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ANALYTIC REGULARITY OF SOLUTIONS TO SPATIALLY HOMOGENEOUS LANDAU EQUATION

WANG Yan-lin, XU Wei-man

(*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*)

Abstract: In this paper, we investigate the smoothness effect for the Cauchy problem of Landau equation with $\gamma \in [0,1]$. Analytic estimate involving time and analytic smoothness effect of the solutions are established under some weak assumptions on the initial data.

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1 Introduction and Main Results

In this paper, we study the Cauchy problem for the spatially homogeneous Landau equation, and investigate the analytic smoothing effect and moreover establish the explicit dependence on the time for the radius of analytic expansion. The Cauchy problem for spatially homogeneous Landau equation reads

$$\begin{cases} \partial_t f = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v-v_*)[f(v_*)\nabla_v f(v) - f(v)\nabla_v f(v_*)]dv_* \right), \\ f(0,v) = f_0(v), \end{cases} \tag{1.1}$$

where $f(t,v) \geq 0$ stands for the density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, and (a_{ij}) is a nonnegative symmetric matrix given by

$$a_{ij}(v) = \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right) |v|^{\gamma+2}. \tag{1.2}$$

We only consider here the condition $\gamma \in [0,1]$, which is called the hard potential case when $\gamma \in (0,1]$ and the Maxwellian molecules case when $\gamma = 0$. Set

$$c = \sum_{i,j=1}^{3} \partial_{v_i v_j} a_{ij} = -2(\gamma+3)\,|v|^{\gamma}$$

and

$$\bar{a}_{ij}(t,v) = (a_{ij} * f)(t,v) = \int_{\mathbb{R}^3} a_{ij}(v-v_*) f(t,v_*) dv_*, \quad \bar{c} = c * f.$$

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Biography: Wang Yanlin (1990–), male, born at Nanyang, Henan, master, major in partial differential equations of fluid dynamics.

Then the Cauchy problem (1.1) can be rewritten as

$$\begin{cases} \partial_t f = \sum\limits_{i,j=1}^{3} \bar{a}_{ij}\partial_{v_i v_j} f - \bar{c} f, \\ f(0,v) = f_0(v). \end{cases} \tag{1.3}$$

It is well-known that the Cauchy problem behaves similarly as heat equation, admitting smoothing effect due to the following intrinsic diffusion property of the Landau collision operator (c.f. [1, 2])

$$\forall\, \xi \in \mathbb{R}^3, \sum_{1\le i,j\le 3} \bar{a}_{ij}(t,v)\xi_i\xi_j \ge K(1+|v|^2)^{\gamma/2}\,|\xi|^2\,.$$

There were extensive work to study the smoothing effect in different spaces. It was proved in [3] that when $\gamma = 0$, the solutions to (1.3) enjoy ultra-analytic regularity for quite general initial data; that is statement of results in [13].

As far as the hard potential case is concerned, it seems natural to expect analytic regularity (see [5]), since the coefficients a_{ij} is only analytic function. This is different from the case $\gamma = 0$, where the a_{ij} are polynomials and thus ultra-analytic. We mention that in [5] the dependence on the time t for the analytic radius is implicit. In this work we concentrate on the analysis of the explicit behavior as time varies. Our main result can be stated as follows.

Theorem 1.1 Let f_0 be the initial datum with finite mass, energy and entropy and $f(t,v)$ be any solution of the Cauchy problem (1.3). Then for all time $t > 0$, $f(t,v)$, as a real function of v variable, is analytic in $\mathbb{R}^3_v$. Moreover, for all time t in the interval $[0,T]$, where T is an arbitrary nonnegative constant, there exists a constant C, depending only on M_0, E_0, H_0, γ and T such that for all $t \geqslant 0$,

$$t^{|\alpha|}\|\partial^\alpha f(t,v)\|_{L^2(\mathbb{R}^3_v)} \leqslant C^{|\alpha|+1}[(|\alpha|-2)!],$$

where α is an arbitrary multi-indices in $\mathbb{N}^3$.

The Landau equation can be obtained as a limit of the Boltzmann equation when the collisions become grazing, cf. [6] and references therein for more details. There were many results about the regularity of solutions for Boltzmann equation with singular cross sections and Landau equation (see [7–9]) for the Sobolev smoothness results, and [10–13] for the Gevrey smoothness results for Boltzmann equation. And a lot of results were obtained on the Sobolev regularity and Gevrey regularity for the solutions of Landau equations, cf. [2, 4, 10, 14–16] and references therein. In this paper, we mainly concern the analytic smoothness of the solutions for the spatially homogeneous Landau equation. Recently, Morimoto and Xu [3] proved the ultra-analytic effect for the Cauchy problem (1.3) for the Maxwellian molecules case, i.e., $f(t,\cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^d)$. Chen, Li and Xu [5] proved the analytic smoothness effect for the solutions of Cauchy problem (1.3) in the potential case, which need a strict limit on the weak solutions of Cauchy problem (1.3), i.e., $\sup\limits_{t\ge t_0}\|f(t,\cdot)\|_{H^m_\gamma} \leqslant C$ with $t_0 \ge 0$

and C depending only on M_0, E_0, H_0, γ and t_0. Here we refer [5] and prove the analytic smoothness effect for the solutions of Cauchy problem (1.3) in the potential case without strict limit of solutions. And then we give a relatively better estimate form of $f(t,v)$.

Now we give some notations in the paper. For a mult-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, denote

$$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3, \quad \alpha! = \alpha_1!\alpha_2!\alpha_3!, \quad \partial^\alpha = \partial_{v_1}^{\alpha_1}\partial_{v_2}^{\alpha_2}\partial_{v_3}^{\alpha_3}.$$

We say $\beta = (\beta_1, \beta_2, \beta_3) \le (\alpha_1, \alpha_2, \alpha_3) = \alpha$ if $\beta_i \le \alpha_i$ for each i. For a multi-index α and a nonnegative integer k with $k \le |\alpha|$, if no confusion occurs, we shall use $\alpha - k$ to denote some multi-index $\bar{\alpha}$ satisfying $\bar{\alpha} \le \alpha$ and $|\bar{\alpha}| = |\alpha| - k$. As in [2], we denote by $M(f(t)), E(f(t))$ and $H(f(t))$ respectively the mass, energy and entropy of the function $f(t,v)$, i.e.,

$$M(f(t)) = \int_{\mathbb{R}^3} f(t,v)\,dv, \quad E(f(t)) = \frac{1}{2}\int_{\mathbb{R}^3} f(t,v)\,|v|^2\,dv,$$
$$H(f(t)) = \int_{\mathbb{R}^3} f(t,v)\log f(t,v)\,dv,$$

and denote $M_0 = M(f(0))$, $E_0 = E(f(0))$ and $H_0 = H(f(0))$. It's known that the solutions of the Landau equation satisfy the formal conservation laws

$$M(f(t)) = M_0,\ E(f(t)) = E_0,\ H(f(t)) \le H_0,\ \forall\, t \ge 0.$$

Here we adopt the following notations,

$$\|\partial^\alpha f(t,\cdot)\|_{L^p_s} = \left(\int_{\mathbb{R}^3} |\partial^\alpha f(t,v)|^p \left(1+|v|^2\right)^{s/2} dv\right)^{\frac{1}{p}}, \quad p \ge 1,$$
$$\|f(t,\cdot)\|^2_{H^m_s} = \sum_{|\alpha|\le m} \|\partial^\alpha f(t)\|^2_{L^2_s}.$$

In the sequel, for simplicity of representation we always write $\|f(t)\|_{L^p_s}$ instead of $\|f(t,\cdot)\|_{L^p_s}$.

Before stating our main theorem, we introduce the fact that in hard potential case, the existence, uniqueness and Sobolev regularity of the weak solution was studied by Desvillettes-Villani (see Theorems 5–7 of [2]).

We also state the definition of the analytic smoothness of f as follows.

Definition 1.2 f is called analytic function in $\mathbb{R}^n$, if $f \in C^\infty(\mathbb{R}^n)$, and there exists $C > 0$,$N_0 > 0$ such that

$$\|\partial^\alpha f\|_{L^2} \leqslant C^{|\alpha|+1}\alpha!,\ \forall \alpha \in \mathbb{N}^n,\ |\alpha| \geqslant N_0.$$

Starting from the smooth solution, we state our main result on the analytic regularity as follows.

Remark 1.3 If $f(t,v)$ in above still satisfies that for all time $t_0 > 0$, and all integer $m \geqslant 0$, $\sup\limits_{t\geqslant t_0} \|f(t,v)\|_{H^m_\gamma} \leqslant c$ with c a constant depending only on M_0, E_0, H_0, γ, m and t_0. Then for all $t > 0$, $f(t,v)$, as a real function of v variable, is analytic in $\mathbb{R}^3_v$ (see [5]).

Remark 1.4 The result of Theorem 1.1 can be extended to any space dimensional case.

The arrangement of this paper is as follows: Section 2 is devoted to the proof of the main result. In Section 3, we present the proof of Proposition 2.3 in Section 2, which is crucial to the proof of main result here.

2 Proof of Main Result

This section is devoted to the proof of main result. To simplify the notations, in the following we always use $\sum\limits_{1\leqslant|\beta|\leqslant|\mu|}$ to denote the summation over all the multi-indices β with $\beta\leqslant\mu$ and $1\leqslant|\beta|\leqslant|\mu|$. Likewise $\sum\limits_{1\leqslant|\beta|\leqslant|\mu|-1}$ denotes the summation over all the multi-indices β with $\beta\leqslant\mu$ and $1\leqslant|\beta|\leqslant|\mu|-1$. We begin with the following lemma.

Lemma 2.1 For all multi-indices $\mu\in\mathbb{N}^3$, $|\mu|\geqslant 2$, we have

$$\sum_{1\leqslant|\beta|\leqslant|\mu|-1}\frac{|\mu|}{|\beta|^4(|\mu|-|\beta|)}\leqslant 24 \tag{2.1}$$

and

$$\sum_{1\leqslant|\beta|\leqslant|\mu|-1}\frac{|\mu|}{|\beta|^3(|\mu|-|\beta|)^2}\leqslant 24. \tag{2.2}$$

This lemma was proved in [5].

Next, we introduce the following crucial lemma, which is important in the proof of the Proposition 2.3. And the detailed proof of this lemma was given in [5]. Throughout the paper we always assume $(-i)!=1$ for nonnegative integer i.

Lemma 2.2 There exist positive constants B, C_1, and $C_2>0$ with B depending only on the dimension and C_1, C_2 depending only on M_0, E_0, H_0, and γ such that for all multi-indices $\mu\in\mathbb{N}^3$ with $|\mu|\geqslant 2$ and all $t>0$, we have

$$\begin{aligned}&\frac{d}{dt}\|\partial^\mu f(t)\|_{L^2}^2+C_1\|\nabla_v\partial^\mu f(t)\|_{L^2_\gamma}^2\\ \leqslant& C_2|\mu|^2\|\nabla_v\partial^{\mu-1}f(t)\|_{L^2_\gamma}^2\\ &+C_2\sum_{2\leqslant|\beta|\leqslant|\mu|}C_\mu^\beta\|\nabla_v\partial^{\mu-\beta+1}f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\mu-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\beta-2}\\ &+C_2\sum_{0\leqslant|\beta|\leqslant|\mu|}C_\mu^\beta\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\mu-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\mu-\beta},\end{aligned}$$

where $C_\mu^\beta=\frac{\mu!}{(\alpha-\beta)!\beta!}$ is the binomial coefficients, $[G(f(t))]_\omega=\|\partial^\omega f(t)\|_{L^2}+B^{|\omega|}(|\omega|-3)!$ and $\mu-l$ denotes some multi-index $\tilde\mu$ satisfying $\tilde\mu\leqslant\mu$ and $|\tilde\mu|=|\mu|-l$.

Proposition 2.3 Let f_0 be the initial datum with finite mass, energy and entropy and $f(t,v)$ be any solution of the Cauchy problem (1.3). Then for all t in the interval $[0,T]$ with T being an arbitrary nonnegative constant, there exists a constant A, depending only on M_0, E_0, H_0, γ, and T such that the following estimate

$$\sup_{t\in[0,T]}t^{|\alpha|}\|\partial^\alpha f(t,v)\|_{L^2}+\Big(\int_0^T t^{2|\alpha|}\|\nabla_v\partial^\alpha f(t)\|_{L^2_\gamma}^2dt\Big)^{\frac12}\leqslant A^{|\alpha|+1}[(|\alpha|-2)!] \tag{2.3}$$

holds for any multi-indices $\alpha \in \mathbb{N}^3$.

This proposition will be proved in Section 3.

Now we present the proof of the main result.

Proof of Theorem 1.1 Given $T \geqslant 0$, for any multi-indices α, using estimate (2.3) in Proposition 2.3, there exists a constant A depending only on M_0, E_0, H_0, γ, and T, such that

$$\sup_{t\in[0,T]} t^{|\alpha|}\|\partial^\alpha f(t,v)\|_{L^2} \leqslant A^{|\alpha|+1}[|(\alpha|-2)!],$$

since

$$\Big(\int_0^T t^{2|\alpha|}\|\nabla_v \partial^\alpha f(t)\|^2_{L^2_\gamma} dt\Big)^{\frac{1}{2}} \geqslant 0$$

holds for any $T \geqslant 0$ and any multi-indices α. In addition, for each t in the interval $[0,T]$ with $T \geqslant 0$, one has

$$t^{|\alpha|}\|\partial^\alpha f(t,v)\|_{L^2} \leqslant \sup_{t\in[0,T]} t^{|\alpha|}\|\partial^\alpha f(t,v)\|_{L^2} \leqslant A^{|\alpha|+1}[|(\alpha|-2)!].$$

The proof of Theorem 1.1 is completed.

3 Proof of Proposition 2.3

This section is devoted to the proof of Proposition 2.3, which is important for the proof of the main result.

Proof of Proposition 2.3 We use induction on $|\alpha|$ to prove estimate (2.3). First, when we take

$$A = \sup_{t\in[0,T]} \|f(t,v)\|_{L^2} + T \sup_{t\in[0,T]} \|\nabla_v f(t)\|^2_{L^2_\gamma},$$

it is easy to find that estimate (2.3) is valid for $|\alpha| = 0$.

Now we assume estimate (2.3) holds for all $|\alpha|$ with $|\alpha| \leqslant k-1$, where integer $k \geqslant 1$. Next, we need to prove the validity of (2.3) for $|\alpha| = k$, which is equivalent to show the following two estimates: for any $|\alpha| = k$,

$$\sup_{t\in[0,T]} t^{|\alpha|}\|\partial^\alpha f(t,v)\|_{L^2} \leqslant \frac{1}{2}A^{|\alpha|+1}[(|\alpha|-2)!] \tag{3.1}$$

and

$$\Big(\int_0^T t^{2|\alpha|}\|\nabla_v \partial^\alpha f(t)\|^2_{L^2_\gamma} dt\Big)^{\frac{1}{2}} \leqslant \frac{1}{2}A^{|\alpha|+1}[(|\alpha|-2)!]. \tag{3.2}$$

To begin with, we prove estimate (3.1). By means of Lemma 2.2 in Section 2, we obtain that

$$\begin{aligned}&\frac{d}{dt}\|\partial^\alpha f(t)\|_{L^2}^2+C_1\|\nabla_v\partial^\alpha f(t)\|_{L^2_\gamma}^2\\ \leqslant& C_2|\alpha|^2\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}^2\\ &+C_2\sum_{2\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\nabla_v\partial^{\alpha-\beta+1}f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\beta-2}\\ &+C_2\sum_{0\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\alpha-\beta}.\end{aligned}$$

Rewriting the last term of the right-hand side of the inequality as

$$\begin{aligned}&C_2\|f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_\alpha\\ &+C_2\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\alpha-\beta},\end{aligned}$$

we get the following inequality

$$\begin{aligned}&\frac{d}{dt}\|\partial^\alpha f(t)\|_{L^2}^2+C_1\|\nabla_v\partial^\alpha f(t)\|_{L^2_\gamma}^2\\ \leqslant& C_2|\alpha|^2\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}^2\\ &+C_2\|f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_\alpha\\ &+C_2\sum_{2\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\nabla_v\partial^{\alpha-\beta+1}f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\beta-2}\\ &+C_2\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\alpha-\beta}.\end{aligned}$$

We multiply the term $t^{2|\alpha|}$ in the both sides of the equality, leading to new inequality as follows

$$\begin{aligned}&\frac{d}{dt}\Big(t^{2|\alpha|}\|\partial^\alpha f(t)\|_{L^2}^2\Big)+C_1t^{2|\alpha|}\|\nabla_v\partial^\alpha f(t)\|_{L^2_\gamma}^2\\ \leqslant&2|\alpha|t^{2|\alpha|-1}\|\partial^\alpha f(t)\|_{L^2}^2+C_2t^{2|\alpha|}|\alpha|^2\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}^2\\ &+C_2t^{2|\alpha|}\|f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_\alpha\\ &+C_2t^{2|\alpha|}\sum_{2\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\nabla_v\partial^{\alpha-\beta+1}f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\beta-2}\\ &+C_2t^{2|\alpha|}\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\alpha-\beta}.\end{aligned}$$

Integrating the inequality above with respect to t over the interval $[0,T]$ to get that

$$\begin{aligned}&t^{2|\alpha|}\|\partial^\alpha f(t)\|_{L^2}^2+\int_0^T C_1t^{2|\alpha|}\|\nabla_v\partial^\alpha f(t)\|_{L^2_\gamma}^2\,dt\\ \leqslant&\int_0^T 2|\alpha|t^{2|\alpha|-1}\|\partial^\alpha f(t)\|_{L^2}^2\,dt\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T C_2 t^{2|\alpha|}|\alpha|^2\|\nabla_v\partial^{\alpha-1}f(t)\|^2_{L^2_\gamma}\,dt\\
&+\int_0^T C_2 t^{2|\alpha|}\|f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_\alpha\,dt\\
&+\int_0^T C_2 t^{2|\alpha|}\sum_{2\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\nabla_v\partial^{\alpha-\beta+1}f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\beta-2}dt\\
&+\int_0^T C_2 t^{2|\alpha|}\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\alpha-\beta}\,dt.\\
&\overset{\text{def}}{=}(S_1)+(S_2)+(S_3)+(S_4)+(S_5).
\end{aligned}
$$

As a result, one has

$$
t^{2|\alpha|}\|\partial^\alpha f(t)\|^2_{L^2}\leqslant(S_1)+(S_2)+(S_3)+(S_4)+(S_5),
$$

since the fact that

$$
\int_0^T C_1 t^{2|\alpha|}\|\nabla_v\partial^\alpha f(t)\|^2_{L^2_\gamma}\,dt\geqslant 0.
$$

To estimate these terms from (S_1) to (S_5) for all $|\alpha|=k$, we need the following estimates which can be deduced directly from the induction hypothesis. The validity of (2.3) for all $|\alpha|\leqslant k-1$ implies that

$$
\sup_{t\in[0,T]}t^{|\alpha|}\|\partial^\alpha f(t,v)\|_{L^2}\leqslant A^{|\alpha|+1}[(|\alpha|-2)!],\qquad 0\leqslant|\alpha|\leqslant k-1, \tag{3.3}
$$

$$
\Big(\int_0^T t^{2|\alpha|}\|\nabla_v\partial^\alpha f(t)\|^2_{L^2_\gamma}dt\Big)^{\frac12}\leqslant A^{|\alpha|+1}[(|\alpha|-2)!],\qquad 0\leqslant|\alpha|\leqslant k-1, \tag{3.4}
$$

and

$$
\Big(\int_0^T t^{2|\alpha|-2}\|\partial^\alpha f(t)\|^2_{L^2_\gamma}dt\Big)^{\frac12}\leqslant A^{|\alpha|}[(|\alpha|-3)!],\qquad 0\leqslant|\alpha|\leqslant k, \tag{3.5}
$$

where A depends only on M_0, E_0, H_0, γ and T. Inequality (3.5) follows from estimate (3.4), and the fact that $\|\partial^\alpha f(t,v)\|_{L^2_\gamma}\leqslant\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}$ for any multi-indices α with $1\leqslant|\alpha|\leqslant k$. For simplicity of presentation, we set

$$
[G(f(t))]_{T,\omega}=\sup_{t\in[0,T]}t^{|\omega|}\|\partial^\omega f(t,v)\|_{L^2}+T^{|\omega|}B^{|\omega|}[(|\omega|-3)!]. \tag{3.6}
$$

Consequently, if we take A large enough such that $A\geqslant TB$, then utilizing (3.3)–(3.6) with the fact $\|\partial^\omega f(t,v)\|_{L^2}\leqslant\|\partial^\omega f(t)\|_{L^2_\gamma}$, one has

$$
t^{|\omega|}[G(f(t))]_\omega\leqslant[G(f(t))]_{T,\omega}\leqslant 2A^{|\omega|+1}[(|\omega|-2)!],0\leqslant|\omega|\leqslant k-1 \tag{3.7}
$$

and

$$\begin{aligned}\int_0^T t^{2|\omega|}[G(f(t))]_\omega^2\,dt &= \int_0^T t^{2|\omega|}[\|\partial^\omega f(t)\|_{L^2} + B^{|\omega|}((|\omega|-3)!]^2\,dt\\ &\leqslant 2\int_0^T t^{2|\omega|}\|\partial^\omega f(t)\|_{L^2}^2\,dt + 2T^{2|\omega|+1}B^{2|\omega|}[(|\omega|-3)!]^2\\ &\leqslant 2T^2\int_0^T t^{2|\omega|-2}\|\partial^\omega f(t)\|_{L^2}^2\,dt + 2T^{2|\omega|+1}B^{2|\omega|}[(|\omega|-3)!]^2\\ &\leqslant 2T^2A^{2|\omega|}[(|\omega|-3)!]^2 + 2TA^{2|\omega|}[(|\omega|-3)!]^2\\ &\leqslant C_TA^{2|\omega|}[(|\omega|-3)!]^2, \qquad 0\leqslant|\omega|\leqslant k.\end{aligned}$$

That is,

$$\Big(\int_0^T t^{2|\omega|}[G(f(t))]_\omega^2 dt\Big)^{\frac12} \leqslant \tilde C_T A^{|\omega|}[(|\omega|-3)!], 0\leqslant|\omega|\leqslant k, \tag{3.8}$$

where C_T and $\tilde C_T$ depends only on M_0, E_0, H_0, γ and T. Now we are ready to treat terms (S_j) with $|\alpha|=k$ for $1\leqslant j\leqslant 5$. In the following process, we use the notation C_i, $3\leqslant i\leqslant 14$ to denote different constants which are larger than 1 and depending only on M_0, E_0, H_0, γ and T. Using the fact $\|\partial^\omega f(t,v)\|_{L^2}\leqslant\|\nabla_v\partial^\omega f(t)\|_{L^2_\gamma}$ and (3.5), we can show that

$$\begin{aligned}(S_1) &= \int_0^T 2|\alpha|t^{2|\alpha|-1}\|\partial^\alpha f(t)\|_{L^2}^2\,dt \leqslant 2|\alpha|T\int_0^T t^{2|\alpha|-2}\|\partial^\alpha f(t)\|_{L^2}^2\,dt\\ &\leqslant 2|\alpha|T\int_0^T t^{2|\alpha|-2}\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}^2\,dt \leqslant C_3|\alpha|\Big(A^{|\alpha|}[(|\alpha|-3)!]\Big)^2\\ &\leqslant C_4\Big(A^{|\alpha|}[(|\alpha|-2)!]\Big)^2,\end{aligned} \tag{3.9}$$

which is sound for all $|\alpha|=k$.

Next, by virtue of (3.4), one has

$$\begin{aligned}(S_2) &= \int_0^T C_2t^{2|\alpha|}|\alpha|^2\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}^2\,dt \leqslant C_2|\alpha|^2T^2\int_0^T t^{2|\alpha|-2}\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}^2\,dt\\ &\leqslant C_5|\alpha|^2\Big(A^{|\alpha|}[(|\alpha|-3)!]\Big)^2 \leqslant C_6\Big(A^{|\alpha|}[(|\alpha|-2)!]\Big)^2.\end{aligned} \tag{3.10}$$

To estimate term (S_3), by means of (3.4), (3.8) and Cauchy inequality, we obtain

$$\begin{aligned}(S_3) &= \int_0^T C_2t^{2|\alpha|}\|f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_\alpha\,dt\\ &\leqslant C_2T\sup_{t\in[0,T]}\|f(t)\|_{L^2_\gamma}\Big(\int_0^T t^{2|\alpha|-2}\|\nabla_v\partial^{\alpha-1}f(t)\|_{L^2_\gamma}^2dt\Big)^{\frac12}\Big(\int_0^T t^{2|\alpha|}[G(f(t))]_\alpha^2dt\Big)^{\frac12}\\ &\leqslant C_7A^{|\alpha|}[(|\alpha|-3)!]\cdot A^{|\alpha|}[(|\alpha|-2)!]\\ &\leqslant C_7\Big(A^{|\alpha|}[(|\alpha|-2)!]\Big)^2,\end{aligned} \tag{3.11}$$

where $C_7 \geqslant C_2 T \sup_{t\in[0,T]} \|f(t)\|_{L^2_\gamma}$.

Now, we present the detailed estimate of term (S_4), which makes full use of estimates (2.1), (3.4), (3.7),

$$\begin{aligned}
(S_4) =& \int_0^T C_2 t^{2|\alpha|} \sum_{2\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\beta-2} dt \\
\leqslant& C_2 T^2 \int_0^T t^{2|\alpha|-2} \sum_{2\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\beta-2} dt \\
\leqslant& C_8 \sum_{2\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta [G(f(t))]_{T,\beta-2} \int_0^T t^{2|\alpha|-|\beta|} \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} dt \\
\leqslant& C_8 \sum_{2\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta [G(f(t))]_{T,\beta-2} \\
& \Big(\int_0^T t^{2(|\alpha|-|\beta|+1)} \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} dt\Big)^{\frac12} \cdot \Big(\int_0^T t^{2|\alpha|-2} \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} dt\Big)^{\frac12} \\
\leqslant& C_9 \sum_{2\leqslant|\beta|\leqslant|\alpha|} \frac{|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!} A^{|\beta|-1}[(|\beta|-4)!] A^{|\alpha|-|\beta|+2}[(|\alpha|-|\beta|-1)!] A^{|\alpha|}[(|\alpha|-3)!] \\
\leqslant& C_9 A\{A^{|\alpha|}[(|\alpha|-2)!]\}^2 \{\sum_{1\leqslant|\beta|\leqslant|\alpha|-1} \frac{|\alpha|}{|\beta|^4(|\alpha|-|\beta|)} + 1\} \\
\leqslant& C_{10} A\{A^{|\alpha|}[(|\alpha|-2)!]\}^2.
\end{aligned} \tag{3.12}$$

Similarly, owing to estimates (2.2), (3.4), (3.5), (3.7), and Cauchy inequality, we obtain

$$\begin{aligned}
(S_5) =& \int_0^T C_2 t^{2|\alpha|} \sum_{1\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\alpha-\beta}\, dt \\
\leqslant& C_2 T^2 \sum_{1\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta [G(f(t))]_{T,\alpha-\beta} \\
& \Big(\int_0^T t^{2|\beta|-2} \|\partial^\beta f(t)\|_{L^2_\gamma} dt\Big)^{\frac12} \cdot \Big(\int_0^T t^{2|\alpha|-2} \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} dt\Big)^{\frac12} \\
\leqslant& C_{11} \sum_{1\leqslant|\beta|\leqslant|\alpha|} \frac{|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!} A^{|\beta|}[(|\beta|-3)!] A^{|\alpha|}[(|\alpha|-3)!] A^{|\alpha|-|\beta|+1}[(|\alpha|-|\beta|-2)!] \\
\leqslant& C_{12} A\{A^{|\alpha|}[(|\alpha|-2)!]\}^2 \{\sum_{1\leqslant|\beta|\leqslant|\alpha|-1} \frac{|\alpha|}{|\beta|^3(|\alpha|-|\beta|)^2} + 1\} \\
\leqslant& C_{13} A\{A^{|\alpha|}[(|\alpha|-2)!]\}^2.
\end{aligned} \tag{3.13}$$

Consequently, it follows from the combining of estimates (3.9)–(3.13) that

$$t^{2|\alpha|} \|\partial^\alpha f(t)\|^2_{L^2} \leqslant \sum_{j=1}^5 (S_j) \leqslant C_{14} A\{A^{|\alpha|}[(|\alpha|-2)!]\}^2. \tag{3.14}$$

Taking A large enough such that

$$A \geqslant 4\max\{TB, C_{14}, \sup_{t\in[0,T]} \|f(t,v)\|_{L^2} + T \sup_{t\in[0,T]} \|\nabla_v f(t)\|^2_{L^2_\gamma}\},$$

we obtain finally $t^{2|\alpha|}\|\partial^\alpha f(t)\|^2_{L^2} \leqslant \{\frac{1}{2}A^{|\alpha|}[(|\alpha|-2)!]\}^2$, $\forall t \in [0,T]$, where A depends only on M_0, E_0, H_0, γ and T. That is, the proof of estimate (3.1) is completed.

Now, it remains to prove estimate (3.2), for $|\alpha| = k$, which can be handled similarly as the proof of estimate (3.1). Reviewing the process of the proof of estimate (3.1), we can find

$$\begin{aligned}
&t^{2|\alpha|}\|\partial^\alpha f(t)\|^2_{L^2} + \int_0^T C_1 t^{2|\alpha|}\|\nabla_v \partial^\alpha f(t)\|^2_{L^2_\gamma}\, dt \\
\leqslant& \int_0^T 2|\alpha| t^{2|\alpha|-1}\|\partial^\alpha f(t)\|^2_{L^2}\, dt + \int_0^T C_2 t^{2|\alpha|}|\alpha|^2\|\nabla_v \partial^{\alpha-1} f(t)\|^2_{L^2_\gamma}\, dt \\
&+ \int_0^T C_2 t^{2|\alpha|}\|f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_\alpha\, dt \\
&+ \int_0^T C_2 t^{2|\alpha|} \sum_{2\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\beta-2} dt \\
&+ \int_0^T C_2 t^{2|\alpha|} \sum_{1\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\alpha-\beta}\, dt. \\
\overset{\text{def}}{=}& (S_1) + (S_2) + (S_3) + (S_4) + (S_5).
\end{aligned}$$

Using the fact that $t^{2|\alpha|}\|\partial^\alpha f(t)\|^2_{L^2} \geqslant 0$, we have $\displaystyle\int_0^T C_1 t^{2|\alpha|}\|\nabla_v \partial^\alpha f(t)\|^2_{L^2_\gamma}\, dt \leqslant \sum_{j=1}^5 (S_j)$, by virtue of (3.14), which leads to

$$C_1 \int_0^T t^{2|\alpha|}\|\nabla_v \partial^\alpha f(t)\|^2_{L^2_\gamma}\, dt \leqslant C_{14} A\{A^{|\alpha|}[(|\alpha|-2)!]\}^2.$$

Taking A as above, together with the fact that $A \geqslant 4\frac{C_{14}}{C_1}$, we obtain

$$\int_0^T t^{2|\alpha|}\|\nabla_v \partial^\alpha f(t)\|^2_{L^2_\gamma}\, dt \leqslant \frac{1}{4}\{A^{|\alpha|+1}[(|\alpha|-2)!]\}^2,$$

namely, estimate (3.2) is valid. The proof of Proposition 2.3 is completed.

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空间齐次朗道方程解的解析正则性

王彦霖, 徐伟嫚

(武汉大学数学与统计学院, 湖北 武汉 430072)

摘要: 本文研究了Landau方程初值问题在$\gamma \in [0,1]$时解的光滑性. 对初值较弱的假设下, 得到了包含时间的解析估计和解的解析正则性.

关键词: Landau方程; Boltzmann方程; 解析正则性; 平滑性

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