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GRADUAL HAUSDORFF METRIC AND ITS APPLICATIONS

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Abstract: This paper is devoted to study of gradual Hausdorff metric introduced by Zhou and Zhang and its application. By means of relationships between gradual numbers and fuzzy numbers, we prove that $(\mathscr{F}_c(\mathbb{R}), \tilde{d}_H)$ is a gradual metric space. And then, we apply gradual Hausdorff metric to fuzzy random variables and obtain new strong law of large numbers for fuzzy random variables, which enrich and deepen the theory of fuzzy numbers and fuzzy random variables.

Keywords: gradual number; gradual Hausdorff metric; fuzzy random variable; strong law of large numbers

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1 Introduction

How to define fuzzy metric is one of the fundamental problems in fuzzy mathematics. There are two approaches in this field till now. One is using fuzzy numbers to define metric on ordinary spaces, first proposed by Kaleva [1], following which fuzzy normed spaces, fuzzy topology induced by fuzzy metric spaces, fixed point theorem and other properties of fuzzy metric spaces are studied by a few researchers (see e.g., [2–5]). The other is using real numbers to measure the distances between fuzzy sets. The references of this approach can be referred to [6–8], etc.

In above literature, the most often proposed fuzzy sets are fuzzy numbers [9]. But it is well known that fuzzy numbers generalize intervals, not numbers and model incomplete knowledge, not the fuzziness per se. Furthermore, fuzzy arithmetic inherit algebraic properties of interval arithmetic, not of numbers. There were some authors who tried to equip fuzzy numbers with a group structure. This kind of attempt suggests a confusion between uncertainty and gradualness. Hence the name "fuzzy number" used by many authors is debatable. Recently, the authors [9] introduced a new concept in fuzzy set theory as "gradual

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Biography: Zhou Caili (1977–), female, born at Tangshan, Hebei, associate professor, major in fuzzy mathematics.

numbers". Gradual numbers express fuzziness only, without imprecision, which are unique generalization of real numbers and equipped with the same algebraic structures as real numbers (addition is a group, etc.). In the brief time since their introduction, gradual numbers were researched by many authors (see e.g., [9–13]). In particular, a fuzzy number can be denoted as a crisp interval of gradual numbers which can be bounded by two special gradual numbers. By virtue of considering such a structural characterization of fuzzy numbers, Zhou and Zhang [14] generalized classical Hausdorff metric to the space of fuzzy numbers and introduced the gradual Hausdorff metric. This paper is a continuity of [14]. In this paper, we investigate further properties for gradual Hausdorff metric and apply this metric to fuzzy random variables.

The organization of the paper is as follows. In Section 2, we state some basic results about gradual numbers, fuzzy numbers, fuzzy random variables, gradual Hausdorff metric, etc. In Section 3, we first investigate further properties for gradual Hausdorff metrics. And then, we apply gradual Hausdorff metric to fuzzy random variables and obtain a new strong law of large numbers for fuzzy random variables.

2 Preliminaries

In this section, we state some basic concepts about gradual numbers, fuzzy numbers, gradual Hausdorff metric, fuzzy random variables.

Definition 2.1 [9] A gradual number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}}$ from $(0,1]$ to the real numbers $\mathbb{R}$. Naturally a nonnegative gradual number is defined by its assignment function from $(0,1]$ to $[0,+\infty)$.

In the sequel, the set of all gradual numbers (resp. nonnegative gradual numbers) is denoted by $\mathbb{R}(I)$ (resp. $\mathbb{R}^*(I)$). It is clear that any real number $b \in \mathbb{R}$ can be viewed as an constant assignment function $A_{\tilde{b}}(\alpha) = b$ for each $\alpha \in (0,1]$. We call such elements in $\mathbb{R}(I)$ constant gradual numbers.

Definition 2.2 [9] Suppose that $*$ is any operation in real numbers and $\tilde{r}_1$ and $\tilde{r}_2$ are two any gradual numbers with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$, respectively. Then $\tilde{r}_1 * \tilde{r}_2$ is the gradual number with an assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ defined by

$$A_{\tilde{r}_1 * \tilde{r}_2}(\alpha) = A_{\tilde{r}_1}(\alpha) * A_{\tilde{r}_2}(\alpha)$$

for each $\alpha \in (0,1]$, where $* \in \{+,-,\times,\div\}$.

Definition 2.3 [14] Let $\tilde{r}$, $\tilde{s} \in \mathbb{R}(I)$.

(1) The relations $\leq$ of $\tilde{r}$ and $\tilde{s}$ can be defined as follows:

$$\tilde{r} \leq \tilde{s} \Longleftrightarrow \forall \alpha \in (0,1], A_{\tilde{r}}(\alpha) \leq A_{\tilde{s}}(\alpha).$$

(2) The maximum and minimum operations of $\tilde{r}$ and $\tilde{s}$ are defined as follows: for all $\alpha \in (0,1]$,

$$A_{\max\{\tilde{r},\tilde{s}\}}(\alpha) = \max\left\{A_{\tilde{r}}(\alpha), A_{\tilde{s}}(\alpha)\right\}, \ A_{\min\{\tilde{r},\tilde{s}\}}(\alpha) = \min\left\{A_{\tilde{r}}(\alpha), A_{\tilde{s}}(\alpha)\right\}.$$

(3) Let $\tilde{r}$ be in $\mathbb{R}(I)$. The mapping $|\tilde{r}|:(0,1]\to\mathbb{R}^*(I)$ defined by

$$A_{|\tilde{r}|}(\alpha)=|A_{\tilde{r}}(\alpha)|,\forall\alpha\in(0,1]$$

is called the absolute value of $\tilde{r}$.

In the following, we describe some basic results for fuzzy numbers. A fuzzy number is a normal, convex, upper semicontinuous and compactly supported fuzzy set on $\mathbb{R}$. In the sequel, let $\mathscr{F}_c(\mathbb{R})$ denote the family of all fuzzy numbers. According to Fortin, Dubois and Fargier [9], a fuzzy number $\tilde{A}$ can be viewed as a particular gradual interval $\tilde{A}=[\tilde{a}^-,\tilde{a}^+]$, where $\tilde{a}^-$ and $\tilde{a}^+$ are defined by

$$A_{\tilde{a}^-}(\alpha)=\inf\{x:\tilde{A}(x)\geq\alpha\} \text{ and } A_{\tilde{a}^+}(\alpha)=\sup\{x:\tilde{A}(x)\geq\alpha\}$$

for each $\alpha\in(0,1]$, respectively. A gradual number $\tilde{r}$ as a degenerate fuzzy number $\{\tilde{r}\}$.

In the same way as defining crisp interval, we can define equality, sum and scalar multiplication on the space of fuzzy numbers as follows: let $\tilde{A}=[\tilde{a}^-,\tilde{a}^+]$ and $\tilde{B}=[\tilde{b}^-,\tilde{b}^+]$ be in $\mathscr{F}_c(\mathbb{R})$ and $\gamma\in\mathbb{R}$. Define

(1) $\tilde{A}=\tilde{B}$ if and only if $\tilde{a}^-=\tilde{b}^-$ and $\tilde{a}^+=\tilde{b}^+$;

(2) $\tilde{A}\oplus\tilde{B}=[\tilde{a}^-+\tilde{b}^-,\tilde{a}^++\tilde{b}^+]$;

(3) $\gamma\tilde{A}=[\gamma\tilde{a}^-,\gamma\tilde{a}^+]$ if $\gamma\geq 0$ and $\gamma\tilde{A}=[\gamma\tilde{a}^+,\gamma\tilde{a}^-]$ if $\gamma<0$.

Definition 2.4 [14] Let $\tilde{A}=[\tilde{a}^-,\tilde{a}^+]$ and $\tilde{B}=[\tilde{b}^-,\tilde{b}^+]$ be in $\mathscr{F}_c(\mathbb{R})$. Define

$$\tilde{d}_H(\tilde{A},\tilde{B})=\max\left\{|\tilde{a}^--\tilde{b}^-|,|\tilde{a}^+-\tilde{b}^+|\right\}.$$

We call $\tilde{d}_H$ gradual Hausdorff metric on $\mathscr{F}_c(\mathbb{R})$. In particular, we define

$$\|\tilde{A}\|=\tilde{d}_H(\tilde{A},\hat{0})=\max\{|\tilde{a}^-|,|\tilde{a}^+|\},$$

where $\hat{0}$ is the fuzzy number $\{\tilde{0}\}$.

Definition 2.5 [14] Let $\{\tilde{A}_n\}_{n\in\mathbb{N}}\subseteq\mathscr{F}_c(\mathbb{R})$ and $\tilde{A}\in\mathscr{F}_c(\mathbb{R})$. We call that $\{\tilde{A}_n\}_{n\in\mathbb{N}}$ converges to $\tilde{A}$ with respect to the gradual Hausdorff metric $\tilde{d}_H$ if and only if

$$\lim_{n\to\infty}\tilde{d}_H(\tilde{A}_n,\tilde{A})=\tilde{0},$$

i.e., for each $\alpha\in(0,1]$, $\lim\limits_{n\to\infty}A_{\tilde{d}_H(\tilde{A}_n,\tilde{A})}(\alpha)=A_{\tilde{0}}(\alpha)=0$. We denote it by $(\tilde{d}_H)\lim\limits_{n\to\infty}\tilde{A}_n=\tilde{A}$ or $\tilde{A}_n\xrightarrow{\tilde{d}_H}\tilde{A}$.

Theorem 2.6 [14] Let $\{\tilde{A}_n\}_{n\in\mathbb{N}}\subseteq\mathscr{F}_c(\mathbb{R})$. $\{\tilde{A}_n\}_{n\in\mathbb{N}}$ converges to fuzzy number $\tilde{A}$ with respect to $\tilde{d}_H$ if and only if $\tilde{a}_n^-$ converges to $\tilde{a}^-$ and $\tilde{a}_n^+$ converges to $\tilde{a}^+$ simultaneously.

Definition 2.7 [15] A gradual metric space is an ordered pair $(X,\tilde{d})$, where X is any set and $\tilde{d}:X\times X\to R^*(I)$ is a mapping satisfying

(1) $\tilde{d}(x,y)=\tilde{0}$ if and only if $x=y$;

(2) $\tilde{d}(x,y)=\tilde{d}(y,x),\forall x,y\in X$;

(3) $\tilde{d}(x,y)\leq\tilde{d}(x,z)+\tilde{d}(z,y),\forall x,y,z\in X$.

Definition 2.8 [16] A mapping $\tilde{x} : \Omega \to \mathbb{R}(I)$ is said to be gradual random variable associated with a measurable space $(\Omega, \mathscr{A})$ if for each $\alpha \in (0, 1]$, the real-valued mapping $\tilde{x}_\alpha : \Omega \to \mathbb{R}$, defined by $\tilde{x}_\alpha(\omega) = A_{\tilde{x}(\omega)}(\alpha)$, $\forall \omega \in \Omega$, is a classical random variable associated with $(\Omega, \mathscr{A})$.

Definition 2.9 [17] Given a probability space $(\Omega, \mathscr{A}, P)$, a mapping $\tilde{X} : \Omega \to \mathscr{F}_c(\mathbb{R})$ is said to be a fuzzy random variable (or FRV for short) if for all $\alpha \in (0, 1]$ the two real-valued mappings

$$\inf \tilde{X}_\alpha : \Omega \to \mathbb{R}, \ \sup \tilde{X}_\alpha : \Omega \to \mathbb{R}$$

(defined so that for all $\omega \in \Omega$ we have that $\tilde{X}_\alpha(\omega) = [\inf(\tilde{X}(\omega))_\alpha, \sup(\tilde{X}(\omega))_\alpha])$ are real-valued random variables.

If we denote $\tilde{X}(\omega) = \{x^-_\alpha(\omega), x^+_\alpha(\omega) | 0 \le \alpha \le 1\}$, then it is well known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0, 1]$, x^-_α and x^+_α are real-valued random variables in the usual sense. Therefore, $\tilde{X}$ is a fuzzy random variable if and only if $\tilde{x}^-$ and $\tilde{x}^+$ defined by

$$A_{\tilde{x}^-(\omega)}(\alpha) = \tilde{x}^-_\alpha(\omega), \ A_{\tilde{x}^+(\omega)}(\alpha) = \tilde{x}^+_\alpha(\omega), \forall \omega \in \Omega$$

are two gradual random variables. In the following, we denote $\tilde{X}$ by $[\tilde{x}^-, \tilde{x}^+]$.

For more details on gradual numbers, fuzzy numbers, gradual Hausdorff metrics, gradual random variables, fuzzy random variables, we refer the reader to [9, 10, 12–16].

3 Main Results

Theorem 3.1 $(\mathscr{F}_c(\mathbb{R}), \tilde{d}_H)$ is a gradual metric space.

Proof Let $\tilde{A}, \tilde{B}$ and $\tilde{C}$ be in $\mathscr{F}_c(\mathbb{R})$. First, we show that $\tilde{d}_H(\tilde{A}, \tilde{B}) = \tilde{0}$ if and only if $\tilde{A} = \tilde{B}$. If $\tilde{d}_H(\tilde{A}, \tilde{B}) = \tilde{0}$, then

$$\tilde{d}_H(\tilde{A}, \tilde{B}) = \max\left\{|\tilde{a}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{b}^+|\right\} = \tilde{0},$$

which implies that $|\tilde{a}^- - \tilde{b}^-| = \tilde{0}$ and $|\tilde{a}^+ - \tilde{b}^+| = \tilde{0}$. It follows that $\tilde{a}^- = \tilde{b}^-$ and $\tilde{a}^+ = \tilde{b}^+$. Hence, $\tilde{A} = \tilde{B}$. The converse is obvious.

In the following, we show that for any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathscr{F}_c(\mathbb{R})$, $\tilde{d}_H(\tilde{A}, \tilde{B}) \le \tilde{d}_H(\tilde{A}, \tilde{C}) + \tilde{d}_H(\tilde{C}, \tilde{B})$. According to Definition 2.2 and Definition 2.3, we have $|\tilde{a}^- - \tilde{b}^-| \le |\tilde{a}^- - \tilde{c}^-| + |\tilde{c}^- - \tilde{b}^-|$ and $|\tilde{a}^+ - \tilde{b}^+| \le |\tilde{a}^+ - \tilde{c}^+| + |\tilde{c}^+ - \tilde{b}^+|$. It follows that

$$\begin{aligned} \max\{|\tilde{a}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{b}^+|\} &\le \max\{|\tilde{a}^- - \tilde{c}^-| + |\tilde{c}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{c}^+| + |\tilde{c}^+ - \tilde{b}^+|\} \\ &\le \max\{|\tilde{a}^- - \tilde{c}^-|, |\tilde{a}^+ - \tilde{c}^+|\} + \max\{|\tilde{c}^- - \tilde{b}^-|, |\tilde{c}^+ - \tilde{b}^+|\}, \end{aligned}$$

i.e., $\tilde{d}_H(\tilde{A}, \tilde{B}) \le \tilde{d}_H(\tilde{A}, \tilde{C}) + \tilde{d}_H(\tilde{C}, \tilde{B})$.

Finally, for any $\tilde{A}, \tilde{B} \in \mathscr{F}_c(\mathbb{R})$, we have

$$\tilde{d}_H(\tilde{A}, \tilde{B}) = \max\left\{|\tilde{a}^- - \tilde{b}^-|, |\tilde{a}^+ - \tilde{b}^+|\right\} = \max\left\{|\tilde{b}^- - \tilde{a}^-|, |\tilde{b}^+ - \tilde{a}^+|\right\} = \tilde{d}_H(\tilde{B}, \tilde{A}).$$

This completes the proof.

Theorem 3.2 Let $\tilde{X} = [\tilde{x}^-, \tilde{x}^+]$ and $\tilde{Y} = [\tilde{y}^-, \tilde{y}^+]$ be two fuzzy random variables. Then $\tilde{d}_H(\tilde{X}, \tilde{Y}) : \Omega \to \mathbb{R}(I)$ defined by

$$\tilde{d}_H(\tilde{X}, \tilde{Y})(\omega) = \tilde{d}_H(\tilde{X}(\omega), \tilde{Y}(\omega)), \forall \omega \in \Omega$$

is a gradual random variable. Especially, $\|\tilde{X}\| = \tilde{d}_H(\tilde{X}, \hat{0})$ is a gradual random variable.

Proof Since $\tilde{X} = [\tilde{x}^-, \tilde{x}^+]$ and $\tilde{Y} = [\tilde{y}^-, \tilde{y}^+]$ are fuzzy random variables, $\tilde{x}^-, \tilde{x}^+, \tilde{y}^-$ and $\tilde{y}^+$ are gradual random variables. It follows from Theorem 3.7 [16] that $|\tilde{x}^- - \tilde{y}^-|$ and $|\tilde{x}^+ - \tilde{y}^+|$ are gradual random variables. Then, $\tilde{d}_H(\tilde{X}, \tilde{Y}) = \max\{|\tilde{x}^- - \tilde{y}^-|, |\tilde{x}^+ - \tilde{y}^+|\}$ is a gradual random variable. Especially, if $\tilde{Y} = \hat{0}$, $\|\tilde{X}\| = \tilde{d}_H(\tilde{X}, \hat{0}) = \max\{|\tilde{x}^-|, |\tilde{x}^+|\}$ is a gradual random variable.

Definition 3.3 A fuzzy random variable $\tilde{X} = [\tilde{x}^-, \tilde{x}^+]$ is called integrable if $\tilde{x}^-$ and $\tilde{x}^+$ are integrable simultaneously. In this case, we call the expectation $E[\tilde{X}]$ of $\tilde{X}$ exists if $E[\tilde{X}] \in \mathscr{F}_c(\mathbb{R})$ and $E[\tilde{X}] = [E[\tilde{x}^-], E[\tilde{x}^+]]$.

From Definition 3.3, we can show that $[E[\tilde{X}]]^- = E[\tilde{x}^-], [E[\tilde{X}]]^+ = E[\tilde{x}^+]$ if $E[\tilde{X}]$ exists.

Theorem 3.4 Let $\tilde{X} = [\tilde{x}^-, \tilde{x}^+]$ and $\tilde{Y} = [\tilde{y}^-, \tilde{y}^+]$ be two fuzzy random variables.

(1) $E(\tilde{A}) = \tilde{A}$, where $\tilde{A}$ denote the fuzzy random variable which have the same outcome $\tilde{A}$ for all $\omega \in \Omega$, i.e., $\tilde{A}(\omega) = \tilde{A}, \forall \omega \in \Omega$;

(2) If $E[\tilde{X}]$ and $E[\tilde{Y}]$ exist and $\lambda, \gamma \in \mathbb{R}^+$, then $E[\lambda\tilde{X} \oplus \gamma\tilde{Y}]$ exists and

$$E[\lambda\tilde{X} \oplus \gamma\tilde{Y}] = \lambda E[\tilde{X}] \oplus \gamma E[\tilde{Y}].$$

Proof (1) Let $\tilde{A} = [\tilde{a}^-, \tilde{a}^+]$. If for any $\omega \in \Omega$, $\tilde{A}(\omega) = \tilde{A}$, then $\tilde{a}^-(\omega) = \tilde{a}^-$ and $\tilde{a}^+(\omega) = \tilde{a}^+$, i.e., $\tilde{a}^-$ and $\tilde{a}^+$ are constant valued gradual random variables. It follows from Theorem 3.10 [16] that $E[\tilde{a}^-] = \tilde{a}^-$ and $E[\tilde{a}^+] = \tilde{a}^+$. Thus, we have

$$E(\tilde{A}) = \big[E[\tilde{a}^-], E[\tilde{a}^+]\big] = [\tilde{a}^-, \tilde{a}^+] = \tilde{A}.$$

(2) Since $E[\tilde{X}]$ and $E[\tilde{Y}]$ exist, $E[\tilde{X}], E[\tilde{Y}] \in \mathscr{F}_c(\mathbb{R})$. It follows that $\lambda E[\tilde{X}] \oplus \gamma E[\tilde{Y}] \in \mathscr{F}_c(\mathbb{R})$ for any $\lambda, \gamma \in \mathbb{R}^+$ and

$$\lambda E[\tilde{X}] = [\lambda E[\tilde{x}^-], \lambda E[\tilde{x}^+]], \ \ \gamma E[\tilde{Y}] = [\gamma E[\tilde{y}^-], \gamma E[\tilde{y}^+]].$$

It follows that

$$\begin{aligned}
\lambda E[\tilde{X}] \oplus \gamma E[\tilde{Y}] &= [\lambda E[\tilde{x}^-], \lambda E[\tilde{x}^+]] \oplus [\gamma E[\tilde{y}^-], \gamma E[\tilde{y}^+]] \\
&= [\lambda E[\tilde{x}^-] + \gamma E[\tilde{y}^-], \lambda E[\tilde{x}^+] + \gamma E[\tilde{y}^+]] \\
&= [E[\lambda\tilde{x}^- + \gamma\tilde{y}^-], E[\lambda\tilde{x}^+ + \gamma\tilde{y}^+]] = E[\lambda\tilde{X} \oplus \gamma\tilde{Y}],
\end{aligned}$$

which implies that $E[\lambda\tilde{X} \oplus \gamma\tilde{Y}]$ exists and $E[\lambda\tilde{X} \oplus \gamma\tilde{Y}] = \lambda E[\tilde{X}] \oplus \gamma E[\tilde{Y}]$. This completes the proof.

Definition 3.5 Let $\{\tilde{X}_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy random variables. If $\{\tilde{x}_n^-\}_{n\in\mathbb{N}}$ and $\{\tilde{x}_n^+\}_{n\in\mathbb{N}}$ are sequences of independent and identically distributed gradual random variables, then $\{\tilde{X}_n\}_{n\in\mathbb{N}}$ is called a sequence of independent and identically distributed fuzzy random variables.

Theorem 3.6 Let $\{\tilde{X}_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed fuzzy random variables on the same probability space $(\Omega, \mathscr{A}, P)$ and $\tilde{X}_n, n = 1, 2, \cdots$ have the same expected value. Then we have

$$\tilde{d}_H\left(\frac{1}{n}\bigoplus_{i=1}^{n}\tilde{X}_i, E[\tilde{X}_1]\right) \to \tilde{0} \text{ a.s..}$$

Proof To end the proof, we use Theorem 2.6. We have

$$\frac{1}{n}\bigoplus_{i=1}^{n}\tilde{X}_i = \left[\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i^-, \frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i^+\right]$$

and $E[\tilde{X}_1] = \left[E[\tilde{x}_1^-], E[\tilde{x}_1^+]\right]$. Since $\{\tilde{X}_n\}_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed fuzzy random variables, $\{\tilde{x}_n^-\}_{n\in\mathbb{N}}$ and $\{\tilde{x}_n^+\}_{n\in\mathbb{N}}$ are sequences of independent and identically distributed gradual random variables. It follows that $\{\tilde{x}_{n\alpha}^-\}_{n\in\mathbb{N}}$ and $\{\tilde{x}_{n\alpha}^+\}_{n\in\mathbb{N}}$ are sequences of independent and identically distributed classical random variables for each $\alpha \in (0, 1]$. Since $\tilde{X}_n$ $(n = 1, 2, \cdots)$ have the same expected value, $E[\tilde{X}_n] = E[\tilde{X}_1]$ for each $n \in \mathbb{N}$, which implies that $E[\tilde{x}_n^-] = E[\tilde{x}_1^-]$ and $E[\tilde{x}_n^+] = E[\tilde{x}_1^+]$. Then we have

$$E[\tilde{x}_{n\alpha}^-] = A_{E[\tilde{x}_n^-]}(\alpha) = A_{E[\tilde{x}_1^-]}(\alpha) = E[\tilde{x}_{1\alpha}^-]$$

and

$$E[\tilde{x}_{n\alpha}^+] = A_{E[\tilde{x}_n^+]}(\alpha) = A_{E[\tilde{x}_1^+]}(\alpha) = E[\tilde{x}_{1\alpha}^+]$$

for each $\alpha \in (0, 1]$. Therefore, according to strong laws of large numbers for classical random variable, we have $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\tilde{x}_{i\alpha}^- = E[\tilde{x}_{1\alpha}^-]$, a.s. and $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\tilde{x}_{i\alpha}^+ = E[\tilde{x}_{1\alpha}^+]$, a.s. for each $\alpha \in (0, 1]$. It follows that $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\tilde{x}_i^- = E[\tilde{x}_1^-]$, a.s. and $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\tilde{x}_i^+ = E[\tilde{x}_1^+]$, a.s.. Then, by Theorem 2.6, we have

$$\tilde{d}_H(\frac{1}{n}\bigoplus_{i=1}^{n}\tilde{X}_i, E[\tilde{X}_1]) \to \tilde{0} \text{ a.s..}$$

This completes the proof.

Conclusions

In the paper, we firstly investigate further properties for gradual Hausdorff metric introduced by Zhou and Zhang. And then, we apply gradual Hausdorff metric to fuzzy random variables and obtain a new strong law of large numbers for fuzzy random variables. In all applications which involve metric and random variable, when data are fuzzy-valued, the conclusions given in this paper can be applied.

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梯度豪斯道夫度量及其应用

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摘要: 本文研究了Zhou 和Zhang 引入的梯度豪斯道夫度量的性质及其应用问题. 利用梯度数和模糊数之间的关系, 证明了$(\mathscr{F}_c(\mathbb{R}), \tilde{d}_H)$ 是一个梯度度量空间, 并把梯度豪斯道夫度量应用到模糊随机变量, 获得了模糊随机变量的新的强大数定律. 本文所得结果丰富和深化了模糊数及模糊随机变量相关理论.

关键词: 梯度数; 梯度豪斯道夫度量; 模糊随机变量; 强大数定律

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