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CONCENTRATION IN THE FLUX APPROXIMATION LIMIT OF RIEMANN SOLUTIONS TO THE EXTENDED CHAPLYGIN GAS EQUATIONS

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Abstract: In this paper, two kinds of occurrence mechanism on the phenomenon of concentration and the formation of delta shock waves in the flux approximation limit of Riemann solutions to the extended Chaplygin gas equations are analyzed. By phase plane analysis and generalized characteristic analysis, we construct the Riemann solution to the extended Chaplygin gas equations completely and obtain two results: on one hand, as the pressure vanishes, any two-shock Riemann solution to the extended Chaplygin gas equations tends to a δ-shock solution to the transportation equation; on the other hand, as the pressure approaches the generalized Chaplygin pressure, any two-shock Riemann solution tends to a δ-shock solution to the generalized Chaplygin gas equations, which generalize to the extended Chaplygin gas.

Keywords: extended Chaplygin gas; δ-shock wave; flux approximation limit; Riemann solutions; transportation equations; generalized Chaplygin gas

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1 Introduction

The extended Chaplygin gas equations can be expressed as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P)_x = 0, \end{cases} \tag{1.1}$$

where ρ, u and P represent the density, the velocity and the scalar pressure, respectively, and

$$P = A\rho^n - \frac{B}{\rho^\alpha}, \quad 1 \le n \le 3, \quad 0 < \alpha \le 1 \tag{1.2}$$

with two parameters $A, B > 0$.

This model was proposed by Naji [1] to study the evolution of dark energy. For $n = 2$, this model can also be seen as the magnetogasdynamics with generalized Chaplygin pressure [2]. When $B = 0$ in (1.2), $P = A\rho^n$ is the standard state equation for perfect fluid. Up

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to now, various kinds of theoretical models have been proposed to interpret the behavior of dark energy. Specially, when $n = 1$ in (1.2), it reduces to the state equation for modified Chaplygin gas, which was originally proposed by Benaoum in 2002 [3]. As an exotic fluid, such a gas can explain the current accelerated expansion of the universe. Whereas when $A = 0$ in (1.2), $P = -\frac{B}{\rho^\alpha}$ is called the pressure for the generalized Chaplygin gas [4]. Furthermore, when $\alpha = 1$, $P = -\frac{B}{\rho}$ is called the pressure for (pure) Chaplygin gas which was introduced by Chaplygin [5], Tsien [6] and von Karman [7] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. It has also been advertised as a possible model for dark energy [8].

When two parameters A, $B \to 0$, the limit system of (1.1) with (1.2) formally becomes the following transportation equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.3}$$

which is also called the zero-pressure gas dynamics, and can be used to describe some important physical phenomena, such as the motion of free particles sticking together under collision and the formation of large scale structures in the universe [9–11].

The transportation equation (1.3) were studied extensively since 1994. The existence of measure solutions of the Riemann problem was first proved by Bouchut [12] and the existence of the global weak solutions was obtained by Brenier and Grenier [13] and Rykov and Sinai [10]. Sheng and Zhang [14] discovered that the δ-shock and vacuum states do occur in the Riemann solutions to the transportation equations (1.3) by the vanishing viscosity method. Huang and Wang [15] proved the uniqueness of the weak solution when the initial data is a Radon measure. Also see [14, 16–19] for more related results.

δ-shock is a kind of nonclassical nonlinear waves on which at least one of the state variables becomes a singular measure. Korchinski [20] first introduced the concept of the δ-function into the classical weak solution in his unpublished Ph. D. thesis. Tan, Zhang and Zheng [21] considered some 1-D reduced system and discovered that the form of δ-functions supported on shocks was used as parts in their Riemann solutions for certain initial data. LeFloch et al. [22] applied the approach of nonconservative product to consider nonlinear hyperbolic systems in the nonconservative form. Recently, the weak asymptotic method was widely used to study the δ-shock wave type solution by Danilov and Shelkovich et al. [23–25].

As for delta shock waves, one research focus is to explore the phenomena of concentration and cavitation and the formation of delta shock waves and vacuum states in solutions. In [26], Chen and Liu considered the Euler equations for isentropic fluids, i.e., in (1.1), they took the prototypical pressure function as follows:

$$P = \varepsilon \frac{\rho^\gamma}{\gamma}, \quad \gamma > 1. \tag{1.4}$$

They analyzed and identified the phenomena of concentration and cavitation and the formation of δ-shocks and vacuum states as $\varepsilon \to 0$, which checked the numerical observation for the 2-D case by Chang, Chen and Yang [27, 28]. They also pointed out that the occurrence of δ-shocks and vacuum states in the process of vanishing pressure limit can be regarded as a phenomenon of resonance between the two characteristic fields. Moreover, they made a further step to generalize this result to the nonisentropic fluids in [29]. Besides, the results were extended to the relativistic Euler equations for polytropic gases in [30], the perturbed Aw-Rascle model in [31], the magnetogasdynamics with generalized Chaplygin pressure in [2], the modified Chaplygin gas equations in [32, 33], etc.

In this paper, we focus on the extended Chaplygin gas equations (1.1) to discuss the phenomena of concentration and cavitation and the formation of delta shock waves and vacuum states in Riemann solutions as the double parameter pressure vanishes wholly or partly, which corresponds to a two parameter limit of Riemann solutions in contrast to the previous works in [2, 26, 29–31]. Equivalently, we study the limit behavior of Riemann solutions to the extended Chaplygin gas equations as the pressure vanishes, or tends to the generalized Chaplygin pressure.

It is noticed that, When $A,\ B \to 0$, system (1.1) with (1.2) formally becomes the transportation equations (1.3). For fixed B, when $A \to 0$, system (1.1) with (1.2) formally becomes the generalized Chaplygin gas equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \frac{B}{\rho^\alpha})_x = 0. \end{cases} \tag{1.5}$$

When $\alpha = 1$, it is just the Chaplygin gas equations. In 1998, Brenier [34] first studied the 1-D Riemann problem and obtained the solutions with concentration when initial data belongs to a certain domain in the phase plane. Recently, Guo, Sheng and Zhang [35] abandoned this constrain and constructively obtained the general solutions of the 1-D Riemann problem in which the δ-shock wave developed. Moreover, in that paper, they also systematically studied the 2-D Riemann problem for isentropic Chaplygin gas equations. In [36], Wang solved the Riemann problem of (1.5) by the weak asymptotic method. It has been shown that, in their results, δ-shocks do occur in the Riemann solutions, but vacuum states do not. For more results about Chaplygin gas, one can refer to [37–40].

In this paper, we first solve the Riemann problem of system (1.1) with Riemann initial data

$$(\rho, u)(x, 0) = (\rho_\pm, u_\pm), \qquad \pm x > 0, \tag{1.6}$$

where $\rho_\pm > 0$, $u_\pm$ are arbitrary constants. With the phase plane analysis method, we construct the Riemann solutions with four different structures: R_1R_2, R_1S_2, S_1R_2 and S_1S_2.

Then we analyze the formation of δ-shocks and vacuum states in the Riemann solutions to the extended Chaplygin gas equations as the pressure vanishes. It is shown that, as the pressure vanishes, any two-shock Riemann solution tends to a δ-shock solution to the transportation equations, and the intermediate density between the two shocks tends to a weighted δ-measure that forms the δ-shock; by contrast, any two-rarefaction-wave Riemann solution tends to a two-contact-discontinuity solution to the transportation equations, and the nonvacuum intermediate state between the two rarefaction waves tends to a vacuum state, even when the initial data stays away from the vacuum. As a result, the delta shocks for the transportation equations result from a phenomenon of concentration, while the vacuum states results from a phenomenon of cavitation in the vanishing pressure limit process. These results are completely consistent with that in [26], and also cover those obtained in [2, 32, 33].

In addition, we also prove that as the pressure tends to the generalized Chaplygin pressure ($A \to 0$), any two-shock Riemann solution to the extended Chaplygin gas equations tends to a δ-shock solution to the generalized Chaplygin gas equations, and the intermediate density between the two shocks tends to a weighted δ-measure that forms the δ-shock. Consequently, the delta shocks for the generalized Chaplygin gas equations result from a phenomenon of concentration in the partly vanishing pressure limit process.

From the above analysis, we can find two kinds of occurrence mechanism on the phenomenon of concentration and the formation of delta shock wave. On one hand, since the strict hyperbolicity of the limiting system (1.3) fails, see Section 4, the delta shock wave forms in the limit process as the pressure vanishes. This is consistent with those results obtained in [2, 26, 29–32]. On the other hand, the strict hyperbolicity of the limiting system (1.5) is preserved, see Section 5, the formation of delta shock waves still occur as the pressure partly vanishes. In this regard, it is different from those in [2, 26, 29–32]. In any case, the phenomenon of concentration and the formation of delta shock wave can be regarded as a process of resonance between two characteristic fields.

The paper is organized as follows. In Section 2, we restate the Riemann solutions to transportation equations (1.3) and the generalized Chaplygin gas equations (1.5). In Section 3, we investigate the Riemann problem of the extended Chaplygin gas equations (1.1)–(1.2) and examine the dependence of the Riemann solutions on the two parameters $A, B > 0$. In Section 4, we analyze the limit of Riemann solutions to the extended Chaplygin gas equations (1.1)–(1.2) with (1.6) as the pressure vanishes. In Section 5, we discuss the limit of Riemann solutions to the extended Chaplygin gas equations (1.1)–(1.2) with (1.6) as the pressure approaches to the generalized Chaplygin pressure. Finally, conclusions are drawn and discussions are made in Section 6.

2 Preliminaries

2.1 Riemann Problem for the Transportation Equations

In this section, we restate the Riemann solutions to the transportation equations (1.3) with initial data (1.6), see [14] for more details.

The transportation equations (1.3) have a double eigenvalue $\lambda = u$ and only one right eigenvectors $\vec{r} = (1, 0)^T$. Furthermore, we have $\nabla\lambda \cdot \vec{r} = 0$, which means that λ is linearly degenerate. The Riemann problem (1.3) and (1.6) can be solved by contact discontinuities, vacuum or δ-shocks connecting two constant states $(\rho_\pm, u_\pm)$.

By taking the self-similar transformation $\xi = \frac{x}{t}$, the Riemann problem is reduced to the boundary value problem of the ordinary differential equations:

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u^2)_\xi = 0 \end{cases} \tag{2.1}$$

with $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$.

For the case $u_- < u_+$, there is no characteristic passing through the region $\{\xi : u_- < \xi < u_+\}$, so the vacuum should appear in the region. The solution can be expressed as

$$(\rho, u)(\xi) = \begin{cases} (\rho_-, u_-), & -\infty < \xi \le u_-, \\ (0, \xi), & u_- < \xi < u_+, \\ (\rho_+, u_+), & u_+ \le \xi < \infty. \end{cases} \tag{2.2}$$

For the case $u_- = u_+$, it is easy to see that the constant states $(\rho_\pm, u_\pm)$ can be connected by a contact discontinuity.

For the case $u_- > u_+$, a solution containing a weighted δ-measure supported on a curve will be constructed. Let $x = x(t)$ be a discontinuity curve, we consider a piecewise smooth solution of (1.3) in the form

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+), & x > x(t). \end{cases} \tag{2.3}$$

To define the measure solutions, a two-dimensional weighted δ-measure $p(s)\delta_S$ supported on a smooth curve $S = \{(x(s), t(s)) : a < s < b\}$ can be defined as

$$\langle p(s)\delta_S, \psi(x(s), t(s))\rangle = \int_a^b p(s)\psi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2}ds \tag{2.4}$$

for any $\psi \in C_0^\infty(R \times R_+)$.

For convenience, we usually select the parameter $s = t$ and use $w(t) = \sqrt{1 + x'(t)^2}p(t)$ to denote the strength of the δ shock wave from now on.

As shown in [14], for any $\psi \in C_0^\infty(R \times R_+)$, the δ-measure solution (2.3) constructed above satisfies

$$\begin{cases} \langle \rho, \psi_t\rangle + \langle \rho u, \psi_x\rangle = 0, \\ \langle \rho u, \psi_t\rangle + \langle \rho u^2, \psi_x\rangle = 0, \end{cases} \tag{2.5}$$

in which

$$\langle \rho, \psi\rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0\psi dxdt + \langle w_1(\cdot)\delta_S, \psi(\cdot,\cdot)\rangle,$$

$$\langle \rho u, \psi\rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0 u_0\psi dxdt + \langle w_2(\cdot)\delta_S, \psi(\cdot,\cdot)\rangle,$$

where

$$\rho_0 = \rho_- + [\rho]H(x - \sigma t), \quad \rho_0 u_0 = \rho_- u_- + [\rho u]H(x - \sigma t)$$

and

$$w_1(t) = \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho u]), \quad w_2(t) = \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho u] - [\rho u^2]).$$

Here, $H(x)$ is the Heaviside function given by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$.

Substituting (2.3) into (2.5), one can derive the generalized Rankine-Hugoniot conditions

$$\begin{cases} \dfrac{dx(t)}{dt} = u_\delta(t), \\ \dfrac{dw(t)}{dt} = [\rho]u_\delta(t) - [\rho u], \\ \dfrac{d(w(t)u_\delta(t))}{dt} = [\rho u]u_\delta(t) - [\rho u^2], \end{cases} \tag{2.6}$$

where $[\rho] = \rho_+ - \rho_-$, etc.

Through solving (2.6) with $x(0) = 0,\ w(t) = 0$, we obtain

$$\begin{cases} u_\delta(t) = \sigma = \dfrac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ x(t) = \sigma t, \\ w(t) = -\sqrt{\rho_-\rho_+}(u_+ - u_-)t. \end{cases} \tag{2.7}$$

Moreover, the δ-measure solution (2.3) with (2.7) satisfies the δ-entropy condition

$$u_+ < \sigma < u_-,$$

which means that all the characteristics on both sides of the δ-shock are incoming.

2.2 Riemann Problem for the Generalized Chaplygin Gas Equations

In this section, we solve the Riemann problem for the generalized Chaplygin gas equations (1.5) with (1.6), which one can also see in [35, 36].

It is easy to see that (1.5) has two eigenvalues

$$\lambda_1^B = u - \sqrt{\alpha B}\rho^{-\frac{\alpha+1}{2}}, \quad \lambda_2^B = u + \sqrt{\alpha B}\rho^{-\frac{\alpha+1}{2}}$$

with corresponding right eigenvectors

$$\overrightarrow{r_1}^B = (-\sqrt{\alpha B}\rho^{-\frac{\alpha+1}{2}}, \rho)^T, \quad \overrightarrow{r_2}^B = (\sqrt{\alpha B}\rho^{-\frac{\alpha+1}{2}}, \rho)^T.$$

So (1.5) is strictly hyperbolic for $\rho > 0$. Moreover, when $0 < \alpha < 1$, we have $\bigtriangledown\lambda_i^B \cdot \overrightarrow{r_i}^B \neq 0$, $i = 1, 2$, which implies that λ_1^B and λ_2^B are both genuinely nonlinear and the associated waves are rarefaction waves and shock waves. When $\alpha = 1$, $\bigtriangledown\lambda_i^B \cdot \overrightarrow{r_i}^B = 0$, $i = 1, 2$, which implies that λ_1^B and λ_2^B are both linearly degenerate and the associated waves are both contact discontinuities, see [41].

Since system (1.5) and the Riemann initial data (1.6) are invariant under stretching of coordinates $(x, t) \to (\beta x, \beta t)$ (β is constant), we seek the self-similar solution

$$(\rho, u)(x, t) = (\rho, u)(\xi), \quad \xi = \frac{x}{t}.$$

Then the Riemann problem (1.5) and (1.6) is reduced to the following boundary value problem of the ordinary differential equations

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u^2 - \frac{B}{\rho^\alpha})_\xi = 0 \end{cases} \tag{2.8}$$

with $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$.

Besides the constant solution, it provides the backward rarefaction wave

$$\overleftarrow{R}(\rho_-, u_-): \begin{cases} \xi = \lambda_1^B = u - \sqrt{\alpha B}\rho^{-\frac{\alpha+1}{2}}, \\ u - \frac{2\sqrt{\alpha B}}{1+\alpha}\rho^{-\frac{\alpha+1}{2}} = u_- - \frac{2\sqrt{\alpha B}}{1+\alpha}\rho_-^{-\frac{\alpha+1}{2}}, \quad \rho < \rho_-, \end{cases} \tag{2.9}$$

and the forward rarefaction wave

$$\overrightarrow{R}(\rho_-, u_-): \begin{cases} \xi = \lambda_2^B = u + \sqrt{\alpha B}\rho^{-\frac{\alpha+1}{2}}, \\ u + \frac{2\sqrt{\alpha B}}{1+\alpha}\rho^{-\frac{\alpha+1}{2}} = u_- + \frac{2\sqrt{\alpha B}}{1+\alpha}\rho_-^{-\frac{\alpha+1}{2}}, \quad \rho > \rho_-. \end{cases} \tag{2.10}$$

When $\alpha = 1$, the backward (forward) rarefaction wave becomes the backward (forward) contact discontinuity.

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot conditions hold:

$$\begin{cases} -\sigma^B[\rho] + [\rho u] = 0, \\ -\sigma^B[\rho u] + [\rho u^2 - \dfrac{B}{\rho^\alpha}] = 0, \end{cases} \tag{2.11}$$

where $[\rho] = \rho - \rho_-$, etc. Together with the Lax shock inequalities, (2.11) gives the backward shock wave

$$\overleftarrow{S}(\rho_-, u_-): \begin{cases} \sigma_1^B = \dfrac{\rho u - \rho_- u_-}{\rho - \rho_-}, \\ u - u_- = -\sqrt{B(\frac{1}{\rho} - \frac{1}{\rho_-})(\frac{1}{\rho^\alpha} - \frac{1}{\rho_-^\alpha})}, \quad \rho > \rho_-, \end{cases} \tag{2.12}$$

and the forward shock wave

$$\overrightarrow{S}(\rho_-, u_-): \begin{cases} \sigma_2^B = \dfrac{\rho u - \rho_- u_-}{\rho - \rho_-}, \\ u - u_- = -\sqrt{B(\frac{1}{\rho} - \frac{1}{\rho_-})(\frac{1}{\rho^\alpha} - \frac{1}{\rho_-^\alpha})}, \quad \rho < \rho_-. \end{cases} \tag{2.13}$$

When $\alpha = 1$, the backward (forward) shock wave becomes the backward (forward) contact discontinuity.

Furthermore, for a given left state (ρ_-, u_-), the backward shock wave $\overleftarrow{S}(\rho_-, u_-)$ has a straight line $u = u_- - \sqrt{B}\rho_-^{-\frac{\alpha+1}{2}}$ as its asymptote, and for a given right state (ρ_+, u_+), the forward shock wave $\overrightarrow{S}(\rho_+, u_+)$ has a straight line $u = u_+ + \sqrt{B}\rho_+^{-\frac{\alpha+1}{2}}$ as its asymptote.

It is easy to see that, when $u_+ + \sqrt{B}\rho_+^{-\frac{\alpha+1}{2}} \leq u_- - \sqrt{B}\rho_-^{-\frac{\alpha+1}{2}}$, the backward shock wave $\overleftarrow{S}(\rho_-, u_-)$ can not intersect with the forward shock wave $\overrightarrow{S}(\rho_+, u_+)$, a delta shock wave must develop in solutions. Under definition (2.4), a delta shock wave can be introduced to construct the solution of (1.5)–(1.6), which can be expressed as

$$(\rho,u)(x,t)=\begin{cases}(\rho_-,u_-), & x<\sigma^Bt,\\ (w^B(t)\delta(x-\sigma^Bt),\sigma^B), & x=\sigma^Bt,\\ (\rho_+,u_+), & x>\sigma^Bt\end{cases} \tag{2.14}$$

with

$$\frac{B}{\rho^\alpha}=\begin{cases}\dfrac{B}{\rho_-^\alpha}, & x<\sigma^Bt,\\ 0, & x=\sigma^Bt,\\ \dfrac{B}{\rho_+^\alpha}, & x>\sigma^Bt,\end{cases}$$

see [34].

By the weak solution definition in Subsection 2.1, for system (1.5), we can get the following generalized Rankine-Hugoniot conditions

$$\begin{cases}\dfrac{dx^B(t)}{dt}=u_\delta^B(t)=\sigma^B,\\ \dfrac{dw^B(t)}{dt}=u_\delta^B(t)[\rho]-[\rho u],\\ \dfrac{d(w^B(t)u_\delta^B(t))}{dt}=u_\delta^B(t)[\rho u]-[\rho u^2-\dfrac{B}{\rho^\alpha}],\end{cases} \tag{2.15}$$

where $x^B(t)$, $w^B(t)$ and $u_\delta^B(t)$ are respectively denote the location, weight and propagation speed of the δ-shock, $[\rho]=\rho(x^B(t)+0,t)-\rho(x^B(t)-0,t)$ denotes the jump of the function ρ across the δ-shock.

Then by solving (2.15) with initial data $x(0)=0$, $w^B(0)=0$, under the entropy condition

$$u_++\sqrt{\alpha B}\rho_+^{-\frac{\alpha+1}{2}}<\sigma^B<u_--\sqrt{\alpha B}\rho_-^{-\frac{\alpha+1}{2}}, \tag{2.16}$$

we can obtain

$$w^B(t)=w_0^Bt, \tag{2.17}$$

$$\sigma^B=\frac{\rho_+u_+-\rho_-u_-+w_0^B}{\rho_+-\rho_-}, \tag{2.18}$$

when $\rho_+\neq\rho_-$, where

$$w_0^B=\big\{\rho_+\rho_-\big((u_+-u_-)^2-(\frac{1}{\rho_+}-\frac{1}{\rho_-})(\frac{B}{\rho_+^\alpha}-\frac{B}{\rho_-^\alpha})\big)\big\}^{\frac12}, \tag{2.19}$$

and

$$w^B(t)=(\rho_-u_--\rho_+u_+)t, \tag{2.20}$$

$$\sigma^B=\frac12(u_++u_-), \tag{2.21}$$

when $\rho_+=\rho_-$.

In the phase plane ($\rho > 0,\ u \in R$), given a constant state (ρ_-, u_-), we draw the elementary wave curves (2.9)–(2.10) and (2.12)–(2.13) passing through this point, which are denoted by $\overleftarrow{R}, \overrightarrow{R}, \overleftarrow{S}$ and $\overrightarrow{S}$, respectively. The backward shock wave $\overleftarrow{S}$ has an asymptotic line $u = u_- - \sqrt{B}\rho_-^{-\frac{\alpha+1}{2}}$. In addition, we draw a S_δ curve, which is determined by

$$u + \sqrt{B}\rho^{-\frac{\alpha+1}{2}} = u_- - \sqrt{B}\rho_-^{-\frac{\alpha+1}{2}},\ \rho > 0. \tag{2.22}$$

Then, the phase plane can be divided into five parts $\mathrm{I}(\rho_-, u_-)$, $\mathrm{II}(\rho_-, u_-)$, $\mathrm{III}(\rho_-, u_-)$, $\mathrm{IV}(\rho_-, u_-)$ and $\mathrm{V}(\rho_-, u_-)$ (see Fig.1).

By the analysis method in the phase plane, one can construct the Riemann solutions for any given (ρ_+, u_+) as follows:

(1) $(\rho_+, u_+) \in \mathrm{I}(\rho_-, u_-)$: $\overleftarrow{R} + \overrightarrow{R}$;

(2) $(\rho_+, u_+) \in \mathrm{II}(\rho_-, u_-)$: $\overleftarrow{R} + \overrightarrow{S}$;

(3) $(\rho_+, u_+) \in \mathrm{III}(\rho_-, u_-)$: $\overleftarrow{S} + \overrightarrow{R}$;

(4) $(\rho_+, u_+) \in \mathrm{IV}(\rho_-, u_-)$: $\overleftarrow{S} + \overrightarrow{S}$;

(5) $(\rho_+, u_+) \in \mathrm{V}(\rho_-, u_-)$: δ-shock.

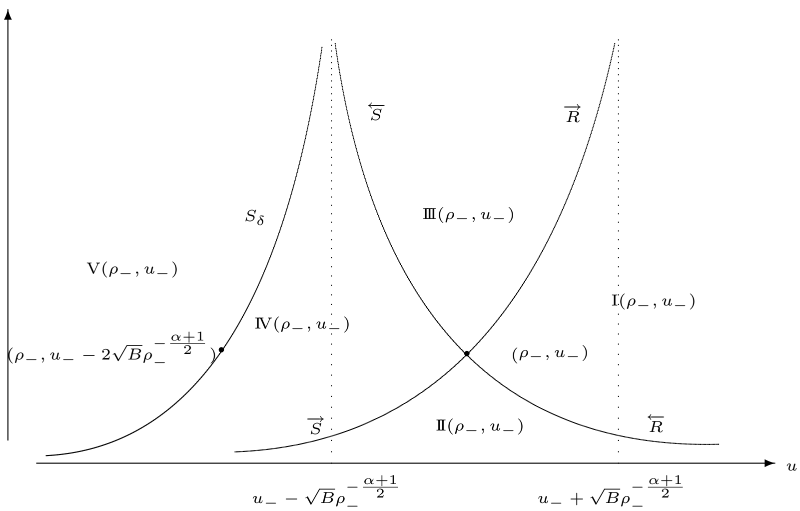

Fig.1 The (ρ, u) phase plane for the genreralized Chaplygin gas equations (1.5).

3 Riemann Problem for the Extended Chaplygin Gas Equations

In this section, we first solve the elementary waves and construct solutions to the Riemann problem of (1.1)–(1.2) with (1.6), and then examine the dependence of the Riemann solutions on the two parameters $A, B > 0$.

The eigenvalues of the system (1.1)–(1.2) are

$$\lambda_1^{AB} = u - \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}},\quad \lambda_2^{AB} = u + \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}$$

with corresponding right eigenvectors

$$\vec{r}_1^{AB} = (-\rho, \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}})^T, \quad \vec{r}_2^{AB} = (\rho, \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}})^T.$$

Moreover, we have

$$\nabla\lambda_i^{AB} \cdot \vec{r}_i^{AB} = \frac{An(n+1)\rho^{n+\alpha} + (1-\alpha)\alpha B}{2\sqrt{(An\rho^{n+\alpha} + \alpha B)\rho^{\alpha+1}}} > 0, \ \ i = 1, 2.$$

Thus λ_1^{AB} and λ_2^{AB} are genuinely nonlinear and the associated elementary waves are shock waves and rarefaction waves.

For (1.1)–(1.2) with (1.6) are invariant under uniform stretching of coordinates: $(x,t) \to (\beta x, \beta t)$ with constant $\beta > 0$, we seek the self-similar solution

$$(\rho, u)(x,t) = (\rho(\xi), u(\xi)), \quad \xi = \frac{x}{t}.$$

Then the Riemann problem (1.1)–(1.2) with (1.6) is reduced to the boundary value problem of the following ordinary differential equations

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u^2 + P)_\xi = 0, \quad P = A\rho^n - \frac{B}{\rho^\alpha} \end{cases} \tag{3.1}$$

with $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$.

Any smooth solutions of (3.1) satisfies

$$\begin{pmatrix} u - \xi & \rho \\ An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}} & \rho(u-\xi) \end{pmatrix} \begin{pmatrix} d\rho \\ du \end{pmatrix} = 0. \tag{3.2}$$

It provides either the constant state solutions $(\rho, u)(\xi) =$ constant, or the rarefaction wave which is a continuous solutions of (3.2) in the form $(\rho, u)(\xi)$. Then, according to [41], for a given left state (ρ_-, u_-), the rarefaction wave curves in the phase plane, which are the sets of states that can be connected on the right by a 1-rarefaction wave or 2-rarefaction wave, are as follows

$$R_1(\rho_-, u_-): \begin{cases} \xi = \lambda_1 = u - \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}, \\ u - u_- = -\int_{\rho_-}^{\rho} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho, \end{cases} \tag{3.3}$$

and

$$R_2(\rho_-, u_-): \begin{cases} \xi = \lambda_2 = u + \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}, \\ u - u_- = \int_{\rho_-}^{\rho} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho. \end{cases} \tag{3.4}$$

From (3.3) and (3.4), we obtain that

$$\frac{d\lambda_1^{AB}}{d\rho}=\frac{\partial\lambda_1^{AB}}{\partial u}\frac{du}{d\rho}+\frac{\partial\lambda_1^{AB}}{\partial\rho}=-\frac{An(n+1)\rho^{n-1}+\dfrac{\alpha(1-\alpha)B}{\rho^{\alpha+1}}}{2\rho\sqrt{An\rho^{n-1}+\dfrac{\alpha B}{\rho^{\alpha+1}}}}<0, \tag{3.5}$$

$$\frac{d\lambda_2^{AB}}{d\rho}=\frac{\partial\lambda_2^{AB}}{\partial u}\frac{du}{d\rho}+\frac{\partial\lambda_2^{AB}}{\partial\rho}=\frac{An(n+1)\rho^{n-1}+\dfrac{\alpha(1-\alpha)B}{\rho^{\alpha+1}}}{2\rho\sqrt{An\rho^{n-1}+\dfrac{\alpha B}{\rho^{\alpha+1}}}}>0, \tag{3.6}$$

which implies that the velocity of 1-rarefaction (2-rarefaction) wave λ_1^{AB} (λ_2^{AB}) is monotonic decreasing (increasing) with respect to ρ.

With the requirement $\lambda_1^{AB}(\rho_-,u_-)<\lambda_1^{AB}(\rho,u)$ and $\lambda_2^{AB}(\rho_-,u_-)<\lambda_2^{AB}(\rho,u)$, noticing (3.5) and (3.6), we get that

$$R_1(\rho_-,u_-):\begin{cases}\xi=\lambda_1=u-\sqrt{An\rho^{n-1}+\frac{\alpha B}{\rho^{\alpha+1}}},\\ u-u_-=-\int_{\rho_-}^{\rho}\frac{\sqrt{An\rho^{n-1}+\frac{\alpha B}{\rho^{\alpha+1}}}}{\rho}d\rho,\quad \rho<\rho_-,\end{cases} \tag{3.7}$$

and

$$R_2(\rho_-,u_-):\begin{cases}\xi=\lambda_2=u+\sqrt{An\rho^{n-1}+\frac{\alpha B}{\rho^{\alpha+1}}},\\ u-u_-=\int_{\rho_-}^{\rho}\frac{\sqrt{An\rho^{n-1}+\frac{\alpha B}{\rho^{\alpha+1}}}}{\rho}d\rho,\quad \rho>\rho_-.\end{cases} \tag{3.8}$$

For the 1-rarefaction wave, through differentiating u respect to ρ in the second equation in (3.7), we get

$$u_\rho=-\frac{\sqrt{An\rho^{n-1}+\dfrac{\alpha B}{\rho^{\alpha+1}}}}{\rho}<0, \tag{3.9}$$

$$u_{\rho\rho}=\frac{-An(n-3)\rho^{n+\alpha}+\alpha(\alpha+3)B}{2\rho^2\sqrt{An\rho^{n+\alpha}+\alpha B\rho^{\alpha+1}}}. \tag{3.10}$$

Thus, it is easy to get $u_{\rho\rho}>0$ for $1\le n\le 3$, i.e., the 1-rarefaction wave is convex for $1\le n\le 3$ in the upper half phase plane ($\rho>0$).

In addition, from the second equation of (3.7), we have

$$u-u_-=\int_\rho^{\rho_-}\frac{\sqrt{An\rho^{n-1}+\frac{\alpha B}{\rho^{\alpha+1}}}}{\rho}d\rho\ge\int_\rho^{\rho_-}\sqrt{\alpha B}\rho^{-\frac{\alpha+1}{2}-1}d\rho=\frac{2\sqrt{\alpha B}}{\alpha+1}(\rho^{-\frac{\alpha+1}{2}}-\rho_-^{-\frac{\alpha+1}{2}}),$$

which means that $\lim\limits_{\rho\to 0}u=+\infty$.

By a similar computation, we have that, for the 2-rarefaction wave, $u_\rho>0$, $u_{\rho\rho}<0$ for $1\le n\le 3$ and $\lim\limits_{\rho\to+\infty}u=+\infty$. Thus, we can draw the conclusion that the 2-rarefaction wave is concave for $1\le n\le 3$ in the upper half phase plane ($\rho>0$).

Now we consider the discontinuous solution. For a bounded discontinuity at $\xi=\sigma$, the Rankine-Hugoniot condition holds

$$\begin{cases} \sigma^{AB}[\rho]=[\rho u], \\ \sigma^{AB}[\rho u]=[\rho u^2+P], \quad P=A\rho^n-\frac{B}{\rho^\alpha}, \end{cases} \tag{3.11}$$

where $[\rho]=\rho_+-\rho_-$, etc.

Eliminating σ from (3.11), we obtain

$$u-u_-=\pm\sqrt{\frac{\rho-\rho_-}{\rho\rho_-}\Big(A(\rho^n-\rho_-^n)-B(\frac{1}{\rho^\alpha}-\frac{1}{\rho_-^\alpha})\Big)}. \tag{3.12}$$

Using the Lax entropy condition, the 1-shock satisfies

$$\sigma^{AB}<\lambda_1^{AB}(\rho_-,u_-), \quad \lambda_1^{AB}(\rho,u)<\sigma^{AB}<\lambda_2^{AB}(\rho,u), \tag{3.13}$$

while the 1-shock satisfies

$$\lambda_1^{AB}(\rho_-,u_-)<\sigma^{AB}<\lambda_2^{AB}(\rho_-,u_-), \quad \lambda_2^{AB}(\rho,u)<\sigma^{AB}. \tag{3.14}$$

From the first equation in (3.11), we have

$$\sigma^{AB}=\frac{\rho u-\rho_-u_-}{\rho-\rho_-}=u+\frac{\rho_-(u-u_-)}{\rho-\rho_-}=u_-+\frac{\rho(u-u_-)}{\rho-\rho_-}. \tag{3.15}$$

Thus, by a simple calculation, (3.13) is equivalent to

$$-\rho\sqrt{An\rho^{n-1}+\frac{\alpha B}{\rho^{\alpha+1}}}<\frac{\rho\rho_-(u-u_-)}{\rho-\rho_-}<-\rho_-\sqrt{An\rho_-^{n-1}+\frac{\alpha B}{\rho_-^{\alpha+1}}}, \tag{3.16}$$

and (3.14) is equivalent to

$$\rho\sqrt{An\rho^{n-1}+\frac{\alpha B}{\rho^{\alpha+1}}}<\frac{\rho\rho_-(u-u_-)}{\rho-\rho_-}<\rho_-\sqrt{An\rho_-^{n-1}+\frac{\alpha B}{\rho_-^{\alpha+1}}}. \tag{3.17}$$

(3.16) and (3.17) imply that $\rho>\rho_-$, $u<u_-$ and $\rho<\rho_-$, $u<u_-$, respectively.

Through the above analysis, for a given left state (ρ_-,u_-), the shock curves in the phase plane, which are the sets of states that can be connected on the right by a 1-shock or 2-shock, are as follows

$$S_1(\rho_-,u_-):\begin{cases} \sigma_1=\dfrac{\rho u-\rho_-u_-}{\rho-\rho_-}, \\ u-u_-=-\sqrt{\frac{\rho-\rho_-}{\rho\rho_-}\Big(A(\rho^n-\rho_-^n)-B(\frac{1}{\rho^\alpha}-\frac{1}{\rho_-^\alpha})\Big)}, \quad \rho>\rho_- \end{cases} \tag{3.18}$$

and

$$S_2(\rho_-,u_-):\begin{cases} \sigma_2=\dfrac{\rho u-\rho_-u_-}{\rho-\rho_-}, \\ u-u_-=-\sqrt{\frac{\rho-\rho_-}{\rho\rho_-}\Big(A(\rho^n-\rho_-^n)-B(\frac{1}{\rho^\alpha}-\frac{1}{\rho_-^\alpha})\Big)}, \quad \rho<\rho_-. \end{cases} \tag{3.19}$$

For the 1-shock wave, through differentiating u respect to ρ in the second equation in (3.18), we get

$$2(u-u_-)u_\rho=\frac{1}{\rho^2}\Big(A(\rho^n-\rho_-^n)-B(\frac{1}{\rho^\alpha}-\frac{1}{\rho_-^\alpha})\Big)+\frac{\rho-\rho_-}{\rho\rho_-}(An\rho^{n-1}+\frac{\alpha B}{\rho^{\alpha+1}})>0, \quad (3.20)$$

which means that $u_\rho<0$ for the 1-shock wave and that the 1-shock wave curve is starlike with respect to (ρ_-,u_-) in the region $\rho>\rho_-$. Similarly, we can get $u_\rho>0$ for the 2-shock wave and that the 2-shock wave curve is starlike with respect to (ρ_-,u_-) in the region $\rho<\rho_-$. In addition, it is easy to check that $\lim\limits_{\rho\to+\infty}u=-\infty$ for the 1-shock wave and $\lim\limits_{\rho\to 0}u=-\infty$ for the 2-shock wave.

Through the analysis above, for a given left state (ρ_-,u_-), the sets of states connected with (ρ_-,u_-) on the right in the phase plane consist of the 1-rarefaction wave curve $R_1(\rho_-,u_-)$, the 2-rarefaction wave curve $R_2(\rho_-,u_-)$, the 1-shock curve $S_1(\rho_-,u_-)$ and the 2-shock curve $S_2(\rho_-,u_-)$. These curves divide the upper half plane into four parts $R_1R_2(\rho_-,u_-)$, $R_1S_2(\rho_-,u_-)$, $S_1R_2(\rho_-,u_-)$ and $S_1S_2(\rho_-,u_-)$. Now, we put all of these curves together in the upper half plane ($\rho>0$, $u\in R$) to obtain a picture as Fig.2.

By the phase plane analysis method, it is easy to construct Riemann solutions for any given right state (ρ_+,u_+) as follows

(1) $(\rho_+,u_+)\in \mathrm{R_1R_2}(\rho_-,\mathrm{u}_-):\mathrm{R_1}+\mathrm{R_2}$;

(2) $(\rho_+,u_+)\in \mathrm{R_1S_2}(\rho_-,\mathrm{u}_-):\mathrm{R_1}+\mathrm{S_2}$;

(3) $(\rho_+,u_+)\in \mathrm{S_1R_2}(\rho_-,\mathrm{u}_-):\mathrm{S_1}+\mathrm{R_2}$;

(4) $(\rho_+,u_+)\in \mathrm{S_1S_2}(\rho_-,\mathrm{u}_-):\mathrm{S_1}+\mathrm{S_2}$.

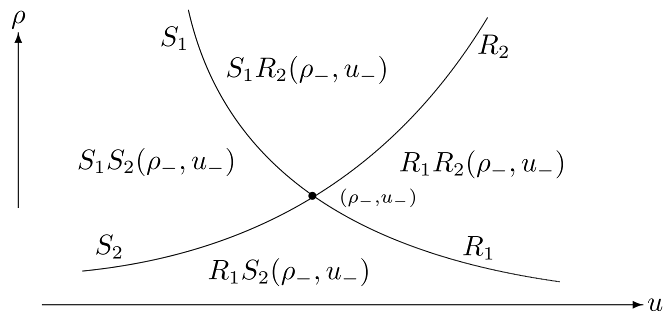

Fig. 2 The (ρ,u) phase plane for the extended Chaplygin gas equations (1.1)–(1.2).

4 Formation of δ-Shocks and Vacuum States as $A,B\to 0$

In this section, we will study the vanishing pressure limit process, i.e., $A,B\to 0$. Since the two regions $S_1R_2(\rho_-,u_-)$ and $R_1S_2(\rho_-,u_-)$ in the (ρ,u) plane have empty interior when $A,B\to 0$, it suffices to analyze the limit process for the two cases $(\rho_+,u_+)\in S_1S_2(\rho_-,u_-)$ and $(\rho_+,u_+)\in R_1R_2(\rho_-,u_-)$.

First, we analyze the formation of δ-shocks in Riemann solutions to the extended Chaplygin gas equations (1.1)–(1.2) with (1.6) in the case $(\rho_+, u_+) \in S_1S_2(\rho_-, u_-)$ as the pressure vanishes.

4.1 δ-Shocks and Concentration

When $(\rho_+, u_+) \in S_1S_2(\rho_-, u_-)$, for fixed $A, B > 0$, let (ρ_*^{AB}, u_*^{AB}) be the intermediate state in the sense that (ρ_-, u_-) and (ρ_*^{AB}, u_*^{AB}) are connected by 1-shock S_1 with speed σ_1^{AB}, (ρ_*^{AB}, u_*^{AB}) and (ρ_+, u_+) are connected by 2-shock S_2 with speed σ_2^{AB}. Then it follows

$$S_1: \begin{cases} \sigma_1^{AB} = \dfrac{\rho_*^{AB}u_*^{AB} - \rho_- u_-}{\rho_*^{AB} - \rho_-}, \\ u_*^{AB} - u_- = -\sqrt{\frac{\rho_*^{AB}-\rho_-}{\rho_*^{AB}\rho_-}\Big(A((\rho_*^{AB})^n - \rho_-^n) - B(\frac{1}{(\rho_*^{AB})^\alpha} - \frac{1}{\rho_-^\alpha})\Big)}, \quad \rho_*^{AB} > \rho_-, \end{cases} \tag{4.1}$$

$$S_2: \begin{cases} \sigma_2^{AB} = \dfrac{\rho_+u_+ - \rho_*^{AB}u_*^{AB}}{\rho_+ - \rho_*^{AB}}, \\ u_+ - u_*^{AB} = -\sqrt{\frac{\rho_+-\rho_*^{AB}}{\rho_+\rho_*^{AB}}\Big(A(\rho_+^n - (\rho_*^{AB})^n) - B(\frac{1}{\rho_+^\alpha} - \frac{1}{(\rho_*^{AB})^\alpha})\Big)}, \quad \rho_+ < \rho_*^{AB}. \end{cases} \tag{4.2}$$

In the following, we give some lemmas to show the limit behavior of the Riemann solutions of system (1.1)–(1.2) with (1.6) as $A, B \to 0$.

Lemma 4.1 $\lim\limits_{A,B\to 0} \rho_*^{AB} = +\infty$.

Proof Eliminating u_*^{AB} in the second equation of (4.1) and (4.2) gives

$$\begin{aligned} u_+ - u_- = & -\sqrt{\frac{\rho_*^{AB} - \rho_-}{\rho_*^{AB}\rho_-}\Big(A((\rho_*^{AB})^n - \rho_-^n) - B(\frac{1}{(\rho_*^{AB})^\alpha} - \frac{1}{\rho_-^\alpha})\Big)} \\ & -\sqrt{\frac{\rho_+ - \rho_*^{AB}}{\rho_+\rho_*^{AB}}\Big(A(\rho_+^n - (\rho_*^{AB})^n) - B(\frac{1}{\rho_+^\alpha} - \frac{1}{(\rho_*^{AB})^\alpha})\Big)}. \end{aligned} \tag{4.3}$$

If $\lim\limits_{A,B\to 0} \rho_*^{AB} = M \in (\max\{\rho_-, \rho_+\}, +\infty)$, then by taking the limit in (4.3) as $A, B \to 0$, we obtain that $u_+ - u_- = 0$, which contradicts with $u_+ < u_-$. Therefore we must have $\lim\limits_{A,B\to 0} \rho_*^{AB} = +\infty$.

By Lemma 4.1, from (4.3) we immediately have the following lemma.

Lemma 4.2 $\lim\limits_{A,B\to 0} A(\rho_*^{AB})^n = \dfrac{\rho_-\rho_+}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}(u_- - u_+)^2$.

Lemma 4.3

$$\lim_{A,B\to 0} u_*^{AB} = \lim_{A,B\to 0} \sigma_1^{AB} = \lim_{A,B\to 0} \sigma_2^{AB} = \sigma. \tag{4.4}$$

Proof From the first equation of (4.1) and (4.2) for S_1 and S_2, by Lemma 4.1, we have

$$\lim_{A,B\to 0}\sigma_1^{AB}=\lim_{A,B\to 0}\frac{\rho_*^{AB}u_*^{AB}-\rho_-u_-}{\rho_*^{AB}-\rho_-}=\lim_{A,B\to 0}\frac{u_*^{AB}-\dfrac{\rho_-u_-}{\rho_*^{AB}}}{1-\dfrac{\rho_-}{\rho_*^{AB}}}=\lim_{A,B\to 0}u_*^{AB},$$

$$\lim_{A,B\to 0}\sigma_2^{AB}=\lim_{A,B\to 0}\frac{\rho_+u_+-\rho_*^{AB}u_*^{AB}}{\rho_+-\rho_*^{AB}}=\lim_{A,B\to 0}\frac{\dfrac{\rho_+u_+}{\rho_*^{AB}}-u_*^{AB}}{\dfrac{\rho_+}{\rho_*^{AB}}-1}=\lim_{A,B\to 0}u_*^{AB},$$

which immediately leads to $\lim\limits_{A,B\to 0}u_*^{AB}=\lim\limits_{A,B\to 0}\sigma_1^{AB}=\lim\limits_{A,B\to 0}\sigma_2^{AB}$.

From the second equation of (4.1), by Lemmas 4.1–4.2, we get

$$\begin{aligned}\lim_{A,B\to 0}u_*^{AB} &= u_- -\lim_{A,B\to 0}\sqrt{\frac{\rho_*^{AB}-\rho_-}{\rho_*^{AB}\rho_-}\Big(A((\rho_*^{AB})^n-\rho_-^n)-B(\frac{1}{(\rho_*^{AB})^\alpha}-\frac{1}{\rho_-^\alpha})\Big)}\\ &= u_- -\sqrt{\frac{1}{\rho_-}\frac{\rho_-\rho_+}{(\sqrt{\rho_-}+\sqrt{\rho_+})^2}(u_--u_+)^2}\\ &= u_- -\frac{\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(u_--u_+)\\ &= \frac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}=\sigma.\end{aligned}$$

The proof is completed.

Lemma 4.4

$$\lim_{A,B\to 0}\int_{x_1^{AB}}^{x_2^{AB}}\rho_*^{AB}dx=\sqrt{\rho_+\rho_-}(u_--u_+)t, \tag{4.5}$$

$$\lim_{A,B\to 0}\int_{x^{AB}}^{x_2^{AB}}\rho_*^{AB}u_*^{AB}dx=\sigma\sqrt{\rho_+\rho_-}(u_--u_+)t. \tag{4.6}$$

Proof Here we only prove the case for $\rho_+\neq\rho_-$. The first equation of the Rankine-Hugoniot condition (3.11) for S_1 and S_2 read

$$\begin{cases}\sigma_1^{AB}(\rho_*^{AB}-\rho_-)=\rho_*^{AB}u_*^{AB}-\rho_-u_-,\\ \sigma_2^{AB}(\rho_+-\rho_*^{AB})=\rho_+u_+-\rho_*^{AB}u_*^{AB},\end{cases} \tag{4.7}$$

from which we have

$$\lim_{A,B\to 0}\rho_*^{AB}(\sigma_2^{AB}-\sigma_1^{AB})=\lim_{A,B\to 0}(-\sigma_1^{AB}\rho_-+\sigma_2^{AB}\rho_+-\rho_+u_++\rho_-u_-)=\sqrt{\rho_+\rho_-}(u_--u_+). \tag{4.8}$$

Similarly, from the second equations of the Rankine-Hugoniot condition (3.11) for S_1 and S_2,

$$\begin{cases}\sigma_1^{AB}(\rho_*^{AB}u_*^{AB}-\rho_-u_-)=\rho_*^{AB}(u_*^{AB})^2-\rho_-u_-^2+A((\rho_*^{AB})^n-\rho_-^n)-B(\frac{1}{(\rho_*^{AB})^\alpha}-\frac{1}{\rho_-^\alpha}),\\ \sigma_2^{AB}(\rho_+u_+-\rho_*^{AB}u_*^{AB})=\rho_+u_+^2-\rho_*^{AB}(u_*^{AB})^2+A(\rho_+^n-(\rho_*^{AB})^n)-B(\frac{1}{\rho_+^\alpha}-\frac{1}{(\rho_*^{AB})^\alpha}),\end{cases} \tag{4.9}$$

we obtain

$$\begin{aligned}
&\lim_{A,B\to 0}\rho_*^{AB}u_*^{AB}(\sigma_2^{AB}-\sigma_1^{AB})\\
=&\lim_{A,B\to 0}(-\sigma_1^{AB}\rho_-u_-+\sigma_2^{AB}\rho_+u_+-\rho_+u_+^2+\rho_-u_-^2-A(\rho_+^n-\rho_-^n)+B(\frac{1}{\rho_+^\alpha}-\frac{1}{\rho_-^\alpha}))\\
=&\sigma\sqrt{\rho_+\rho_-}(u_--u_+). \qquad (4.10)
\end{aligned}$$

Thus, from (4.8) and (4.10) we immediately get (4.5) and (4.6). For the case $\rho_+=\rho_-$, the conclusion is obviously true, so we omit it. The proof is finished.

The above Lemmas 4.1–4.4 show that, as $A,B\to 0$, the curves of the shock wave S_1 and S_2 will coincide and the delta shock waves will form. Next we will arrange the values which give the exact position, propagation speed and strength of the delta shock wave according to Lemmas 4.3 and 4.4.

From (4.5) and (4.6), we let

$$w(t)=\sqrt{\rho_+\rho_-}(u_--u_+)t, \qquad (4.11)$$

$$w(t)u_\delta(t)=\sigma\sqrt{\rho_+\rho_-}(u_--u_+)t, \qquad (4.12)$$

then

$$u_\delta(t)=\sigma. \qquad (4.13)$$

Furthermore, by letting $\frac{dx(t)}{dt}=u_\delta(t)$, we have

$$x(t)=\sigma t. \qquad (4.14)$$

From (4.11)–(4.14), we can see that the quantities defined above are exactly consistent with those given by (2.7). Thus, it uniquely determines that the limits of the Riemann solutions to system (1.1)–(1.2) and (1.6) when $A,B\to 0$ in the case $(\rho_+,u_+)\in$ Ⅳ and $u_->u_+$ is just the delta shock solution of (1.3) and (1.6). So we get the following results which characterizes the vanishing pressure limit in the case $(\rho_+,u_+)\in$ Ⅳ and $u_->u_+$.

Theorem 4.1 If $u_->u_+$, for each fixed A,B, $(\rho_+,u_+)\in$ Ⅳ, assuming that (ρ^{AB},u^{AB}) is a two-shock wave solution of (1.1)–(1.2) and (1.6) which is constructed in Section 3, it is obtained that when $A,B\to 0$, (ρ^{AB},u^{AB}) converges to a delta shock wave solution to the transportation equations (1.3) with the same initial data.

4.2 Formation of Vacuum States

In this subsection, we show the formation of vacuum states in the Riemann solutions to (1.1)–(1.2) with (1.6) in the case $(\rho_+,u_+)\in R_1R_2(\rho_-,u_-)$ with $u_-<u_+$ and $\rho_\pm>0$ as the pressure vanishes.

At this moment, for fixed $A,B>0$, let (ρ_*^{AB},u_*^{AB}) be the intermediate state in the sense that (ρ_-,u_-) and (ρ_*^{AB},u_*^{AB}) are connected by 1-rarefaction wave R_1 with speed λ_1^{AB},

$(\rho_*^{AB}, u_*^{AB})$and (ρ_+, u_+) are connected by 2-rarefaction wave R_2 with speed λ_2^{AB}. Then it follows

$$R_1: \begin{cases} \xi = \lambda_1^{AB} = u - \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}, \\ u - u_- = -\displaystyle\int_{\rho_-}^{\rho} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho, \quad \rho_*^{AB} \le \rho \le \rho_-, \end{cases} \tag{4.15}$$

$$R_2: \begin{cases} \xi = \lambda_2^{AB} = u + \sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}, \\ u_+ - u = \displaystyle\int_{\rho}^{\rho_+} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho, \quad \rho_*^{AB} \le \rho \le \rho_+. \end{cases} \tag{4.16}$$

Now, from the second equations of (4.15) and (4.16), using the following integral identity

$$\begin{aligned} & \int_{\rho}^{\rho_-} \frac{\sqrt{An\rho_-^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho \\ = \ & \frac{2}{\alpha+1}\Big(- \sqrt{An\rho_-^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}} + \sqrt{An\rho_-^{n-1}} \ln\big(\sqrt{An\rho_-^{n-1}\rho^{\alpha+1} + \alpha B} + \sqrt{An\rho_-^{n-1}\rho^{\alpha+1}}\big)\Big)\Big|_{\rho}^{\rho_-}, \end{aligned}$$

it follows that the intermediate state (ρ_*^{AB}, u_*^{AB}) satisfies

$$\begin{aligned} & u_+ - u_- \\ = \ & \int_{\rho_*^{AB}}^{\rho_-} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho + \int_{\rho_*^{AB}}^{\rho_+} \frac{\sqrt{An\rho^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho \\ \le \ & \int_{\rho_*^{AB}}^{\rho_-} \frac{\sqrt{An\rho_-^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho + \int_{\rho_*^{AB}}^{\rho_+} \frac{\sqrt{An\rho_+^{n-1} + \frac{\alpha B}{\rho^{\alpha+1}}}}{\rho} d\rho \\ = \ & \frac{2}{\alpha+1}\Big(- \sqrt{An\rho_-^{n-1} + \frac{\alpha B}{\rho_-^{\alpha+1}}} + \sqrt{An\rho_-^{n-1}} \ln\big(\sqrt{An\rho_-^{n-1}\rho_-^{\alpha+1} + \alpha B} + \sqrt{An\rho_-^{n-1}\rho_-^{\alpha+1}}\big) \\ & + \sqrt{An\rho_-^{n-1} + \frac{\alpha B}{(\rho_*^{AB})^{\alpha+1}}} - \sqrt{An\rho_-^{n-1}} \ln\big(\sqrt{An\rho_-^{n-1}(\rho_*^{AB})^{\alpha+1} + \alpha B} + \sqrt{An\rho_-^{n-1}(\rho_*^{AB})^{\alpha+1}}\big) \\ & - \sqrt{An\rho_+^{n-1} + \frac{\alpha B}{\rho_+^{\alpha+1}}} + \sqrt{An\rho_+^{n-1}} \ln\big(\sqrt{An\rho_+^{n-1}\rho_+^{\alpha+1} + \alpha B} + \sqrt{An\rho_+^{n-1}\rho_+^{\alpha+1}}\big) \\ & + \sqrt{An\rho_+^{n-1} + \frac{\alpha B}{(\rho_*^{AB})^{\alpha+1}}} - \sqrt{An\rho_+^{n-1}} \ln\big(\sqrt{An\rho_+^{n-1}(\rho_*^{AB})^{\alpha+1} + \alpha B} + \sqrt{An\rho_+^{n-1}(\rho_*^{AB})^{\alpha+1}}\big)\Big), \end{aligned} \tag{4.17}$$

which implies the following result.

Theorem 4.2 Let $u_- < u_+$ and $(\rho_+, u_+) \in \mathrm{I}(\rho_-, \mathrm{u}_-)$. For any fixed $A, B > 0$, assume that (ρ^{AB}, u^{AB}) is the two-rarefaction wave Riemann solution of (1.1)–(1.2) with Riemann data $(\rho_\pm, u_\pm)$ constructed in Section 3. Then as $A, B \to 0$, the limit of the Riemann solution (ρ^{AB}, u^{AB}) is two contact discontinuities connecting the constant states $(\rho_\pm, u_\pm)$ and the intermediate vacuum state as follows

$$(\rho, u)(\xi) = \begin{cases} (\rho_-, u_-), & -\infty < \xi \le u_-, \\ (0, \xi), & u_- \le \xi \le u_+, \\ (\rho_+, u_+), & u_+ \le \xi < \infty, \end{cases}$$

which is exactly the Riemann solution to the transport equations (1.3) with the same Riemann data $(\rho_\pm, u_\pm)$.

Indeed, if $\lim\limits_{A,B\to 0}\rho_*^{AB} = K \in (0, \min\{\rho_-, \rho_+\})$, then (4.17) leads to $u_+ - u_- = 0$, which contradicts with $u_- < u_+$. Thus $\lim\limits_{A,B\to 0}\rho_*^{AB} = 0$, which just means vacuum occurs. Moreover, as $A, B \to 0$, one can directly derive from (4.15) and (4.16) that $\lambda_1^{AB},\ \lambda_2^{AB} \to u$ and two rarefaction waves R_1 and R_2 tend to two contact discontinuities $\xi = \frac{x}{t} = u_\pm$, respectively. These reach the desired conclusion.

5 Formation of δ-Shocks as $A \to 0$

In this section, we study the formation of the delta shock waves in the limit of Riemann solutions of (1.1)–(1.2) with (1.6) as $A \to 0$ in the case $(\rho_+, u_+) \in \mathrm{V}(\rho_-, \mathrm{u}_-)$, i.e., $u_+ + \sqrt{B}\rho_+^{-\frac{\alpha+1}{2}} \le u_- - \sqrt{B}\rho_-^{-\frac{\alpha+1}{2}}$ (see Fig. 3).

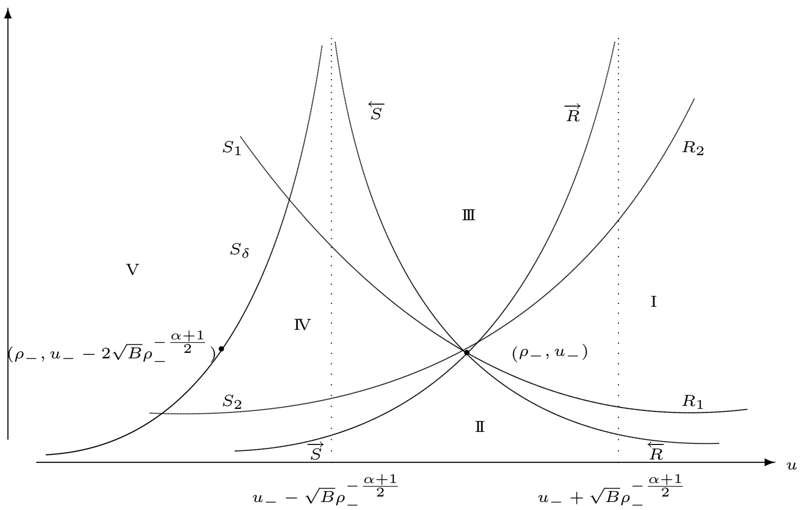

Fig.3 The (ρ, u) phase plane for the generalized Chaplygin gas equations (1.5) and the extended Chaplygin gas equations (1.1)–(1.2).

Lemma 5.1 When $(\rho_+, u_+) \in \mathrm{V}(\rho_-, \mathrm{u}_-)$, there exists a positive parameter A_0 such that $(\rho_+, u_+) \in S_1S_2(\rho_-, u_-)$ when $0 < A < A_0$.

Proof From $(\rho_+, u_+) \in \mathrm{V}(\rho_-, \mathrm{u}_-)$, we have

$$u_+ + \sqrt{B}\rho_+^{-\frac{\alpha+1}{2}} \le u_- - \sqrt{B}\rho_-^{-\frac{\alpha+1}{2}}, \tag{5.1}$$

then

$$\begin{aligned}(u_- - u_+)^2 &\ge \big(\sqrt{B}\rho_+^{-\frac{\alpha+1}{2}} + \sqrt{B}\rho_-^{-\frac{\alpha+1}{2}}\big)^2\\ &= B(\rho_+^{-\alpha-1} + \rho_-^{-\alpha-1} + 2\rho_+^{-\frac{\alpha+1}{2}}\rho_-^{-\frac{\alpha+1}{2}})\\ &> B(\rho_+^{-\alpha-1} + \rho_-^{-\alpha-1} - \rho_+^{-1}\rho_-^{-\alpha} - \rho_-^{-1}\rho_+^{-\alpha})\\ &= B(\frac{1}{\rho_+} - \frac{1}{\rho_-})(\frac{1}{\rho_+^\alpha} - \frac{1}{\rho_-^\alpha}).\end{aligned} \tag{5.2}$$

All the states (ρ, u) connected with (ρ_-, u_-) by a backward shock wave S_1 or a forward shock wave S_2 satisfy

$$u - u_- = -\sqrt{\frac{\rho-\rho_-}{\rho\rho_-}\Big(A(\rho^n - \rho_-^n) - B(\frac{1}{\rho^\alpha} - \frac{1}{\rho_-^\alpha})\Big)}, \quad \rho > \rho_-, \tag{5.3}$$

or

$$u - u_- = -\sqrt{\frac{\rho-\rho_-}{\rho\rho_-}\Big(A(\rho^n - \rho_-^n) - B(\frac{1}{\rho^\alpha} - \frac{1}{\rho_-^\alpha})\Big)}, \quad \rho < \rho_-. \tag{5.4}$$

When $\rho_+ = \rho_-$, the conclusion is obviously true. When $\rho_+ \neq \rho_-$, by taking

$$(u_+ - u_-)^2 = \frac{\rho_+ - \rho_-}{\rho_+\rho_-}\Big(A_0(\rho_+^n - \rho_-^n) - B(\frac{1}{\rho_+^\alpha} - \frac{1}{\rho_-^\alpha})\Big), \tag{5.5}$$

we have

$$A_0 = \frac{\rho_+\rho_-}{(\rho_+ - \rho_-)(\rho_+^n - \rho_-^n)}\Big((u_+ - u_-)^2 - B(\frac{1}{\rho_+} - \frac{1}{\rho_-})(\frac{1}{\rho_+^\alpha} - \frac{1}{\rho_-^\alpha})\Big), \tag{5.6}$$

which together with (5.2) gives the conclusion. The proof is completed.

When $0 < A < A_0$, the Riemann solution of (1.1)–(1.2) with (1.6) includes a backward shock wave S_1 and a forward shock wave S_2 with the intermediate state (ρ_*^A, u_*^A) besides two constant states $(\rho_\pm, u_\pm)$. We then have

$$S_1: \begin{cases} \sigma_1^A = \dfrac{\rho_*^A u_*^A - \rho_- u_-}{\rho_*^A - \rho_-}, \\ u_*^A - u_- = -\sqrt{\frac{\rho_*^A - \rho_-}{\rho_*^A\rho_-}\Big(A((\rho_*^A)^n - \rho_-^n) - B(\frac{1}{(\rho_*^A)^\alpha} - \frac{1}{\rho_-^\alpha})\Big)}, \quad \rho_*^A > \rho_- \end{cases} \tag{5.7}$$

and

$$S_2: \begin{cases} \sigma_2^A = \dfrac{\rho_+ u_+ - \rho_*^A u_*^A}{\rho_+ - \rho_*^A}, \\ u_+ - u_*^A = -\sqrt{\frac{\rho_+ - \rho_*^A}{\rho_+\rho_*^A}\Big(A(\rho_+^n - (\rho_*^A)^n) - B(\frac{1}{\rho_+^\alpha} - \frac{1}{(\rho_*^A)^\alpha})\Big)}, \quad \rho_+ < \rho_*^A, \end{cases} \tag{5.8}$$

here σ_1^A and σ_2^A are the propagation speed of S_1 and S_2, respectively. Similar to that in Section 4, in the following, we give some lemmas to show the limit behavior of the Riemann solutions of system (1.1)–(1.2) with (1.6) as $A \to 0$.

Lemma 5.2 $\lim\limits_{A\to 0} \rho_*^A = +\infty$.

Proof Eliminating u_*^A in the second equation of (5.7) and (5.8) gives

$$\begin{aligned} u_- - u_+ = & \sqrt{\frac{\rho_*^A - \rho_-}{\rho_*^A\rho_-}\Big(A((\rho_*^A)^n - \rho_-^n) - B(\frac{1}{(\rho_*^A)^\alpha} - \frac{1}{\rho_-^\alpha})\Big)} \\ & + \sqrt{\frac{\rho_+ - \rho_*^A}{\rho_+\rho_*^A}\Big(A(\rho_+^n - (\rho_*^A)^n) - B(\frac{1}{\rho_+^\alpha} - \frac{1}{(\rho_*^A)^\alpha})\Big)}. \end{aligned} \tag{5.9}$$

If $\lim\limits_{A\to 0}\rho_*^A = K \in (\max\{\rho_-,\rho_+\},+\infty)$, then by taking the limit in (5.9) as $A\to 0$, we obtain that

$$\begin{aligned} u_- - u_+ &= \sqrt{B}\Big(\sqrt{(\frac{1}{\rho_-}-\frac{1}{K})(\frac{1}{\rho_-^\alpha}-\frac{1}{K^\alpha})}+\sqrt{(\frac{1}{\rho_+}-\frac{1}{K})(\frac{1}{\rho_+^\alpha}-\frac{1}{K^\alpha})}\Big)\\ &< \sqrt{B}\Big(\sqrt{\frac{1}{\rho_-}\frac{1}{\rho_-^\alpha}}+\sqrt{\frac{1}{\rho_+}\frac{1}{\rho_+^\alpha}}\Big)\\ &= \sqrt{B}\rho_+^{-\frac{\alpha+1}{2}}+\sqrt{B}\rho_-^{-\frac{\alpha+1}{2}}, \end{aligned}\tag{5.10}$$

which contradicts with (5.2). Therefore we must have $\lim\limits_{A\to 0}\rho_*^A=+\infty$. The proof is completed.

By Lemma 5.2, from (5.9) we immediately have the following lemma.

Lemma 5.3 $\lim\limits_{A\to 0} A(\rho_*^A)^n < \rho_-(u_- - u_+)^2$.

Lemma 5.4 Let $\lim\limits_{A\to 0} u_*^A = \widehat{\sigma^B}$, then

$$\lim_{A\to 0} u_*^A = \lim_{A\to 0}\sigma_1^A = \lim_{A\to 0}\sigma_2^A = \widehat{\sigma^B} \in \Big(u_+ + \sqrt{\alpha B}\rho_+^{-\frac{\alpha+1}{2}}, u_- - \sqrt{\alpha B}\rho_-^{-\frac{\alpha+1}{2}}\Big).\tag{5.11}$$

Proof From the second equation of (5.7) for S_1, by Lemmas 4.2 and 4.3, we have

$$\begin{aligned} \lim_{A\to 0} u_*^A &= u_- - \lim_{A\to 0}\sqrt{\frac{\rho_*^A-\rho_-}{\rho_*^A\rho_-}\Big(A((\rho_*^A)^n-\rho_-^n)-B(\frac{1}{(\rho_*^A)^\alpha}-\frac{1}{\rho_-^\alpha})\Big)}\\ &= u_- - \sqrt{\frac{1}{\rho_-}\Big(\lim_{A\to 0}A(\rho_*^A)^n+\frac{B}{\rho_-^\alpha}\Big)}\\ &< u_- - \sqrt{\alpha B}\rho_-^{-\frac{\alpha+1}{2}}. \end{aligned}\tag{5.12}$$

Similarly, from the second equation of (5.8) for S_2, we have

$$\begin{aligned} \lim_{A\to 0} u_*^A &= u_+ + \lim_{A\to 0}\sqrt{\frac{\rho_+-\rho_*^A}{\rho_+\rho_*^A}\Big(A(\rho_+^n-(\rho_*^A)^n)-B(\frac{1}{\rho_+^\alpha}-\frac{1}{(\rho_*^A)^\alpha})\Big)}\\ &= u_+ + \sqrt{\frac{1}{\rho_+}\Big(\lim_{A\to 0}A(\rho_*^A)^n+\frac{B}{\rho_+^\alpha}\Big)}\\ &> u_+ + \sqrt{\alpha B}\rho_+^{-\frac{\alpha+1}{2}}. \end{aligned}\tag{5.13}$$

Furthermore, similar to the analysis in Lemma 4.3, we can obtain $\lim\limits_{A\to 0}u_*^A = \lim\limits_{A\to 0}\sigma_1^A = \lim\limits_{A\to 0}\sigma_2^A = \widehat{\sigma^B}$. The proof is completed.

Lemma 5.5 For $\widehat{\sigma^B}$ mentioned in Lemma 5.4,

$$\widehat{\sigma^B} = \sigma^B = \frac{\rho_+u_+-\rho_-u_-+\big\{\rho_+\rho_-\big((u_+-u_-)^2-(\frac{1}{\rho_+}-\frac{1}{\rho_-})(\frac{B}{\rho_+^\alpha}-\frac{B}{\rho_-^\alpha})\big)\big\}^{\frac{1}{2}}}{\rho_+-\rho_-}\tag{5.14}$$

as $\rho_+ \neq \rho_-$ and

$$\widehat{\sigma^B} = \sigma^B = \frac{u_+ + u_-}{2} \tag{5.15}$$

as $\rho_+ = \rho_-$.

Proof Letting $\lim\limits_{A\to 0} A(\rho_*^A)^n = L$, by Lemma 5.4, from (5.12) and (5.13) we have

$$\lim_{A\to 0} u_*^A = u_- - \sqrt{\frac{1}{\rho_-}\Big(L + \frac{B}{\rho_-^\alpha}\Big)} = u_+ + \sqrt{\frac{1}{\rho_+}\Big(L + \frac{B}{\rho_+^\alpha}\Big)} = \widehat{\sigma^B},$$

which leads to

$$L + \frac{B}{\rho_+^\alpha} = \rho_-(u_- - \widehat{\sigma^B})^2, \tag{5.16}$$

$$L + \frac{B}{\rho_-^\alpha} = \rho_+(u_+ - \widehat{\sigma^B})^2. \tag{5.17}$$

Eliminating L from (5.16) and (5.17), we have

$$(\rho_+ - \rho_-)(\widehat{\sigma^B})^2 - 2(\rho_+ u_+ - \rho_- u_-)\widehat{\sigma^B} + \rho_+ u_+^2 - \rho_- u_-^2 - B(\frac{1}{\rho_+^\alpha} - \frac{1}{\rho_-^\alpha}) = 0. \tag{5.18}$$

From (5.18), noticing $\widehat{\sigma^B} \in \Big(u_+ + \sqrt{\alpha B}\rho_+^{-\frac{\alpha+1}{2}}, u_- - \sqrt{\alpha B}\rho_-^{-\frac{\alpha+1}{2}}\Big)$, we immediately get (5.14) and (5.15). The proof is finished.

Similar to Lemma 4.4, we have the following lemma.

Lemma 5.6

$$\lim_{A\to 0} \int_{x_1^A}^{x_2^A} \rho_*^A dx = w_0^B t, \tag{5.19}$$

$$\lim_{A\to 0} \int_{x_1^A}^{x_2^A} \rho_*^A u_*^A dx = w_0^B \sigma^B t. \tag{5.20}$$

Proof Here we only prove the case for $\rho_+ \neq \rho_-$. Similar to the proof of Lemma 4.4, taking account into (3.11) and (5.18), we have

$$\begin{aligned}
\lim_{A\to 0} \rho_*^A(\sigma_2^A - \sigma_1^A) &= \lim_{A\to 0}(-\sigma_1^A \rho_- + \sigma_2^A \rho_+ - (\rho_+ u_+ - \rho_- u_-) \\
&= \sigma^B(\rho_+ - \rho_-) - (\rho_+ u_+ - \rho_- u_-) = w_0^B,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{A\to 0} \rho_*^A v_*^A(\sigma_2^A - \sigma_1^A) \\
= {}& \lim_{A\to 0}(-\sigma_1^A \rho_- u_- + \sigma_2^A \rho_+ u_+ - \rho_+ u_+(u_+ + \beta t) + \rho_- u_-(u_- + \beta t) \\
& -A(\rho_+^n - \rho_-^n) + B(\frac{1}{\rho_+^\alpha} - \frac{1}{\rho_-^\alpha})) \\
= {}& \sigma^B(\rho_+ u_+ - \rho_- u_-) - (\rho_+ u_+^2 - \rho_- u_-^2) + B(\frac{1}{\rho_+^\alpha} - \frac{1}{\rho_-^\alpha}) \\
= {}& (\rho_+ - \rho_-)(\sigma^B)^2 - (\rho_+ u_+ - \rho_- u_-)\sigma^B \\
= {}& \sigma^B(\sigma^B(\rho_+ - \rho_-) - (\rho_+ u_+ - \rho_- u_-)) = \sigma^B w_0^B.
\end{aligned}$$

So

$$\lim_{A\to 0}\int_{x_1^A(t)}^{x_2^A(t)}\rho_*^A dx == \lim_{A\to 0}\int_0^t \rho_*^A(\sigma_2^A-\sigma_1^A)dt = w_0^B t, \tag{5.21}$$

$$\lim_{A\to 0}\int_{x_1^A(t)}^{x_2^A(t)}\rho_*^A u_*^A dx = \lim_{A\to 0}\int_0^t \rho_*^A u_*^A(\sigma_2^A-\sigma_1^A)dt = \sigma^B w_0^B t. \tag{5.22}$$

For the case $\rho_+ = \rho_-$, the conclusion is obviously true, so we omit it. The proof is finished.

The above Lemmas 5.1–5.6 show that, as $A\to 0$, the curves of the shock wave S_1^A and S_2^A will coincide and the delta shock waves will form. Next, we will arrange the values which give the exact position, propagation speed and strength of the delta shock wave according to Lemmas 5.4 and 5.6.

From (5.21) and (5.22), when $\rho_+\neq\rho_-$, we let

$$w^B(t) = w_0^B t, \tag{5.23}$$

$$w^B(t)u_\delta^B(t) = (\sigma_0^B+\beta t)w_0^B t, \tag{5.24}$$

then

$$u_\delta^B(t) = \sigma_0^B+\beta t, \tag{5.25}$$

which is equal to $\sigma^B(t)$. Furthermore, by letting $\frac{dx^B(t)}{dt}=\sigma^B(t)$, we have

$$x^B(t) = \sigma_0^B t + \frac{1}{2}\beta t^2. \tag{5.26}$$

From (5.23)–(5.26), we can see that the quantities defined above are exactly consistent with those given by (2.17)–(2.20). When $\rho_+=\rho_-$, similar results can be obtained. Thus, it uniquely determines that the limits of Riemann solutions to system (1.1)–(1.2) and (1.6) when $A\to 0$ in the case $(\rho_+,u_+)\in\mathrm{V}$ and $u_->u_+$ is just the delta shock solution of (1.5) and (1.6). So we get the following results which characterizes the vanishing pressure limit in the case $(\rho_+,u_+)\in\mathrm{V}$ and $u_->u_+$.

Theorem 5.1 If $u_->u_+$, for each fixed A, B, $(\rho_+,u_+)\in\mathrm{V}$, assuming that (ρ^A,u^A) is a two-shock wave solution of (1.1)–(1.2) and (1.6) which is constructed in Section 3, it is obtained that when $A\to 0$, (ρ^A,u^A) converges to a delta shock wave solution to the generalized Chaplygin gas equations (1.5) with the same initial data.

6 Conclusions and Discussions

In this paper, we have considered two kinds of flux approximation limits of Riemann solutions to the extended Chaplygin gas equations and studied the concentration and the formation of delta shocks during the limit process. Moreover, we have proved that the vanishing pressure limit of Riemann solutions to extended Chaplygin gas equations is just the corresponding ones to transportation equations, and when the extended Chaplygin pressure approaches the generalized Chaplygin pressure, the limit of Riemann solutions to the extended Chaplygin gas equations is just the corresponding ones to the generalized Chaplygin gas equations.

On the other hand, recently, Shen and Sun have studied the Riemann problem for the nonhomogeneous transportation equations, and the nonhomogeneous (generalized) Chaplygin gas equations with coulomb-like friction, see [38, 39, 42]. Similarly, we will also consider the Riemann problem for the nonhomogeneous extended Chaplygin gas equations with coulomb-like friction. Furthermore, we will consider the formation of delta shock waves in its flux approximation limit and analyze the relations of Riemann solutions among the nonhomogeneous extended Chaplygin gas equations, the nonhomogeneous generalized Chaplygin gas equations and the nonhomogeneous transportation equations. These will be left for our future work.

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延拓Chaplygin气体黎曼解的流逼近极限过程中的集中性研究

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摘 要: 本文研究了延拓Chaplygin气体的黎曼解在流逼近极限过程中的集中现象和狄拉克激波的两种形成机制问题. 利用相平面分析法和广义特征分析法, 构造出了延拓Chaplygin气体的整体黎曼解, 并获得了两个结果: 当压力消失时, 延拓Chaplygin气体的包含两个激波的解收敛到输运方程的狄拉克激波解; 当压力项趋近于广义Chaplygin压力项时, 延拓Chaplygin气体的包含两个激波的解收敛到广义Chaplygin气体的狄拉克激波解. 结论推广到了延拓Chaplygin气体.

关键词: 延拓Chaplygin气体; 狄拉克激波; 流逼近极限; 黎曼解; 输运方程; 广义Chaplygin气体

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