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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR CAPUTO-HADAMARD TYPE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we study a class of Caputo-Hadamard fractional differential equations with boundary value problems. By using Banach fixed point theorem and the method of upper and lower solutions method, the existence and uniqueness results of the solutions are obtained, which generalizes some results about ordinary differential equations with boundary value problems. As an application, two examples are given to illustrate our main results.

Keywords: fractional differential equations; Caputo-Hadamard derivatives; Banach fixed point theorem; upper and lower solutions method

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1 Introduction

Over the past few decades, the fractional calculus made great progress, and it was widely used in various fields of science and engineering. There were numbers applications in electromagnetics, control theory, viscoelasticity and so on. There was a high-speed development in fractional differential equations in recent years, and we referred the reader to the monographs Podlubny [1], Kilbas et al. [2] and Zhou [3]. In the current theory of fractional differential equations, much of the work is based on Riemann-Liouville and Caputo fractional derivatives, but the research of Caputo-Hadamard fractional derivatives of differential equations is very few, which includes logarithmic function and arbitrary exponents. Motivated by this fact, we consider a class of Caputo-Hadamard fractional differential equations with boundary value problems (BVPs).

Nowadays, some authors studied the existence and uniqueness of solutions for nonlinear fractional differential equation with boundary value problems. For the recent development of the topic, we referred the reader to a series papers by Ahmad et al. [4–6], Mahmudov et al. [7] and the references therein. Details and properties of the Hadamard fractional derivative and integral can be found in [8–12].

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Wafa Shammakh [13] studied the existence and uniqueness results for the following three-point BVPs

$$\begin{cases} {}^C_H D^\alpha x(t) + f(t, x(t)) = 0, \ 1 \le t \le e, \ 1 \le \alpha \le 2, \\ x(1) = 0, \ {}^C_H Dx(e) = \gamma {}^C_H Dx(\xi), \end{cases}$$

where ${}^C_H D^\alpha$ is the Caputo-Hadamard fractional derivative of order $1 \le \alpha \le 2$, $0 \le \gamma < 1$, $\xi \in (1, e)$, ${}^C_H D = t\frac{d}{dt}$, and $f : [1, e] \to [0, \infty)$.

Yacine Arioua and Nouredine Benhamidouche [14] studied the existence of solutions for the following BVPs of nonlinear fractional differential equations

$$\begin{cases} {}^C_H D^\alpha_{1+} u(t) + f(t, u(t)) = 0, \ 1 < t < e, \ 2 < \alpha \le 3, \\ u(1) = u'(1) = 0, \ \left({}^C_H D^{\alpha-1}_{1+}\right) u(e) = \left({}^C_H D^{\alpha-2}_{1+}\right) u(e) = 0, \end{cases}$$

where ${}^C_H D^\alpha_{1+}$ is the Caputo-Hadamard fractional derivative of order α, and $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Yunru Bai and Hua Kong [15] used the method of upper and lower solutions, proved the existence of solutions to nonlinear Caputo-Hadamard fractional differential equations

$$\begin{cases} {}^C_H D^\alpha_{a+} x(t) = f(t, x(t), \ {}^H I^\alpha_{a+} x(t)), \ t \in [a, b], \\ x(a) = x_a, \end{cases}$$

where ${}^C_H D^\alpha_{a+}$ and ${}^H I^\alpha_{a+}$ stand for the Caputo-Hadamard fractional derivative and Hadamard integral operators, $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $1 < a < b < \infty$.

The purpose of this paper is to discuss the existence and uniqueness of solutions for nonlinear Caputo-Hadamard fractional differential equations

$$\begin{cases} {}^C_H D^\alpha_{1+} u(t) = f(t, u(t)), \ 1 \le t \le e, \ 2 < \alpha \le 3, \\ u(1) = u'(1) = 0, \ u(e) = \lambda \displaystyle\int_1^e u(s) ds, \ 1 \le \lambda \le 2, \end{cases} \tag{1.1}$$

where ${}^C_H D^\alpha_{1+}$ is the Caputo-Hadamard fractional derivative of order $2 < \alpha \le 3$, and f is a continuous function.

2 Preliminaries

In this section, we introduce some necessary definitions, lemmas and notations that will be used later.

Definition 2.1 [2] The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ for a continuous function $g : [1, \infty) \to \mathbb{R}$ is given by

$${}^H I^\alpha_{1+} g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds, \alpha > 0,$$

where $\Gamma(\cdot)$ stands for the Gamma function.

Definition 2.2 [2] The Hadamard fractional derivative of order $\alpha \in \mathbb{R}_+$ for a continuous function $g : [1, \infty) \to \mathbb{R}$ is given by

$$^{H}D^{\alpha}_{1+}g(t) = \delta^n(^{H}I^{\alpha}_{1+}g)(t) = \left(t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} \frac{g(s)}{s}ds,$$

where $n-1 < \alpha < n, n = [\alpha]+1, \delta = t\frac{d}{dt}$, and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3 [16, 17] The Caputo-Hadamard fractional derivative of order $\alpha \in \mathbb{R}_+$ for at least n-times differentiable function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$^{C}_{H}D^{\alpha}_{1+}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} \delta^n \frac{g(s)}{s}ds.$$

Lemma 2.4 [16, 17] Let $u \in C^n_\delta([1, e], \mathbb{R})$, then

$$^{H}I^{\alpha}_{1+}(^{C}_{H}D^{\alpha}_{1+}u)(t) = u(t) - \sum_{j=0}^{n-1} c_j(\ln t)^j, \tag{2.1}$$

here $C^n_\delta([1, e], \mathbb{R}) = \{u : [1, e] \to \mathbb{R} : \delta^{n-1}u \in C([1, e], \mathbb{R})\}$.

Lemma 2.5 Let $h \in C([1, e], \mathbb{R}), u \in C^3_\delta([1, e], \mathbb{R})$. Then the unique solution of the linear Caputo-Hadamard fractional differential equation

$$\begin{cases} ^{C}_{H}D^{\alpha}_{1+}u(t) = h(t), 1 \le t \le e,\ 2 < \alpha \le 3, \\ u(1) = u'(1) = 0, u(e) = \lambda \displaystyle\int_1^e u(s)ds,\ 1 \le \lambda \le 2 \end{cases} \tag{2.2}$$

is equivalent to the following integral equation

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)} \\ &\times \left(\lambda \int_1^e \int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1} \frac{h(r)}{r}drds - \int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1} \frac{h(s)}{s}ds\right), \end{aligned} \tag{2.3}$$

where $A = \displaystyle\int_1^e (\ln t)^2 dt = (e-2)$.

Proof In view of Lemma 2.4, applying $^{H}I^{\alpha}_{1+}$ to both sides of (2.2),

$$u(t) = {}^{H}I^{\alpha}_{1+}h(t) + c_0 + c_1(\ln t) + c_2(\ln t)^2,$$

where $c_0, c_1, c_2 \in \mathbb{R}$.

The boundary condition $u(1) = u'(1) = 0$ implies that $c_0 = c_1 = 0$. Thus

$$u(t) = {}^{H}I^{\alpha}_{1+}h(t) + c_2(\ln t)^2. \tag{2.4}$$

In view of the boundary condition $u(e)=\lambda\int_1^e u(s)ds$, we conclude that

$$\begin{aligned}&u(e)={}^HI_{1+}^{\alpha}h(e)+c_2=\lambda\int_1^e {}^HI_{1+}^{\alpha}h(s)ds+\lambda c_2\int_1^e(\ln t)^2dt,\\&c_2=\frac{1}{1-\lambda A}\left(\lambda\int_1^e {}^HI_{1+}^{\alpha}h(s)ds-{}^HI_{1+}^{\alpha}h(e)\right).\end{aligned}\tag{2.5}$$

Substituting (2.5) in (2.4), we obtain (2.3). This completes the proof.

Based on Lemma 2.5, the solution of problems (1.1)–(1.2) can be expressed as

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}ds+\frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)}\\&\times\left(\lambda\int_1^e\int_1^s\left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,u(r))}{r}drds-\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}ds\right).\end{aligned}\tag{2.6}$$

3 Main Results

Let $E:=C([1,e],\mathbb{R})$ be the Banach space of all continuous functions from $[1,e]$ to $\mathbb{R}$ with the norm $\|u\|=\max\limits_{t\in[1,e]}|u(t)|$. Due to Lemma 2.5, we define an operator $\mathbb{A}:E\to E$ as

$$\begin{aligned}\mathbb{A}u(t)=&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}ds+\frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)}\\&\times\left(\lambda\int_1^e\int_1^s\left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,u(r))}{r}drds-\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}ds\right).\end{aligned}\tag{3.1}$$

It should be noticed that BVPs (1.1) has solutions if and only if the operator $\mathbb{A}$ has fixed points.

First, we obtain the existence and uniqueness results via Banach fixed point theorem.

Theorem 3.1 Assume that $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and there exists a constant $L>0$ such that

(H1) $|f(t,u)-f(t,v)|\le L|u-v|,\forall\, t\in[1,e],u,v\in\mathbb{R}$. If

$$L\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1+\lambda(e-1)}{\Gamma(\alpha+1)|1-\lambda A|}\right]<1,\tag{3.2}$$

then problem (1.1) has a unique solution on $[1,e]$.

Proof Denote $Q=\frac{1}{\Gamma(\alpha+1)}+\frac{1+\lambda(e-1)}{\Gamma(\alpha+1)|1-\lambda A|}$, we set $B_r:=\{u\in C([1,e],\mathbb{R}):\|u\|\le r\}$ and choose $r\ge\frac{MQ}{1-LQ}$, where $M=\max\limits_{t\in[1,e]}|f(t,0)|<\infty$.

Obviously it is concluded that

$$|f(s,u(s))|=|f(s,u(s))-f(s,0)+f(s,0)|\le Lr+M.$$

Now, we show that $\mathbb{A}B_r \subseteq B_r$. For any $u \in B_r, t \in [1,e]$, we have

$$\begin{aligned}\|\mathbb{A}u\| = \max_{t\in[1,e]} |(\mathbb{A}u)(t)| \le \max_{t\in[1,e]} \Bigg\{ &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u(s))|}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)|1-\lambda A|} \\ &\times \left(|\lambda|\int_1^e\int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u(r))|}{r}drds + \int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s}ds\right)\Bigg\} \\ \le &(Lr+M)Q \le r,\end{aligned}$$

which implies that $\mathbb{A}B_r \subseteq B_r$. Let $u, v \in B_r$, and for each $t \in [1,e]$, we have

$$\begin{aligned}|(\mathbb{A}u)(t) - (\mathbb{A}v)(t)| \le &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u(s)) - f(s,v(s))|}{s}ds \\ &+ \frac{(\ln t)^2}{\Gamma(\alpha)|1-\lambda A|}\left(|\lambda|\int_1^e\int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{|f(r,u(r)-f(r,v(r))|}{r}drds\right. \\ &\left. + \int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s)-f(s,v(s))|}{s}ds\right) \\ \le &LQ\|u-v\|.\end{aligned}$$

Therefore,

$$\|\mathbb{A}u - \mathbb{A}v\| \le LQ\|u-v\|.$$

From assumption (3.2), it follows that $\mathbb{A}$ is a contraction mapping. Hence problem (1.1) has a unique solution by using Banach fixed point theorem. This completes the proof.

Next, we will use the method of upper and lower solutions to obtain the existence result of BVPs (1.1).

Definition 3.2 Functions $\overline{u}, \underline{u} \in C([1,e],\mathbb{R})$ are called upper and lower solutions of fractional integral equation (2.6), respactively, if it satisfies for any $t \in [1,e]$,

$$\begin{aligned}\underline{u}(t) \le &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)} \\ &\times\left(\lambda\int_1^e\int_1^s\left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{h(r)}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\right), \\ \overline{u}(t) \ge &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)} \\ &\times\left(\lambda\int_1^e\int_1^s\left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{h(r)}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds\right).\end{aligned}$$

Define

$$X_{(\underline{u},\overline{u})} = \{u \in C([1,e],\mathbb{R}) : \underline{u}(t) \le u(t) \le \overline{u}(t), t \in [1,e], u \text{ is the solution of (2.6)}\}.$$

Theorem 3.3 Let $f \in C([1,e]\times\mathbb{R},\mathbb{R})$. Assume that $\overline{u}, \underline{u} \in C([1,e],\mathbb{R})$ are upper and lower solutions of fractional integral equation (2.6) with $\underline{u}(t) \le \overline{u}(t)$ for $t \in [1,e]$. If f is nondecreasing with respect to u that is $f(t,u_1) \le f(t,u_2)$, $u_1 \le u_2$, then there exist maximal and minimal solutions $u_M, u_L \in X_{(\underline{u},\overline{u})}$ in $X_{(\underline{u},\overline{u})}$, moreover, for each $u \in X_{(\underline{u},\overline{u})}$, one has

$$u_L(t) \le u(t) \le u_M(t),\ t \in [1,e].$$

Proof Constructing two sequences $\{p_n\},\{q_n\}$ as follows

$$\begin{cases} p_0 = \underline{u}, \\ p_{n+1}(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,p_n(s))}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)} \\ \times\left(\lambda\displaystyle\int_1^e\int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,p_n(r))}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,p_n(s))}{s}ds\right), n = 0,1\cdots, \end{cases} \tag{3.3}$$

$$\begin{cases} q_0 = \overline{u}, \\ q_{n+1}(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,q_n(s))}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)} \\ \times\left(\lambda\displaystyle\int_1^e\int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,q_n(r))}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,q_n(s))}{s}ds\right), n = 0,1\cdots. \end{cases} \tag{3.4}$$

This proof divides into three steps.

Step 1 Finding the monotonicity of the two sequences, that is, the sequences $\{p_n\},\{q_n\}$ satisfy the following relation

$$\underline{u}(t) = p_0(t) \le p_1(t)\cdots \le p_n(t) \le q_n(t)\cdots \le q_1(t) \le q_0(t) = \overline{u}(t) \tag{3.5}$$

for $t \in [1,e]$.

First, we verify that the sequence $\{p_n\}$ is nondecreasing and satisfies

$$p_n(t) \le q_0(t),\ t\in[1,e],\ \forall\ n \in \mathbb{N}.$$

Since $\overline{u},\underline{u}$ are upper and lower solutions respectively, we know that $\underline{u}(t) = p_0(t) \le \overline{u}(t) = q_0(t)$ for $t \in [1,e]$,

$$\begin{aligned} p_0(t) \le & \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,p_0(s))}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)} \\ & \times\left(\lambda\int_1^e\int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,p_0(r))}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,p_0(s))}{s}ds\right) \\ = & p_1(t),\ t\in[1,e]. \end{aligned}$$

Since f is nondecreasing respect to the second variable, this implies that

$$f(s,p_0(s)) \le f(s,q_0(s)),\ s\in[1,e].$$

This deduces

$$
\begin{aligned}
p_1(t) =& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,p_0(s))}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)}\\
&\times\left(\lambda\int_1^e\int_1^s\left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,p_0(r))}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,p_0(s))}{s}ds\right)\\
\le& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,q_0(s))}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)}\\
&\times\left(\lambda\int_1^e\int_1^s\left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,q_0(r))}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,q_0(s))}{s}ds\right)\\
=& q_0(t),\ t\in[1,e].
\end{aligned}
$$

Therefore, we assume inductively

$$p_{n-1}(t)\le p_n(t)\le q_0(t),\ t\in[1,e].$$

In view of definition of $\{p_n\},\{q_n\}$, we have

$$
\begin{aligned}
p_n(t) =& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,p_{n-1}(s))}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)}\\
&\times\left(\lambda\int_1^e\int_1^s\left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,p_{n-1}(r))}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,p_{n-1}(s))}{s}ds\right),\\
p_{n+1}(t) =& \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,p_n(s))}{s}ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)}\\
&\times\left(\lambda\int_1^e\int_1^s\left(\ln\frac{s}{r}\right)^{\alpha-1}\frac{f(r,p_n(r))}{r}drds - \int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{f(s,p_n(s))}{s}ds\right).
\end{aligned}
$$

By means of the monotonicity of f, it is obvious that

$$p_n(t)\le p_{n+1}(t)\le q_0(t),\ t\in[1,e].$$

We show that

$$p_n(t)\le q_n(t),\ t\in[1,e],\ n\in\mathbb{N}.$$

For $n=0$, it is obvious that $\underline{u}(t)=p_0(t)\le\overline{u}(t)=q_0(t)$ for all $t\in[1,e]$. Now, we also suppose inductively

$$p_n(t)\le q_n(t),\ t\in[1,e].$$

Analogously, we easily conclude from the monotonicity of f with respect to the second variables that

$$p_{n+1}(t)\le q_{n+1}(t).$$

In a similar way, we know that the sequence $\{q_n\}$ is nonincreasing.

Step 2 The sequences constructed by (3.3), (3.4) are both relatively compact in $C([1,e],\mathbb{R})$.

According to that f is continuous and $\overline{u}, \underline{u} \in C([1,e],\mathbb{R})$, from Step 1, we have $\{p_n\}$ and $\{q_n\}$ also belong to $C([1,e],\mathbb{R})$. Moreover, it follows from (3.5) that $\{p_n\}$ and $\{q_n\}$ are uniformly bounded. For any $t_1, t_2 \in [1,e]$, without loss of generality, let $t_1 \leq t_2$, we know that

$$\begin{aligned}
|p_{n+1}(t_1) - p_{n+1}(t_2)| =& \frac{1}{\Gamma(\alpha)}\bigg| \int_1^{t_1} \left[\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \right] \frac{f(s,p_n(s))}{s} ds \\
&+ \int_{t_1}^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \frac{f(s,p_n(s))}{s} ds + \frac{(\ln t_2)^2 - (\ln t_1)^2}{(1-\lambda A)} \\
&\left[\lambda \int_1^e \int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1} \frac{f(r,q_n(r))}{r} drds - \int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1} \frac{f(s,q_n(s))}{s} ds \right] \bigg| \\
\leq& \frac{W}{\Gamma(\alpha)}\bigg| \int_1^{t_1} \frac{1}{s}\left[\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \right] ds \\
&+ \int_{t_1}^{t_2} \frac{1}{s}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} ds + \frac{(\ln t_2)^2 - (\ln t_1)^2}{(1-\lambda A)} \\
&\left[\frac{1}{s}\left(\lambda \int_1^e \int_1^s \frac{1}{s}\left(\ln\frac{s}{r}\right)^{\alpha-1} drds + \int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1} ds \right) \right] \bigg|,
\end{aligned}$$

approaches zero as $t_2 - t_1 \to 0$, where $W > 0$ is a constant independent of n, t_1 and t_2, $|f(t,p_n(t))| \leq W$. It implies that $\{p_n\}$ is equicontinuous in $C([1,e],\mathbb{R})$. By Arzelà-Ascoli theorem, we imply that $\{p_n\}$ is relatively compact in $C([1,e],\mathbb{R})$. In the same way, we conclude that $\{q_n\}$ is also relatively compact in $C([1,e],\mathbb{R})$.

Step 3 There exist maximal and minimal solutions in $X_{(\underline{u},\overline{u})}$.

The sequences $\{p_n\}$ and $\{q_n\}$ are both monotone and relatively compact in $C([1,e],\mathbb{R})$ by Step 1 and Step 2. There exist continuous functions p and q such that $p_n(t) \leq p(t) \leq q(t) \leq q_n(t)$ for all $t \in [1,e]$ and $n \in \mathbb{N}$. $\{p_n\}$ and $\{q_n\}$ converge uniformly to p and q in $C([1,e],\mathbb{R})$, severally. Therefore, p and q are two solutions of (2.6), i.e.,

$$\begin{aligned}
p(t) =& -\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,p(s))}{s} ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)} \\
&\times \left(\lambda \int_1^e \int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1} \frac{f(r,p(r))}{r} drds - \int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1} \frac{f(s,p(s))}{s} ds \right), \\
q(t) =& -\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,q(s))}{s} ds + \frac{(\ln t)^2}{\Gamma(\alpha)(1-\lambda A)} \\
&\times \left(\lambda \int_1^e \int_1^s \left(\ln\frac{s}{r}\right)^{\alpha-1} \frac{f(r,q(r))}{r} drds - \int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1} \frac{f(s,q(s))}{s} ds \right)
\end{aligned}$$

for $t \in [1,e]$. However, fact (3.5) determines that

$$\underline{u}(t) \leq p(t) \leq q(t) \leq \overline{u}(t),\ t \in [1,e].$$

Finally, we shall show that p and q are the minimal and maximal solutions in $X_{(\underline{u},\overline{u})}$, respectively. For any $u \in X_{(\underline{u},\overline{u})}$, then we have

$$\underline{u}(t) \leq u(t) \leq \overline{u}(t),\ t \in [1,e].$$

Because f is nondecreasing with respect to the second parameter, we conclude

$$\underline{u}(t) \le p_n(t) \le u(t) \le q_n(t) \le \overline{u}(t),\ t \in [1,e],\ n \in \mathbb{N}.$$

Taking limits as $n \to \infty$ into the above inequality, we have

$$\underline{u}(t) \le p(t) \le u(t) \le q(t) \le \overline{u}(t),\ t \in [1,e],$$

which means that $u_L = p$ and $u_M = q$ are the minimal and maximal solutions in $X_{(\underline{u},\overline{u})}$. This completes the proof.

Theorem 3.4 Assume that assumptions of Theorem 3.3 are satisfied. Then fractional nonlinear differential equation (1.1) has at least one solution in $C([1,e],\mathbb{R})$.

Proof By the hypotheses and Theorem 3.3, we induct $X_{(\underline{u},\overline{u})} \neq \emptyset$, then the solution set of fractional integral equation (2.6) is nonempty in $C([1,e],\mathbb{R})$. It follows from the solution set of (2.6) together with Lemma 2.5 that problem (1.1) has at least one solution in $C([1,e],\mathbb{R})$. This completes the proof.

4 Examples

In this section, we present two examples to explain our main results.

Example 1 Consider the following nonlinear Caputo-Hadamard fractional differential equation

$$\begin{cases} {}^C_H D^{\frac{5}{2}}_{1+} u(t) = \dfrac{(\sqrt{t}+\ln t)}{(t+3)^3}\dfrac{u^2}{|u|+1},\ 1 \le t \le e, \\ u(1) = u'(1) = 0, u(e) = \displaystyle\int_1^e u(s)ds, \end{cases} \tag{4.1}$$

here $\alpha = \frac{5}{2}$, $\lambda = 1$, $f(t,u) = \frac{(\sqrt{t}+\ln t)}{(t+3)^3}\frac{u^2}{|u|+1}$, $1 \le t \le e$. One can easily calculate $Q = \frac{24}{15\sqrt{\pi}(3-e)} \approx 3.2$. Clearly f is a continuous function and we have

$$|f(t,u) - f(t,v)| \le \frac{1}{64}(\sqrt{t}+1)|u-v| \le \frac{1}{32}|u-v|.$$

Therefore $LQ < 1$. Thus all conditions of Theorem 3.1 satisfy which implies the existence of uniqueness solution of the the boundary value problem (4.1).

Example 2 Consider the problem

$$\begin{cases} {}^C_H D^{\frac{5}{2}}_{1+} u(t) = \dfrac{t^4}{16\sqrt{\pi}}(|u|+1), \\ u(1) = u'(1) = 0,\ u(e) = \dfrac{1}{(6-2e)}\displaystyle\int_1^e u(s)ds. \end{cases} \tag{4.2}$$

Proof Where $\alpha = \frac{5}{2}$, $\lambda = \frac{1}{(6-2e)}$, $f(t,u) = \frac{t^4}{16\sqrt{\pi}}(|u|+1)$, $t \in [1,e]$, f is continuous and nondecreasing with respect to u. Thus

$$u(t) = {}^H I^{\frac{5}{2}}_{1+}(f(t,u(t))) + \frac{(\ln t)^2}{\Gamma\left(\frac{5}{2}\right)\frac{1}{\lambda}}\left[\frac{1}{\lambda}\int_1^e {}^H I^{\frac{5}{2}}_{1+}(f(r,u(r)))dr - {}^H I^{\frac{5}{2}}_{1+}(f(t,u(e)))\right]. \tag{4.3}$$

It is easy to check that $(\underline{u}(t), \overline{u}(t)) = (0, (\ln t)^3)$ is a pair of upper and lower solutions of (4.3) and that all assumptions of Theorem 3.2 are satisfied. So $u_L = p$ and $u_M = q$ are the minimal and maximal solutions of the boundary problem (4.3), and the iteration sequences is as follows

$$\begin{cases} p_0 = \underline{u}(t), \\ p_{n+1} = {}^H I_{1+}^{\frac{5}{2}}(f(t, p_n(t))) + \dfrac{(\ln t)^2}{\Gamma\left(\frac{5}{2}\right)\frac{1}{\lambda}} \left[\dfrac{1}{\lambda}\displaystyle\int_1^e {}^H I_{1+}^{\frac{5}{2}}(f(r, p_n(r)))dr - {}^H I_{1+}^{\frac{5}{2}}(f(t, p_n(e)))\right], \end{cases} \tag{4.4}$$

$$\begin{cases} q_0 = \overline{u}(t), \\ q_{n+1} = {}^H I_{1+}^{\frac{5}{2}}(f(t, q_n(t))) + \dfrac{(\ln t)^2}{\Gamma\left(\frac{5}{2}\right)\frac{1}{\lambda}} \left[\dfrac{1}{\lambda}\displaystyle\int_1^e {}^H I_{1+}^{\frac{5}{2}}(f(r, q_n(r)))dr - {}^H I_{1+}^{\frac{5}{2}}(f(t, q_n(e)))\right], \end{cases} \tag{4.5}$$

$\lim\limits_{n\to\infty} p_n = p$ and $\lim\limits_{n\to\infty} q_n = q$. Applying Theorem 3.4, this boundary value problem (4.2) has at least one solution.

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Caputo-Hadamard型分数阶微分方程边值问题解的存在唯一性

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摘要: 本文研究了一类Caputo-Hadamard型分数阶微分方程边值问题的解. 利用Banach不动点定理和上下解方法, 获得了解的存在性和唯一性, 推广了常微分方程边值问题的一些结果. 作为应用, 给出了两个例子来说明我们的主要结果.

关键词: 分数阶微分方程; Caputo-Hadamard导数; Banach不动点定理; 上下解

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