μ^+ is a positive solution of (1.1);

(iii) $\|u_\mu^+\| \to 0$ as $\mu \to 0^+$.

Proof By Proposition 3.2 (i), there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset N_\mu$ such that

$$I_\mu(u_n) = \alpha + o(1) \text{ and } I'_\mu(u_n) = o(1) \text{ in } (\mathbf{W}_a^{1,p}(\mathbb{R}^N))^{-1}, \tag{3.7}$$

where $o(1) \to 0$ as $n \to \infty$. Since I_μ is coercive on N_μ, we get that $\{u_n\}$ is bounded in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$. If necessary to a subsequence, there exists $u_\mu^+ \in \mathbf{W}_a^{1,p}(\mathbb{R}^N)$ such that as $n \to \infty$,

$$\begin{cases} u_n \rightharpoonup u_\mu^+ & \text{weakly in } \mathbf{W}_a^{1,p}(\mathbb{R}^N), \\ u_n \to u_\mu^+ & \text{a.e. in } \mathbb{R}^N, \\ \nabla u_n \to \nabla u_\mu^+ & \text{a.e. in } \mathbb{R}^N, \\ \dfrac{u_n}{|x|^{a+1}} \rightharpoonup \dfrac{u_\mu^+}{|x|^{a+1}} & \text{a.e. in } L^p(\mathbb{R}^N), \\ \displaystyle\int_{\mathbb{R}^N} g|u_n|^{r-2}u_n v dx \to \int_{\mathbb{R}^N} g|u_\mu^+|^{r-2}u_\mu^+ v dx & \text{for all } v \in \mathbf{W}_a^{1,p}(\mathbb{R}^N). \end{cases} \tag{3.8}$$

Moreover, by the Egorov Theorem and Hölder inequality, we have

$$\int_{\mathbb{R}^N} f_\mu |u_n|^{q-2} u_n v dx = \int_{\mathbb{R}^N} f_\mu |u_\mu^+|^{q-2} u_\mu^+ v dx + o(1).$$

Consequently, passing to the limit in $\langle I'_\mu(u_n), v\rangle$, by (3.7) and (3.8), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (|x|^{-ap}|\nabla u_\mu^+|^{p-2}\nabla u_\mu^+ \nabla v - \lambda \frac{|u_\mu^+|^{p-2}u_\mu^+ v}{|x|^{p(a+1)}}) dx \\ &- \int_{\mathbb{R}^N} f_\mu |u_\mu^+|^{q-2} u_\mu^+ v dx - \int_{\mathbb{R}^N} g|u_\mu^+|^{r-2} u_\mu^+ v dx = 0 \end{aligned}$$

for all $v \in \mathbf{W}_a^{1,p}(\mathbb{R}^N)$. That is, $\langle I'_\mu(u_\mu^+), v\rangle = 0$. Thus, u_μ^+ is a weak solution of (1.1).

Furthermore, since $u_n \in N_\mu$, we can deduce that

$$\int_{\mathbb{R}^N} f_\mu |u_n|^q dx = \frac{q(r-p)}{p(r-q)} \|u_n\|^p - \frac{r \cdot q}{r-q} I_\mu(u_n), \tag{3.9}$$

which implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f_\mu |u_n|^q dx = \int_{\mathbb{R}^N} f_\mu |u_\mu^+|^q dx \geqslant -\frac{r \cdot q}{r-q} \alpha > 0. \tag{3.10}$$

Thus, $u_\mu^+ \in N_\mu$ is a nontrival solution of (1.1).

Next, we will show, up to a subsequence, that $u_n \to u_\mu^+$ strongly in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$ and $I_\mu(u_\mu^+) = \alpha$. In fact, by the Fatou's lemma, it follows that

$$\begin{aligned} \alpha \leqslant I_\mu(u_\mu^+) &= (\frac{1}{p} - \frac{1}{r})\|u_\mu^+\|^p - (\frac{1}{q} - \frac{1}{r}) \int_{\mathbb{R}^N} f_\mu |u_\mu^+|^q dx \\ &\leqslant \lim_{n\to\infty} \inf((\frac{1}{p} - \frac{1}{r})\|u_n\|^p - (\frac{1}{q} - \frac{1}{r}) \int_\Omega f_\mu |u_n|^q dx) \\ &= \lim_{n\to\infty} \inf I_\mu(u_n) = \alpha, \end{aligned}$$

which implies that $I_\mu(u_\mu^+) = \alpha$ and $\lim\limits_{n\to\infty} \|u_n\|^p = \|u_\mu^+\|^p$. Standard argument shows that $u_n \to u_\mu^+$ strongly in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$.

Moreover, we have $u_\mu^+ \in N_\mu^+$. Otherwise, if $u_\mu^+ \in N_\mu^-$, then by Lemma 2.2 and Lemma 2.4, there is a unique $t^- = \frac{1}{\|u_\mu^+\|} t^-(\frac{u_\mu^+}{\|u_\mu^+\|})$ such that $t^- u_\mu^+ \in N_\mu^-$ and so

$$0 > \alpha^+ = \alpha = I_\mu(u_\mu^+) = I_\mu(t^- u_\mu^+) = \sup_{t\geqslant 0} I_\mu(tu) > \alpha^-,$$

which is a contradiction. Since $I_\mu(u_\mu^+) = I_\mu(|u_\mu^+|)$ and $|u_\mu^+| \in N_\mu^+$, we may assume that u_μ^+ is a nontrivial nonnegative solution of (1.1). By Harnack inequality, it follows that $u_\mu^+ > 0$ in $\mathbb{R}^N$.

Finally, by (2.1) and the Hölder inequality, we can obtain

$$\|u_\mu^+\|^{p-q} < \mu \frac{r-q}{r-p} \|f_+\|_{L^{q^*}} S_\lambda^{-\frac{q}{p}},$$

which implies that $\|u_\mu^+\| \to 0$ as $\mu \to 0^+$. This completes the proof.

Let $u_l = u_0(x+le)$, for $l \in \mathbb{R}$ and $e \in \mathbb{S}^{N-1}$, where $u_0(x)$ is a radially symmetric positive solution of (3.1) such that $I^\infty(u_0) = (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}}$ and $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N \,|\, |x| = 1\}$. Then we have the following result.

Lemma 3.4 (i) $\lim\limits_{l\to 0} \|u_l\|^p = S_\lambda^{\frac{r}{r-p}}$ uniformly in $e \in \mathbb{S}^{N-1}$;

(ii) $\lim\limits_{l\to 0} \int_{\mathbb{R}^N} |u_l|^r dx = S_\lambda^{\frac{r}{r-p}}$ uniformly in $e \in \mathbb{S}^{N-1}$;

(iii) $\lim\limits_{l\to 0} I^\infty(u_l) = (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}}$ uniformly in $e \in \mathbb{S}^{N-1}$.

We refer to the argument of Lemma 4.2 in He and Yang (see [21]).

4 The Estimates of Energy

The following statement is paramount to prove our main result.

Proposition 4.1 For $\mu \in (0, L_2)$, we have $\alpha^- < \alpha^+ + (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}}$.

Proof Let $u_\mu^+ \in N_\mu^+$ be a positive solution of (1.1) in Proposition 3.3. Then we obtain

$$\begin{aligned}
I_\mu(u_\mu^+ + tu_l) &= \frac{1}{p}\|u_\mu^+ + tu_l\|^p - \frac{1}{q}\int_{\mathbb{R}^N} f_\mu |u_\mu^+ + tu_l|^q dx - \frac{1}{r}\int_{\mathbb{R}^N} g|u_\mu^+ + tu_l|^r dx \\
&= I_\mu(u_\mu^+) + I^\infty(tu_l) + \frac{1}{p}(\|u_\mu^+ + tu_l\|^p - \|u_\mu^+\|^p - \|tu_l\|^p) \\
&\quad - \frac{1}{q}\int_{\mathbb{R}^N} f_\mu(|u_\mu^+ + tu_l|^q - |u_\mu^+|^q)dx \\
&\quad - \frac{1}{r}\int_{\mathbb{R}^N} g(|u_\mu^+ + tu_l|^r - |u_\mu^+|^r)dx + \frac{1}{r}\int_{\mathbb{R}^N} t^r|u_l|^r dx \\
&\leqslant I_\mu(u_\mu^+) + I^\infty(tu_l) + \frac{1}{p}(\|u_\mu^+ + tu_l\|^p - \|u_\mu^+\|^p - \|tu_l\|^p) \\
&\quad - \int_{\mathbb{R}^N} f_\mu\left\{\int_0^{tu_l}[(u_\mu^+ + \eta)^{q-1} - (u_\mu^+)^{q-1}]d\eta\right\}dx \\
&\quad + \frac{1}{r}\int_{\mathbb{R}^N}(1-g)t^r|u_l|^r dx - \frac{1}{r}\int_{\mathbb{R}^N} g(|u_\mu^+ + tu_l|^r - |u_\mu^+|^r - t^r|u_l|^r)dx \\
&\leqslant \alpha^+ + (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}} + \frac{1}{p}(\|u_\mu^+ + tu_l\|^p - \|u_\mu^+\|^p - \|tu_l\|^p) \\
&\quad + \frac{t^q}{q}\int_{\mathbb{R}^N}|f_-||u_l|^q dx + \frac{t^r}{r}\int_{\mathbb{R}^N}(1-g_1)|u_l|^r dx - \frac{t^r}{r}\int_{\mathbb{R}^N} g_2|u_l|^r dx.
\end{aligned} \tag{4.1}$$

Since

$$I_\mu(u_\mu^+ + tu_l) \to I_\mu(u_\mu^+) = \alpha^+ < 0 \text{ as } t \to 0$$

and

$$I_\mu(u_\mu^+ + tu_l) \to -\infty \text{ as } t \to +\infty.$$

There exist $0 < t_1 < t_2$ such that

$$I_\mu(u_\mu^+ + tu_l) < \alpha^+ + (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}} \text{ for all } t \in [0, t_1) \cup (t_2, +\infty). \tag{4.2}$$

Thus we only need to show that there exists $l_0 > 0$ such that for $l > l_0$, we have

$$\sup_{t_1 \leqslant t \leqslant t_2} I_\mu(u_\mu^+ + tu_l) < \alpha^+ + (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}}. \tag{4.3}$$

Since $u_\mu^+ + tu_l \to u_\mu^+$ as $l \to \infty$, by Brézis-Lieb lemma, we can find $l_0 > 0$ such that for $l > l_0$,

$$\|u_\mu^+ + tu_l\|^p - \|u_\mu^+\|^p - \|tu_l\|^p < \epsilon_l \text{ for } \epsilon_l > 0 \text{ small enough.} \tag{4.4}$$

For $u, v > 0$, we can remark that $(u+v)^r - u^r - v^r \geqslant 0$, and so

$$\int_{\mathbb{R}^N} g(|u_\mu^+ + tu_l|^r - |u_\mu^+|^r - t^r|u_l|^r)dx \geqslant 0. \tag{4.5}$$

From condition (A_1), (A_2) and (3.5), we can obtain

$$\begin{aligned}
\frac{t^q}{q}\int_{\mathbb{R}^N}|f_-||u_l|^q dx &\leqslant c_f\cdot c_4\frac{t^q}{q}\int_{\mathbb{R}^N}|x+le|^{-r_f}(|x+le|^{a(\lambda)/\delta}+|x+le|^{b(\lambda)/\delta})^{-\delta p}dx\\
&\leqslant C_f\int_{|x|<l}|x+le|^{-r_f-p\cdot a(\lambda)}dx+C_f\int_{|x|\geqslant l}|x+le|^{-r_f-p\cdot a(\lambda)}dx\\
&\leqslant C_f l^N\int_{|x|<1}|x+le|^{-r_f-p\cdot a(\lambda)}dx+C_f\int_{|x|\geqslant l}|x+le|^{-r_f-p\cdot a(\lambda)}dx\\
&\leqslant C_f(l+1)^{N-r_f-p\cdot a(\lambda)}\quad\text{for all } l\geqslant 1, \qquad (4.6)\\
\frac{t^r}{r}\int_{\mathbb{R}^N}(1-g_1)|u_l|^r dx &\leqslant c_{g_1}\cdot c_4\int_{\mathbb{R}^N}|x+le|^{-r_{g_1}}(|x+le|^{a(\lambda)/\delta}+|x+le|^{b(\lambda)/\delta})^{-\delta r}dx\\
&\leqslant C_{g_1}\int_{|x|<l}|x+le|^{-r_{g_1}-p\cdot a(\lambda)}dx+C_{g_1}\int_{|x|\geqslant l}|x+le|^{-r_{g_1}-p\cdot a(\lambda)}dx\\
&\leqslant C_{g_1}l^N\int_{|x|<1}|x+le|^{-r_{g_1}-p\cdot a(\lambda)}dx+C_{g_1}\int_{|x|\geqslant l}|x+le|^{-r_{g_1}-p\cdot a(\lambda)}dx\\
&\leqslant C_{g_1}(l+1)^{N-r_{g_1}-p\cdot a(\lambda)}\quad\text{for all } l\geqslant 1 \qquad (4.7)
\end{aligned}$$

and

$$\begin{aligned}
\frac{t^r}{r}\int_{\mathbb{R}^N}g_2|u_l|^r dx=\frac{t^r}{r}\int_{\mathbb{R}^N}g_2(x-le)|u_o|^r dx &\geqslant (\min_{x\in B^N(1)}u_0^r(x))\int_{B^N(1)}g_2(x-le)dx\\
&\geqslant\left(\min_{x\in B^N(1)}u_0^r(x)\right)c_{g_2}\int_{B^N(1)}l^{-r_{g_2}}dx \qquad (4.8)\\
&\geqslant\left(\min_{x\in B^N(1)}u_0^r(x)\right)c_{g_2}l^{-r_{g_2}}.
\end{aligned}$$

Since $0<r_{g_2}<\min\{r_f-N, r_{g_1}-N\}$ and $t_1\leqslant t\leqslant t_2$, by (4.1)–(4.8), we can find $l_0>0$ such that

$$\sup_{t\geqslant 0}I_\mu(u_\mu^+ + tu_l)<\alpha^+ + (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}}\quad\text{for all } l>\max\{l_0,1\}.$$

In order to complete the proof of Proposition 4.1, it remains to show that there exists a positive number t_* such that $u_\mu^+ + t_* u_l\in N_\mu^-$. Let

$$U_1=\left\{u\in\mathbf{W}_a^{1,p}(\mathbb{R}^N)\setminus\{0\}\middle|\frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right)>1\right\}\cup\{0\},$$
$$U_2=\left\{u\in\mathbf{W}_a^{1,p}(\mathbb{R}^N)\setminus\{0\}\middle|\frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right)<1\right\}.$$

Then the manifold N_μ^- divides $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$ into two connected components U_1 and U_2, and $\mathbf{W}_a^{1,p}(\mathbb{R}^N)\setminus N_\mu^- = U_1\cup U_2$. For each $u\in N_\mu^+$, we have $1<t_{\max}(u)<t^-(u)$. Since $t^-(u)=\frac{1}{\|u\|}t^-(\frac{u}{\|u\|})$, we can obtain $N_\mu^+\subset U_1$ and so $u_\mu^+\in U_1$.

Next we claim that there exists $t_0>0$ such that $u_\mu^+ + t_0u_l\in U_2$. In fact, we find a constant $c>0$ such that $0<t^-(\frac{u_\mu^+ + tu_l}{\|u_\mu^+ + tu_l\|})<c$ for each $t>0$. If not, then we may assume that there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$ and $t^-(\frac{u_\mu^+ + t_n u_l}{\|u_\mu^+ + t_n u_l\|}) \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_\mu^+ + t_n u_l}{\|u_\mu^+ + t_n u_l\|}$. Since $t^-(v_n)v_n \in N_\mu^-$ and by the Lebesgue dominated convergence theorem, we can deduce

$$\begin{aligned}\int_{\mathbb{R}^N} g v_n^r dx &= \frac{1}{\|u_\mu^+ + t_n u_l\|^r} \int_{\mathbb{R}^N} g(u_\mu^+ + t_n u_l)^r dx \\ &= \frac{1}{\|\frac{u_\mu^+}{t_n} + u_l\|^r} \int_{\mathbb{R}^N} g(\frac{u_\mu^+}{t_n} + u_l)^r dx \to \frac{\int_{\mathbb{R}^N} g u_l^r dx}{\|u_l\|^r} \quad \text{as } n \to \infty.\end{aligned}$$

Then we have

$$\begin{aligned}I_\mu(t^-(v_n)v_n) &= \frac{1}{p}(t^-(v_n))^p - \frac{(t^-(v_n))^q}{q} \int_{\mathbb{R}^N} f_\mu v_n^q dx - \frac{(t^-(v_n))^r}{r} \int_{\mathbb{R}^N} g v_n^r dx \\ &\to -\infty \text{ as } n \to \infty,\end{aligned}$$

which contradicts the fact that I_μ is bounded below on N_μ. Let

$$t_0 = \left(\frac{(\int_{\mathbb{R}^N} |u_l|^r dx)^{\frac{p}{r}} + 1}{\|u_l\|^p} |c^p - \|u_\mu^+\|^p| \right)^{\frac{1}{p}} + 1. \tag{4.9}$$

By (4.4) and Lemma 3.4, we have, as $l \to \infty$,

$$\begin{aligned}\|u_\mu^+ + t_0 u_l\|^p &= \|u_\mu^+\|^p + t_0^p \|u_l\|^p + o(1) > \|u_\mu^+\|^p + |c^p - \|u_\mu^+\|^p| + o(1) \\ &> c^p + o(1) > (t^-(\frac{u_\mu^+ + t_0 u_l}{\|u_\mu^+ + t_0 u_l\|}))^p + o(1).\end{aligned}$$

Thus there exists $l_0 > 0$ such that for $l > l_0$, we get

$$\frac{1}{\|u_\mu^+ + t_0 u_l\|} t^-(\frac{u_\mu^+ + t_0 u_l}{\|u_\mu^+ + t_0 u_l\|}) < 1$$

or $u_\mu^+ + t_0 u_l \in U_2$. Define a path $\gamma(s) = u_\mu^+ + s t_0 u_l$ for $s \in [0, 1]$, and so

$$\gamma(0) = u_\mu^+ \in U_1, \quad \gamma(1) = u_\mu^+ + t_0 u_l \in U_2.$$

By Lemma 2.4, we have $\frac{1}{\|u\|} t^-(\frac{u}{\|u\|})$ is a continuous function for $u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \setminus \{0\}$ and $\gamma([0, 1])$ is connected. Then there exists $s_0 \in (0, 1)$ such that $u_\mu^+ + s_0 t_0 u_l \in N_\mu^-$. Take $t_* = s_0 t_0$ and this proof is completed.

Then we have the following result.

Theorem 4.2 For $\mu \in (0, L_2)$, (1.1) has a positive solution $u_\mu^- \in N_\mu^-$ such that $I_\mu(u_\mu^-) = \alpha^-$.

Proof By Ekeland's variational principle [22], there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset N_\mu^-$ such that

$$I_\mu(u_n) = \alpha^- + o(1) \quad \text{and} \quad I'_\mu(u_n) = o(1) \quad \text{in } (\mathbf{W}_a^{1,p}(\mathbb{R}^N))^{-1}.$$

Since $\alpha^- < \alpha^+ + (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}}$, by Lemma 2.3 and Proposition 3.2, there exists a subsequence $\{u_n\}_{n\in\mathbb{N}}$ and a non-zero solution $u_\mu^- \in N_\mu^-$ of (1.1) such that as $n \to \infty$, it holds

$$u_n \to u_\mu^- \quad \text{in } \mathbf{W}_a^{1,p}(\mathbb{R}^N).$$

Since $I_\mu(u_\mu^-) = I_\mu(|u_\mu^-|)$ and $|u_\mu^-| \in N_\mu^-$, u_μ^- is a positive solution of (1.1). We finish the proof.

5 Concentration Behavior

In this section, we discuss the concentration behavior of solutions to (1.1) so that we can get the proof of Theorem 1.1 (ii).

Lemma 5.1 We have

$$\inf_{u\in N_0} I_0(u) = \inf_{u\in N^\infty} I^\infty(u) = (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}}. \tag{5.1}$$

Furthermore, (1.1) does not admit any solution $w_0 \in \mathbf{W}_a^{1,p}(\mathbb{R}^N)$ such that $I_0(w_0) = \inf\limits_{u\in N_0} I_0(u)$.

Proof By Lemma 2.4, there exists the unique $t^-(u_l) > 0$ such that $t^-(u_l)u_l \in N_0$ for all $l > 0$, that is,

$$\|t^-(u_l)u_l\|^p = \int_{\mathbb{R}^N} f_-|t^-(u_l)u_l|^q dx + \int_{\mathbb{R}^N} g|t^-(u_l)u_l|^r dx. \tag{5.2}$$

Since

$$\|u_l\|^p = \int_{\mathbb{R}^N} |u_l|^r dx = S_\lambda^{\frac{r}{r-p}} \quad \text{for all } l \geqslant 0 \tag{5.3}$$

and

$$\int_{\mathbb{R}^N} f_-|u_l|^q dx \to 0 \text{ and } \int_{\mathbb{R}^N} (1-g)|u_l|^r dx \to 0 \quad \text{as } l \to \infty. \tag{5.4}$$

By (5.2)–(5.4), we have $t^-(u_l) \to 1$ as $l \to \infty$. Thus

$$\lim_{l\to\infty} I_0(t^-(u_l)u_l) = \lim_{l\to\infty} I^\infty(t^-(u_l)u_l) = (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}}. \tag{5.5}$$

Then we can obtain

$$\inf_{u\in N_0} I_0(u) \leqslant (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}} = \inf_{u\in N^\infty} I^\infty(u). \tag{5.6}$$

For $u \in N_0$, by Lemma 2.4 (i),

$$I_0(u) = I_0(t^-(\frac{u}{\|u\|})\frac{u}{\|u\|}) = \sup_{t\geqslant 0} I(tu). \tag{5.7}$$

Moreover, there exists a unique $t^\infty > 0$ such that $t^\infty u \in N^\infty$. Thus,

$$I_0(u) = I_0(t^\infty u) \geqslant I^\infty(t^\infty u) \geqslant (\frac{1}{p} - \frac{1}{r})S_\lambda^{\frac{r}{r-p}} \tag{5.8}$$

and so $\inf\limits_{u\in N_0} I_0(u) \geqslant (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}}$. Then we have

$$\inf_{u\in N_0} I_0(u) = \inf_{u\in N^\infty} I^\infty(u) = (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}}. \tag{5.9}$$

In order to show that (1.1) does not admit any solution w_0 such that $I_0(w_0) = \inf\limits_{u\in N_0} I_0(u)$, we argue by the contrary. By Lemma 2.4 (i), we have $I_0(w_0) = \sup\limits_{t\geqslant 0} I_0(tw_0)$. Moreover, there exists a unique $t_{w_0} > 0$ such that $t_{w_0}w_0 \in N^\infty$. Thus we obtain

$$(\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}} = \inf_{u\in N_0} I_0(u) = I_0(w_0) = I^\infty(t_{w_0}w_0) - \frac{1}{q}\int_{\mathbb{R}^N} f_-|t_{w_0}w_0|^q dx, \tag{5.10}$$

and this implies $\int_{\mathbb{R}^N} f_-|w_0|^q dx = 0$, that is, $w_0 \equiv 0$ in $\{x\in\mathbb{R}^{\mathbb{N}} | f_-(x) \neq 0\}$ from (A$_1$). Then we can obtain

$$(\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}} = \inf_{u\in N^\infty} I^\infty(u) = I^\infty(t_{w_0}w_0).$$

By the Lagrange multiplier and the maximum principle, we may assume that $t_{w_0}w_0$ is a positive solution of (1.1). This contradiction completes the proof.

Lemma 5.2 Assume that $\{u_n\}$ is a minimizing sequence in N_0 for I_0. Then

(i) $\int_{\mathbb{R}^N} f_-|u_n|^q dx = o(1)$;

(ii) $\int_{\mathbb{R}^N} (1-g)|u_n|^r dx = o(1)$.

Furthermore, $\{u_n\}$ is a (P.S.)$_{(\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}}}$-sequence in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$ for I^∞.

Proof For each n, there exists a unique $t_n > 0$ such that $t_nu_n \in N^\infty$, that is,

$$t_n^p\|u_n\|^p = t_n^r\int_{\mathbb{R}^N} |u_n|^r dx.$$

By Lemma 2.4 (i), we have

$$\begin{aligned} I_0(u_n) \geqslant I_0(t_nu_n) &= I^\infty(t_nu_n) - \frac{t_n^q}{q}\int_{\mathbb{R}^N} f_-|u_n|^q dx + \frac{t_n^r}{r}\int_{\mathbb{R}^N}(1-g)|u_n|^r dx \\ &\geqslant (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}} - \frac{t_n^q}{q}\int_{\mathbb{R}^N} f_-|u_n|^q dx + \frac{t_n^r}{r}\int_{\mathbb{R}^N}(1-g)|u_n|^r dx. \end{aligned}$$

Since $I_0(u_n) = (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}} + o(1)$ from Lemma 5.1, we have, as $n\to\infty$,

$$\frac{t_n^q}{q}\int_{\mathbb{R}^N} f_-|u_n|^q dx = o(1)$$

and

$$\frac{t_n^r}{r}\int_{\mathbb{R}^N}(1-g)|u_n|^r dx = o(1).$$

Next, we will show that there exists $M > 0,\ c_0 > 0$ such that $t_n > c_0$ for $n > M$. Suppose the contrary. Then we may assume $t_n \to 0$ as $n\to\infty$. As in the proof of Lemma 2.3, we know that $\|u_n\|$ is uniformly bounded and so $\|t_n u_n\| \to 0$ or $I^\infty(t_n u_n) \to 0$. This contradicts the fact $I^\infty(t_n u_n) \geqslant (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}} > 0$ from Lemma 5.1. Then we have

$$\int_{\mathbb{R}^N} f_- |u_n|^q dx = o(1)$$

and

$$\int_{\mathbb{R}^N} (1-g)|u_n|^r dx = o(1).$$

This implies

$$\|u_n\|^p = \int_{\mathbb{R}^N} |u_n|^r dx + o(1)$$

and so

$$I^\infty(u_n) = (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}} + o(1),$$

that is, $\{u_n\}$ is a $(\mathrm{P.S.})_{(\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}}}$-sequence in $\mathbf{W}_a^{1,p}(\mathbb{R}^N)$ for I^∞. This completes the proof.

Let

$$N_\mu^-(d) = \{u \in N_\mu^- | I_\mu(u) \leqslant (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}} + d\} \quad \text{for } d < 0,$$

be the filtration of the Nehari manifold N_μ. Then we have the following lemmas.

Lemma 5.3 There exists $d_0 < 0$ such that for $u \in N_0(d_0)$, we have

$$\int_{\mathbb{R}^N} \frac{x}{|x|^{1-ap}} (|x|^{-ap} |\nabla u|^p - \frac{\lambda}{|x|^{p(a+1)}} u^p) dx \neq 0.$$

Proof Suppose the contrary. We may assume that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset N_0$ such that $I_0(u_n) = (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}} + o(1)$ and $\int_{\mathbb{R}^N} \frac{x}{|x|^{1-ap}} (|x|^{1-ap} |\nabla u_n|^p - \frac{\lambda}{|x|^{p(a+1)}} u_n^p) dx = o(1)$. By Proposition 3.2 and the concentration-compactness principle (see [23, Theorem 4.1]), there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\|u_n(x) - u_0(x - x_n)\| \to 0 \quad \text{as } n \to \infty. \tag{5.11}$$

Now we will show that $|x_n| \to \infty$ as $n \to \infty$ by contradiction. We may assume that $\{x_n\}$ is bounded and $x_n \to x_*$ for some $x_* \in \mathbb{R}^N$. Then by (5.11),

$$\begin{aligned}\int_{\mathbb{R}^N} f_- |u_n|^q dx &= \int_{\mathbb{R}^N} f_-(x) |u_0(x - x_n)|^q dx + o(1) \\ &= \int_{\mathbb{R}^N} f_-(x + x_*) |u_0(x)|^q dx + o(1),\end{aligned}$$

this contradicts the result of Lemma 5.2 (i). Hence we may assume $\frac{x_n}{|x_n|} \to e$ as $n \to \infty$, where $e \in \mathbb{S}^{N-1}$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^N} \frac{x}{|x|^{1-ap}} (|x^{1-ap}| |\nabla u_n|^p - \frac{\lambda}{|x|^{-ap}} |u_n|^p) dx \\ &= \int_{\mathbb{R}^N} \frac{x + x_n}{|x + x_n|^{1-ap}} (|x + x_n|^{-ap} |\nabla u_0|^p - \frac{\lambda}{|x + x_n|^{p(a+1)}} |u_0|^p) dx \\ &= \int_{\mathbb{R}^N} e |\nabla u_0|^p dx + o(1). \end{aligned}$$

This contradiction completes the proof.

By (2.1) and Lemma 2.4 (i), for each $u \in N_\mu^-$, there exists the unique $t_0^-(u) > 0$ such that $t_0^-(u)u \in N_0$ and $t_0^-(u) > t_{\max}(u) > 0$. Then we have the following result.

Lemma 5.4 Let

$$T = \frac{r-q}{p-q}(1 + \frac{r-p}{r-q}).$$

For each $\mu \in (0, L_2)$ and $u \in N_\mu^-(\alpha^+)$, we have $t_0^-(u) < T^{\frac{1}{r-p}}$.

Proof For $u \in N_\mu^-(\alpha^+)$, we distinguish from the following distinctive cases.

Case (i) $t_0^-(u) < 1$. Since $T > 1$, we have $t_0^-(u) < 1 < T^{\frac{1}{r-p}}$.

Case (ii) $t_0^-(u) \geqslant 1$. Since

$$\begin{aligned}(t_0^-(u))^r \int_{\mathbb{R}^N} g|u|^r dx &= (t_0^-(u))^p \|u\|^p - (t_0^-(u))^q \int_{\mathbb{R}^N} f_-|u|^q dx \\ &\leqslant (t_0^-(u))^p (\|u\|^p + \int_{\mathbb{R}^N} |f_-||u|^q dx)\end{aligned}$$

and by Lemma 2.2 (iii), we have

$$(t_0^-(u))^{r-p} \geqslant \frac{\|u\|^p + \int_{\mathbb{R}^N} |f_-||u|^q dx}{\int_{\mathbb{R}^N} g|u|^r dx}. \tag{5.12}$$

Moreover, from the argument in the proof of Lemma 2.2, we have

$$\|u\| \leqslant \frac{r-q}{r-p} \int_{\mathbb{R}^N} g|u|^r dx, \tag{5.13}$$

$$\|u\| \geqslant \frac{r-q}{r-p} \int_{\mathbb{R}^N} |f_-||u|^q dx. \tag{5.14}$$

Thus, by (5.12)–(5.14), we have

$$\begin{aligned}(t_0(u))^{r-p} &\leqslant \frac{\|u\|^p}{\int_{\mathbb{R}^N} g|u|^r dx} \cdot \frac{1}{\|u\|^p}(\|u\|^p + \int_{\mathbb{R}^N} |f_-||u|^q dx) \\ &\leqslant \frac{r-q}{p-q}(1 + \frac{\int_{\mathbb{R}^N} |f_-||u|^q dx}{\|u\|^p}) \leqslant \frac{r-q}{p-q}(1 + \frac{r-p}{r-q}).\end{aligned}$$

This completes the proof.

Lemma 5.5 There exists $\mu_0 \in (0, L_2)$ such that for each $\mu \in (0, \mu_0)$ and $u \in N_\mu^-(\alpha^+)$,

$$\int_{\mathbb{R}^N} \frac{x}{|x|^{1-ap}}(|x|^{-ap}|\nabla u|^p - \lambda \frac{|u|^p}{|x|^{p(a+1)}})dx \neq 0.$$

Proof For $u \in N_\mu^-(\alpha^+)$, by Lemma 2.4 (i), there exists $t_0^-(u) > 0$ such that $t_0^-(u)u \in N_0$. Moreover, by Lemma 5.4 and the Hölder inequality and Sobolev embedding theorem, we have

$$I_\mu(u) = \sup_{t\geqslant 0} I_\mu(tu) \geqslant I_\mu(t_0^-(u)u) = I_0(t_0^-(u)u) - \frac{\mu(t_0^-(u))^q}{q}\int_{\mathbb{R}^N} f_+|u|^q dx$$

or so

$$\begin{aligned} I_0(t_0^-(u)u) &\leqslant I_\mu(u) + \frac{\mu(t_0^-(u))^q}{q}\int_{\mathbb{R}^N} f_+|u|^q dx \\ &< \alpha^+ + (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}} + \frac{\mu T^{\frac{q}{r-p}}}{q}\|f_+\|_{L^{q^*}} S_\lambda^{-\frac{q}{p}}\|u\|^q. \end{aligned}$$

Since $I_\mu(u) < \alpha^+ + (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}} < (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}}$, by Lemma 2.1, for $\mu \in (0, L_2)$ and $u \in N_\mu^-(\alpha^+)$, there exists c_* independent of μ such that $\|u\| \leqslant c_*$. Thus,

$$I_0(t_0^-(u)u) < \alpha^+ + (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}} + \frac{\mu T^{\frac{q}{r-p}}}{q}\|f_+\|_{L^{q^*}} S_\lambda^{-\frac{q}{p}} c_*^q.$$

Then by Lemma 5.3, we have

$$\int_{\mathbb{R}^N} \frac{x}{|x|^{1-ap}}(|x|^{-ap}|\nabla(t_0^-(u)u)|^p - \frac{\lambda}{|x|^{p(a+1)}}|t_0^-(u)u|^p)dx \neq 0$$

and this implies

$$\int_{\mathbb{R}^N} \frac{x}{|x|^{1-ap}}(|x|^{-ap}|\nabla u|^p - \frac{\lambda}{|x|^{p(a+1)}}|u|^p)dx \neq 0 \quad \text{for } u \in N_\mu^-(\alpha^+).$$

The proof is completed.

6 Proof of Theorem 1.1

In this section, we will follow an idea in [24] to prove our main result. For $c \in \mathbb{R}^+$, we denote

$$[I_\mu \leqslant c] = \{u \in N_\mu^- | u \geqslant 0,\ I_\mu(u) \leqslant c\}.$$

Then we try to show that for a sufficiently small $\sigma > 0$, we have

$$\mathrm{cat}([I_\mu \leqslant \alpha^+ + (\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}} - \sigma]) \geqslant 2 \tag{6.1}$$

Here ‘cat’ means the Lusternik-Schnirelmann category [15]. First, let us recall its definition.

Definition 6.1 A non-empty, closed subset Y is contractible in a topological space $\mathbf{X}$ if there exists $h \in \mathbf{C}([0,1]\times Y,\ \mathbf{X})$ such that for some $x_0 \in \mathbf{X}$,

$$h(0,x) = x, \quad h(1,x) = x_0.$$

Definition 6.2 Let $Y_1, Y_2, \cdots, Y_k$ be closed subsets of a topological space $\mathbf{X}$. The category of $\mathbf{X}$ is the least integer k such that Y_j is contractible in $\mathbf{X}$ for all j and $\cup_{j=1}^k Y_j = \mathbf{X}$, denoted by $\mathrm{cat}(\mathbf{X})$.

When there do not exist finitely many closed subsets $Y_1, Y_2, \cdots, Y_k \subset \mathbf{X}$ such that Y_j is contractible in $\mathbf{X}$ for all j and $\bigcup_{j=1}^{k} Y_j = \mathbf{X}$, we denote $\mathrm{cat}(\mathbf{X}) = \infty$. We need the following lemmas (see Theorem 2.3 in [25] and Lemma 2.5 in [24]).

Lemma 6.3 Let $\mathbf{X}$ be a Hilbert manifold and $F \in C^1(\mathbf{X}, \mathbb{R})$. Assume that there are $c_0 \in \mathbb{R}$ and $k \in \mathbf{N}$ such that

(i) F satisfies the Palais-Smale condition for energy level $c \leqslant c_0$;

(ii) $\mathrm{cat}(\{x \in \mathbf{X} | F(x) \leqslant c_0\}) \geqslant k$.

Then F has at least k critical points in $\{x \in \mathbf{X} | F(x) \leqslant c_0\}$.

Lemma 6.4 Let $\mathbf{X}$ be a topological space. Assume that there are $\varphi \in C(\mathbb{S}^{N-1}, \mathbf{X})$ and $\psi \in C(\mathbf{X}, \mathbb{S}^{N-1})$ such that $\psi \circ \varphi$ is homotopic to the identity map of $\mathbb{S}^{N-1}$, that is, there exists $h \in C([0,1] \times \mathbb{S}^{N-1}, \mathbb{S}^{N-1})$ such that $h(0,x) = (\psi \circ \varphi)(x),\ \ h(1,x) = x$. Then $\mathrm{cat}(\mathbf{X}) \geqslant 2$.

For $l > l_0$, we define a map $\varphi_\mu : \mathbb{S}^{N-1} \to \mathbf{W}_a^{1,p}(\mathbb{R}^N)$ by

$$\varphi_\mu(e)(x) = u_\mu^+ + t_* u_l \quad \text{for } e \in \mathbb{S}^{N-1},$$

where $u_\mu^+ + t_* u_l$ is as in the proof of Proposition 4.1. Then we have the following result.

Lemma 6.5 There exists a sequence $\{\sigma_l\} \subset \mathbb{R}^+$ with $\sigma_l \to 0$ as $l \to \infty$ such that

$$\varphi_\mu(\mathbb{S}^{N-1}) \subset [I_\mu \leqslant \alpha^+ + (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}} - \sigma_l].$$

Proof By Proposition 4.1, for $l > l_0$, we have $u_\mu^+ + t_* u_l \in N_\mu^-$ and

$$\sup_{t \geqslant 0} I_\mu(u_\mu^+ + t u_l) < \alpha^+ + (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}} \quad \text{uniformly in } e \in \mathbb{S}^{N-1}.$$

Since $\varphi_\mu(\mathbb{S}^{N-1})$ is compact and $I_\mu(u_\mu^+ + t_* u_l) \leqslant \alpha^+ + (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}} - \sigma_l$, the conclusion holds.

From Lemma 5.5, we define a barycenter map, $\psi_\mu : [I_\mu < \alpha^+ + (\frac{1}{p} - \frac{1}{r}) S_\lambda^{\frac{r}{r-p}}] \to \mathbb{S}^{N-1}$ by

$$\psi_\mu(u) = \frac{\displaystyle\int_{\mathbb{R}^N} \frac{x}{|x|^{1-ap}} (|x|^{-ap} |\nabla u|^p - \lambda \frac{|u|^p}{|x|^{p(a+1)}}) dx}{|\displaystyle\int_{\mathbb{R}^N} \frac{x}{|x|^{1-ap}} (|x^{-ap}| |\nabla u|^p - \lambda \frac{|u|^p}{|x|^{p(a+1)}}) dx|}.$$

Then we have the following result.

Lemma 6.6 Let μ_0 be as in Lemma 5.5. Then for $\mu \in (0, \mu_0)$, there exists $l_* > l_0$ such that the map

$$\psi_\mu \circ \varphi_\mu : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1} \quad \text{for } l > l_*$$

is homotopic to the identity operator.

Proof Denote

$$\mathrm{supp}\ \psi_\mu = \{u \in \mathbf{W}_a^{1,p}(\mathbb{R}^N) \backslash \{0\} | \int_{\mathbb{R}^N} \frac{x}{|x|} (|\nabla u|^p - \lambda \frac{|u|^p}{|x|^p}) dx \neq 0\}$$

and define $\widetilde{\psi_\mu}:\text{supp }\psi_\mu\to\mathbb{S}^{N-1}$ by

$$\widetilde{\psi_\mu}(u)=\frac{\displaystyle\int_{\mathbb{R}^N}\frac{x}{|x|}(|\nabla u|^p-\lambda\frac{|u|^p}{|x|^p})dx}{|\displaystyle\int_{\mathbb{R}^N}\frac{x}{|x|}(|\nabla u|^p-\lambda\frac{|u|^p}{|x|^p})dx|}$$

as an extension of ψ_μ. Since $u_l\in\text{supp }\psi_\mu$ for all $e\in\mathbb{S}^{N-1}$ and large enough l, we may assume $\gamma:[s_1,s_2]\to\mathbb{S}^{N-1}$ is a regular geodesic between $\psi_\mu(u_l)$ and $\widetilde{\psi_\mu}(\varphi_\mu(e))$ such that

$$\gamma(s_1)=\psi_\mu(u_l),\quad \gamma(s_2)=\widetilde{\psi_\mu}(\varphi_\mu(e)).$$

By an argument similar to Lemma 5.3, there exists $l_*\geqslant l_0$ such that

$$u_0(x+\frac{l}{2(1-\theta)}e)\in\text{supp }\psi_\mu$$

for all $e\in\mathbb{S}^{N-1},\ l>l_*$ and $\theta\in[\frac{1}{2},1)$. We define

$$h_l(\theta,e):[0,1]\times\mathbb{S}^{N-1}\to\mathbb{S}^{N-1}$$

by

$$h_l(\theta,e)=\begin{cases}\gamma(2\theta(s_1-s_2)+s_2) & \text{for }\theta\in[0,\dfrac{1}{2}),\\ \widetilde{\psi_\mu}(u_0(x+\dfrac{l}{2(1-\theta)}e)) & \text{for }\theta\in[\dfrac{1}{2},1),\\ e & \text{for }\theta=1.\end{cases}$$

Then $h_l(0,e)=\widetilde{\psi_\mu}(\varphi_\mu(e))$ and $h_l(1,e)=e$. By the standard regularity, we have $h_l(\theta,e)\in C(\mathbb{R}^N)$.

Next, we will show that $\lim\limits_{\theta\to1^-}h_l(\theta,e)=e$ and $\lim\limits_{\theta\to\frac{1}{2}^-}h_l(\theta,e)=\widetilde{\psi_\mu}(u_l)$.

(i) $\lim\limits_{\theta\to1^-}h_l(\theta,e)=e$, since

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{x}{|x|}(|\nabla u_0(x+\frac{l}{2(1-\theta)}e)|^p-\frac{\lambda}{|x+\frac{l}{2(1-\theta)}e|^p}u_0^p(x+\frac{l}{2(1-\theta)}e))dx\\ =&\int_{\mathbb{R}^N}\frac{x+\frac{l}{2(1-\theta)}e}{|x+\frac{l}{2(1-\theta)}e|}(|\nabla u_0(x)|^p-\frac{\lambda}{|x+\frac{l}{2(1-\theta)}e|^p}u_0^p(x))dx\\ =&\,e\int_{\mathbb{R}^N}|\nabla u_0|^pdx+o(1)\end{aligned}$$

as $\theta\to1^-$.

(ii) $\lim\limits_{\theta\to\frac{1}{2}^-}h_l(\theta,e)=\widetilde{\psi_\mu}(u_l)$. Since $\widetilde{\psi_\mu}\in C(\text{supp }\psi_\mu,\mathbb{S}^{N-1})$, then we have

$$h_l(\theta,e)\in C([0,1]\times\mathbb{S}^{N-1},\mathbb{S}^{N-1})$$

and

$$h_l(0,e)=\psi_\mu(\varphi_\mu(e)),\quad h_l(1,e)=e,$$

for all $e\in\mathbb{S}^{N-1}$ and $l>l_*$. This completes the proof.

Lemma 6.7 For $\mu\in(0,\mu_0)$ and $l>l_*$, the energy functional I_μ admits at least two critical points in $[I_\mu<\alpha^++(\frac{1}{p}-\frac{1}{r})S_\lambda^{\frac{r}{r-p}}]$.

Proof It is easy to deduce from Lemmas 6.3, 6.4, 6.6 and Proposition 3.2.

Proof of Theorem 1.1 Now we can complete the proof of Theorem 1.1

(i) by Proposition 3.3 and Theorem 4.2;

(ii) from Proposition 3.3 and Lemma 6.7, (1.1) has at least three positive solutions u_μ^+,u_1^-,u_2^-, where $u_\mu^+\in N_\mu^+$ and $u_i^-\in N_\mu^-$ for $i=1,2$.

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一类具有凸凹非线性项与Sobolev-Hardy次临界指标的椭圆方程

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摘要: 本文研究了一类具有凸凹非线性项与Sobolev-Hardy次临界指标的椭圆方程. 利用Lusternik-Schnirelmann畴数理论以及Nehari流形结构与纤维丛映射的关系，改善了方程在Sobolev空间$\mathbf{W}_a^{1,p}(\mathbb{R}^N)$中正解的存在性与多重性.

关键词: 次临界Sobolev-Hardy指标; Nehari流形; 变号位势; 凸凹非线性项

MR(2010)**主题分类号**: 35J30; 35J75 **中图分类号**: O175.25