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A LIMINF RESULT FOR LÉVY'S MODULUS OF CONTINUITY OF A FRACTIONAL BROWNIAN MOTION

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Abstract: In this paper, we investigate functional limit for Lévy's modulus of continuity of a fractional Brownian motion. By using large deviation and small deviation for Brownian motion, a liminf for Lévy's modulus of continuity of a fractional Brownian motion is obtained, which extends the corresponding result of Brownian motion.

Keywords: fractional Brownian motion; Lévy's modulus of continuity; liminf result

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1 Introduction and Main Result

Let $\{X(t); t \geq 0\}$ be a standard α-fractional Brownian motion with $0 < \alpha < 1$ and $X(0) = 0$. The $\{X(t); t \geq 0\}$ has a covariance function

$$R(s,t) = E(X(s)X(t)) = \frac{1}{2}(s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha})$$

for $s, t \geq 0$, and representation

$$X(t) = \int_{R^1} \frac{1}{k_\alpha} \left\{ |x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right\} dB(x),$$

where

(i) $k_\alpha^2 = \int_{R^1} \left\{ |x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right\}^2 dx$,

(ii) $\{B(t); -\infty < t < +\infty\}$ is a Brownian motion,

(iii) $\frac{1}{k_\alpha} \left\{ |x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right\}$ is interpreted to be $I_{(0,t]}$ when $\alpha = \frac{1}{2}$.

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$\{X(t); t \geq 0\}$ has stationary increments with $E(X(s+t)-X(s))^2 = t^{2\alpha}$, $s, t \geq 0$ and is a standard Brownian motion when $\alpha = \frac{1}{2}$.

Let $C_0[0,1]$ be the space of continuous functions from $[0,1]$ to R with value zero at the origin, endowed with usual norm $\|f\| = \sup\limits_{0\leq t\leq 1} |f(t)|$, and

$$H = \left\{ f \in C_0[0,1] : f \text{ is an absolutely continuous function}, \|f\|_H^2 = \int_0^1 (\dot{f}(s))^2 ds < \infty \right\}.$$

Then H is a Hilbert space with respect to the scalar product

$$\langle f, g\rangle = \int_0^1 \dot{f}(x)\dot{g}(x)dx \quad \text{for} \quad f, g \in H.$$

Define a mapping $I : C_0[0,1] \to [0,\infty]$ by

$$I(f) = \begin{cases} \frac{1}{2}\displaystyle\int_0^1 |\dot{f}(t)|^2 dt, & f \in H, \\ +\infty, & \text{otherwise}. \end{cases} \tag{1.1}$$

The limit set associated with functional laws of the iterated logarithm for $\{X(t); t \geq 0\}$ is K_α, the subset of functions in $C_0[0,1]$ with the form

$$f(t) = \int_{R^1} \frac{1}{k_\alpha}\{|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\} g(x)dx, \quad 0 \leq t \leq 1,$$

here the function $g(x)$ ranges over the unit ball of $L^2(R^1)$, and hence $\displaystyle\int_{R^1} g^2(s)ds \leq 1$. The subset K of $C_0[0,1]$ is defined by

$$K = \{f \in H : f \in K_\alpha, 2I(f) \leq 1\}.$$

For $0 < h < 1$, $0 \leq s \leq 1, 0 \leq t \leq 1$, let

$$l(h) = (2h^{2\alpha}\log(h^{-1}))^{\frac{1}{2}}, \ \ \Delta(t,h)(s) = X(t+hs) - X(t).$$

In [1], Monrad and Rootzén gave a Chung's functional law of the iterated logarithm for fractional Brownian motion, as follows, for any $f \in K$, $\langle f, f\rangle < 1$,

$$\liminf_{t\to 0} (\log\log t^{-1})^{\alpha+1/2} \left\| \frac{X(t\cdot)}{(2t^{2\alpha}\log\log t^{-1})^{1/2}} - f \right\| = \gamma(f) \quad \text{a.s.},$$

where $\gamma(f)$ is a constant satifying $2^{-1/2}c^\alpha(1-\langle f,f\rangle)^{-\alpha} \leq \gamma(f) \leq 2^{-1/2}C^\alpha(1-\langle f,f\rangle)^{-\alpha}$, c, C denote the positive constants in (2.13) of [1].

Inspired by the arguments of Monrad and Rootzén, in the present paper, we obtain a liminf result for Lévy's modulus of continuity of a fractional Brownian motion. The main result is stated as follows.

Theorem 1.1 For each $f \in K$ with $\langle f, f\rangle < 1$, then

$$\liminf_{h\to 0}(\log h^{-1})^{\alpha+\frac{1}{2}} \inf_{t\in[0,1]} \left\| \frac{\Delta(t,h)}{l(h)} - f \right\| = b(f) \quad \text{a.s.}, \tag{1.2}$$

where $b(f)$ is a constant satisfying

$$2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha} \le b(f) \le 2^{-1/2}C^{\alpha}(1-\langle f,f\rangle)^{-\alpha},$$

here c and C denote the positive constants in (2.13) of [1].

Remark When $\alpha = \frac{1}{2}$, $\{X(t); t \ge 0\}$ is a standard Brownian motion, in this case $c = C = \frac{\pi^2}{8}$, the result in Theorem 1.1 is the exact approximation rate on the modulus of continuity for Brownian motion.

2 Some Lemmas

Our proofs are based on the following lemmas.

In order to prove (3.1) below, we need the following Lemma 2.1.

Lemma 2.1 (see (3.14) of [2]) Let $\{X(t); t \ge 0\}$ be fractional Brownian motion as above, $\sigma^2(u) = E(X(t+u) - X(t))^2$, we have that for any $\varepsilon > 0$, there exists a positive constant $k_0 = k_0(\varepsilon)$ such that

$$P\left(\sup_{0\le t\le T}\sup_{0\le s\le u}|X(t+s)-X(t)| \ge (1+\varepsilon)x\sigma(u)\right) \le \frac{k_0 T}{u}\exp\left(-\frac{x^2}{2}\right)$$

for any $T, 0 < u \le T$ and $x \ge x_0$ with some $x_0 > 0$.

In order to prove (3.2) below, we need the following Lemma 2.2 and Lemma 2.3.

Lemma 2.2 (see Lemma 2.3 in [3]) Let $0 < \alpha < 1, 0 \le q_0 < 1$ and fix $0 < q_0 < q < \alpha$. Let $d_k = k^{k+(1-r)}$, $s_k = k^{-k}$ for $k \ge 1$ and $0 < r < 1$. Let

$$Y_k(s_k, t) = \int_{|x|\notin I_k} \frac{1}{k_\alpha}\left\{|x - s_k t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\right\} dB(x), \quad 0 \le t \le 1, \tag{2.1}$$

where $I_k = (s_k d_{k-1}, s_k d_r]$. Let $0 < \beta < r$. Then, for

$$\delta = \min\{2\beta(\alpha - q), r-\beta, (1-r)(2-2\alpha), (2\alpha-2q)r\},$$

there is a constant $C' > 0$ depending only on α such that uniformly in t, u, k,

$$\sigma_k^2(t,u) = E\{[Y_k(s_k, t+u) - Y_k(s_k, t)]^2\} \le C' u^{2q} s_k^{2\alpha} k^{-\delta}. \tag{2.2}$$

Lemma 2.3 Let $\{\Gamma(t) : t \ge 0\}$ be a centred Gaussian process with stationary increments and $\Gamma(0) = 0$. We assume that $\sigma^2(u) = E(\Gamma(t+u) - \Gamma(t))^2$. Let $T > 0$, we have, for x large enough, any $\varepsilon > 0$,

$$P\left\{\sup_{0\le s\le T}|\Gamma(s)| > x\sigma(T)\right\} \le C_1 \exp\left(-\frac{x^2}{2+\varepsilon}\right),$$

where $C_1 > 0$ is a constant.

Proof This conclusion is from page 49 in [4].

3 The Proof of Theorem 1.1

We only need to show the following two claims:

$$\liminf_{h\to 0}\left(\log h^{-1}\right)^{\alpha+\frac{1}{2}}\inf_{t\in[0,1]}\|\frac{\Delta(t,h)}{l(h)}-f\|\geq 2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\quad \text{a.s.},\tag{3.1}$$

$$\liminf_{h\to 0}\left(\log h^{-1}\right)^{\alpha+\frac{1}{2}}\inf_{t\in[0,1]}\|\frac{\Delta(t,h)}{l(h)}-f\|\leq 2^{-1/2}C^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\quad \text{a.s.}.\tag{3.2}$$

3.1 The Proof of (3.1)

Let $h_n = n^{-d}$, $\rho(h) = h(\log h^{-1})^{-3-\frac{1}{\alpha}}$. We further set $k_n = [\frac{1}{\rho(h_n)}]$, $t_i = i\rho(h_n), i = 0, 1, \cdots, k_n$. Then

$$\begin{aligned}&\min_{0\leq i\leq k_n+1}\|\frac{\Delta(t_i,h_n)}{l(h_n)}-f\|\\ \leq&2\sup_{t\in[0,2]}\sup_{s\in[0,\rho(h_n)]}\frac{|X(t+s)-X(t)|}{l(h_n)}+\inf_{t\in[0,1]}\|\frac{\Delta(t,h_n)}{l(h_n)}-f\|.\end{aligned}\tag{3.3}$$

For any $0 < \varepsilon < 1$, choose $\delta > 0$, such that $\eta = -\delta + \langle f, f\rangle + \frac{1-\langle f,f\rangle}{(1-\varepsilon)^{1/\alpha}} > 1$. Then we have

$$\begin{aligned}&P\left(\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}}\min_{0\leq i\leq k_n+1}\|\frac{\Delta(t_i,h_n)}{l(h_n)}-f\|\leq(1-\varepsilon)2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\right)\\ \leq&\sum_{0\leq i\leq k_n+1}P\left(\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}}\|\frac{\Delta(t_i,h_n)}{l(h_n)}-f\|\leq(1-\varepsilon)2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\right)\\ \leq&(2+k_n)P\left(\left\|\frac{X(h_n\cdot)}{h_n^{\alpha}}-(2\log h_n^{-1})^{\frac{1}{2}}f\right\|\leq\sqrt{2}(\log h_n^{-1})^{-\alpha}(1-\varepsilon)2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\right).\end{aligned}$$

By Proposition 4.2 in [1], we have for any $\delta > 0$ and n large enough

$$\begin{aligned}&\left(\log h_n^{-1}\right)^{-1}\log P\left(\left\|\frac{X(h_n\cdot)}{h_n^{\alpha}}-(2\log h_n^{-1})^{\frac{1}{2}}f\right\|\leq\sqrt{2}(\log h_n^{-1})^{-\alpha}(1-\varepsilon)2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\right)\\ \leq&\left(\log h_n^{-1}\right)^{-1}\log P\left(\left\|\frac{X(h_n\cdot)}{h_n^{\alpha}}\right\|\leq\sqrt{2}(\log h_n^{-1})^{-\alpha}(1-\varepsilon)2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\right)-\langle f,f\rangle+\delta.\end{aligned}$$

By Corollary 2.2 in [1],

$$\begin{aligned}&\left(\log h_n^{-1}\right)^{-1}\log P\left(\left\|\frac{X(h_n\cdot)}{h_n^{\alpha}}\right\|\leq\sqrt{2}(\log h_n^{-1})^{-\alpha}(1-\varepsilon)2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\right)-\langle f,f\rangle+\delta\\ \leq&\left(\log h_n^{-1}\right)^{-1}\left(-2^{-1/(2\alpha)}c(1-\varepsilon)^{-1/\alpha}(2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha})^{-1/\alpha}\log h_n^{-1}\right)-\langle f,f\rangle+\delta=-\eta.\end{aligned}$$

Thus

$$\begin{aligned}&P\left(\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}}\min_{0\leq i\leq k_n+1}\|\frac{\Delta(t_i,h_n)}{l(h_n)}-f\|\leq(1-\varepsilon)2^{-1/2}c^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\right)\\ \leq&\frac{2+\rho(h_n)}{\rho(h_n)}h_n^{\eta}.\end{aligned}$$

Choose $d > (\eta - 1)^{-1}$, then

$$\sum_{n=1}^{\infty} \left(1 + \frac{2}{\rho(h_n)}\right) h_n^{\eta} < \infty,$$

which implies, by the Borel-Cantelli lemma,

$$\liminf_{n\to\infty} \left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \min_{0\le i\le k_n+1} \|\frac{X(t_i + h_n\cdot) - X(t_i)}{l(h_n)} - f\| \ge \frac{2^{-1/2}c^{\alpha}}{(1 - \langle f, f\rangle)^{\alpha}} \text{ a.s..} \tag{3.4}$$

On the other hand, for any $\delta > 0$,

$$\begin{aligned}
&P\Bigg(\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \sup_{0\le t\le 2} \sup_{0\le s\le \rho(h_n)} \frac{|X(t+s) - X(t)|}{l(h_n)} \ge \delta \Bigg) \\
=&P\Bigg(\left(\frac{(\log h_n^{-1})^{\alpha}}{\sqrt{2}h_n^{\alpha}}\right) \sup_{0\le t\le 2} \sup_{0\le s\le \rho(h_n)} |X(t+s) - X(t)| \ge \delta \Bigg) \\
=&P\Bigg(\left(\log h_n^{-1}\right)^{-2\alpha-1} \sup_{0\le t\le \frac{2}{\rho(hn)}} \sup_{0\le s\le 1} |X(t+s) - X(t)| \ge \sqrt{2}\delta \Bigg) \\
\le&\frac{2+\rho(h_n)}{\rho(h_n)} P\Bigg(\left(\log h_n^{-1}\right)^{-2\alpha-1} \sup_{0\le t\le 1} \sup_{0\le s\le 1} |X(t+s) - X(t)| \ge \sqrt{2}\delta \Bigg) \\
=&\frac{2+\rho(h_n)}{\rho(h_n)} P\Bigg(\sup_{0\le t\le 1} \sup_{0\le s\le 1} |X(t+s) - X(t)| \ge 2\delta \left(\log h_n^{-1}\right)^{2\alpha+1} \Bigg).
\end{aligned}$$

By Lemma 2.1, we have that for any $\varepsilon > 0$, there exists a positive $k_0 = k_0(\varepsilon)$ such that

$$\begin{aligned}
&P\Bigg(\sup_{0\le t\le 1} \sup_{0\le s\le 1} |X(t+s) - X(t)| \ge \sqrt{2}\delta \left(\log h_n^{-1}\right)^{4\alpha+2} \Bigg) \\
\le&k_0 \exp\Bigg(-\left(\frac{\delta}{(1+\varepsilon)}\right)^2 \left(\log h_n^{-1}\right)^{4\alpha+2} \Bigg).
\end{aligned}$$

Taking into account $\log h_n^{-1} \to \infty$ as $n \to \infty$, we have

$$k_0 \sum_{n=1}^{\infty} \frac{2+\rho(h_n)}{\rho(h_n)} h_n^{\left(\frac{\delta}{(1+\varepsilon)}\right)^2 \left(\log h_n^{-1}\right)^{4\alpha+1}} < \infty.$$

By the Borel-Cantelli lemma,

$$\limsup_{n\to\infty} \left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \sup_{0\le t\le 2} \sup_{0\le s\le \rho(h_n)} \frac{|X(t+s) - X(t)|}{l(h_n)} = 0 \text{ a.s..} \tag{3.5}$$

By (3.3)–(3.5), we get

$$\lim_{n\to\infty} \left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \inf_{t\in[0,1]} \|\frac{X(t + h_n\cdot) - X(t)}{l(h_n)} - f\| \ge \frac{2^{-1/2}c^{\alpha}}{(1 - \langle f, f\rangle)^{\alpha}} \text{ a.s..} \tag{3.6}$$

Remark that h_n is ultimately strictly decreasing to 0, so for any small h, there is a unique n such that $h \in (h_{n+1}, h_n]$. Let $\phi_{t,h}(s) = \frac{X(t+hs)-X(t)}{l(h)}$, $s \in [0,1], t \in [0,1]$. We define

$$\xi(h) = (\log h^{-1})^{\frac{1}{2}+\alpha} \inf_{t\in[0,1]} \|\phi_{t,h}(\cdot) - f(\cdot)\|, \;\; \xi_n = \inf_{h_{n+1}\le h\le h_n} \xi(h).$$

By the definition of infimum, for any $\varepsilon > 0$, there exists $h'_n \in (h_{n+1}, h_n]$ such that $\xi_n \geq \xi(h'_n) - \varepsilon$.

For any $r \in [0,1]$, let $x = \frac{rh_{n+1}}{h'_n}$. Then we have $0 \leq x \leq 1$,

$$\begin{aligned}
&\inf_{t\in[0,1]} \|\frac{X(t+h_{n+1}\cdot)-X(t)}{l(h_{n+1})} - f\| \\
=&\inf_{t\in[0,1]} \sup_{0\leq r\leq 1} |\phi_{t,h_{n+1}}(r) - f(r)| \\
\leq&\inf_{t\in[0,1]} \sup_{0\leq x\leq 1} \left|\phi_{t,h_{n+1}}(\frac{h'_n}{h_{n+1}}x) - f(\frac{h'_n}{h_{n+1}}x)\right| \\
\leq&(l(h_{n+1}))^{-1}l(h'_n)(\log h_n'^{-1})^{-\alpha-\frac{1}{2}}\xi(h'_n) + \left\|\frac{l(h_n)}{l(h_{n+1})} - 1\right\| \|f(\cdot)\| + \left\|f(\cdot) - f(\frac{h'_n}{h_{n+1}}\cdot)\right\|.
\end{aligned} \tag{3.7}$$

Noting that

$$\|f(\frac{h'_n}{h_{n+1}}\cdot) - f(\cdot)\| \leq \sqrt{1-\frac{h_{n+1}}{h_n}} \leq \sqrt{1-(1-\frac{1}{n+1})^d}, \tag{3.8}$$

$$\left|\frac{l(h'_n)}{l(h_{n+1})} - 1\right| \leq (1+\frac{1}{n})^{d\alpha/2} - 1. \tag{3.9}$$

By (3.6)–(3.9), we have

$$\liminf_{n\to\infty} \xi(h'_n) \geq \frac{2^{-1/2}c^\alpha}{(1-\langle f,f\rangle)^\alpha} \text{ a.s..}$$

Since $\liminf_{h\to 0} \xi(h) \geq \liminf_{n\to\infty} \xi_n \geq \liminf_{n\to\infty} \xi(h'_n) - \varepsilon$, which ends the proof.

3.2 The Proof of (3.2)

Note that

$$\liminf_{h\to 0} \left(\log h^{-1}\right)^{\alpha+\frac{1}{2}} \inf_{t\in[0,1]} \|\frac{\Delta(t,h)}{l(h)} - f\| \leq \liminf_{h\to 0} \left(\log h^{-1}\right)^{\alpha+\frac{1}{2}} \|\frac{X(h\cdot)}{l(h)} - f\| \quad \text{a.s.,}$$

then it is sufficient to show that

$$\liminf_{n\to\infty} \left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \|\frac{X(h_n\cdot)}{l(h_n)} - f\| \leq 2^{-1/2}C^\alpha(1-\langle f,f\rangle)^{-\alpha} \quad \text{a.s.,} \tag{3.10}$$

where $h_n = \frac{1}{n}$.

For $r = 1, 2, 3, \cdots$, we define

$$Z_r(t) = \int_{|x|\in(d_{r-1},d_r]} \frac{1}{k_\alpha} \left\{|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\right\} dB(x), \tag{3.11}$$

$$\tilde{X}_r(t) = X(t) - Z_r(t) \tag{3.12}$$

for $0 \leq t \leq 1, d_r = r^{r+(1-\gamma)}, s_r = r^{-r}, 0 < \gamma < 1$. Then $\{Z_r(\cdot)\}, r = 1, 2, \cdots$ are independent and

$$\{s_r^\alpha \tilde{X}_r(\cdot)\} \overset{\mathcal{D}}{=} \{Y_r(s_r, \cdot)\}, \tag{3.13}$$

where $Y_r(s_r,\cdot)$ is as in Lemma 2.2.

In order to prove (3.10), we need to prove that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left\{\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \|\frac{Z_n(h_n\cdot)}{l(h_n)} - f\| \le (1+\varepsilon)2^{-1/2}C^{\alpha}(1-\langle f,f\rangle)^{-\alpha}\right\} = \infty \tag{3.14}$$

and

$$\sum_{n=1}^{\infty} P\left\{\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \|\frac{\tilde{X}_n(h_n\cdot)}{l(h_n)}\| \ge \varepsilon\right\} < \infty. \tag{3.15}$$

First of all, we prove (3.15).

Now using the argument as in Lemma 2.2, we have

$$\sigma_n^2(t,u) = E(Y_n(s_n,t+u) - Y_n(s_n,t))^2 \le C' u^{2q} s_n^{2\alpha} n^{-\delta},$$

where q,δ are as in Lemma 2.2. For any $\varepsilon > 0$, we have, by Lemma 2.3,

$$\begin{aligned}
&P\left\{\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \|\frac{\tilde{X}_n(h_n\cdot)}{l(h_n)}\| \ge \varepsilon\right\} \\
=&P\left\{\|\tilde{X}_n(\cdot)\| \ge \sqrt{2}(\log h_n^{-1})^{-\alpha}\varepsilon\right\} \\
=&P\left\{\sup_{0\le t\le 1} |\tilde{X}_n(t)| \ge \sqrt{2}(\log h_n^{-1})^{-\alpha}\varepsilon\right\} \\
=&P\left\{\sup_{0\le t\le 1} |Y_n(s_n,t)| \ge \sqrt{2}(C'^{1/2}s_n^{\alpha}n^{-\delta/2})C'^{-1/2}(\log h_n^{-1})^{-\alpha}n^{\delta/2}\varepsilon\right\} \\
\le&C_1 \exp\left(-\frac{C'' n^{\delta}\varepsilon^2}{2+\varepsilon}\left(\log h_n^{-1}\right)^{-2\alpha}\right).
\end{aligned} \tag{3.16}$$

Taking n sufficiently large such that $\frac{C'' n^{\delta/2}\varepsilon^2}{2+\varepsilon} > 1$, we get (3.15) by the definition of the sequence $\{h_n : n \ge 1\}$

Second, we prove (3.14).

For any $\varepsilon > 0$, choose $\delta > 0$, such that $\eta' = \frac{1-\langle f,f\rangle}{(1+\varepsilon)^{1/\alpha}} + \langle f,f\rangle + \delta < 1$. Let $\beta = 2^{-1/2}C^{\alpha}(1-\langle f,f\rangle)^{-\alpha}$, then

$$\begin{aligned}
&P\left\{\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \|\frac{Z_n(h_n\cdot)}{l(h_n)} - f\| \le (1+2\varepsilon)\beta\right\} \\
\ge&P\left\{\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \|(2\log h_n^{-1})^{-1/2}X(\cdot) - f\| \le (1+\varepsilon)\beta\right\} \\
&- P\left\{\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}} \|(2\log h_n^{-1})^{-1/2}\tilde{X}_n(\cdot)\| \ge \varepsilon\beta\right\} \\
=:&I_1 - I_2.
\end{aligned}$$

By Proposition 4.2 in [1], we have for any $\delta > 0$ and n large enough,

$$\begin{aligned}
&\left(\log h_n^{-1}\right)^{-1} \log P\left(\left\|X(\cdot) - (2\log h_n^{-1})^{\frac{1}{2}} f\right\| \le \sqrt{2}(\log h_n^{-1})^{-\alpha}(1+\varepsilon)\beta\right) \\
\ge& \left(\log h_n^{-1}\right)^{-1} \log P\left(\|X(\cdot)\| \le \sqrt{2}(\log h_n^{-1})^{-\alpha}(1+\varepsilon)\beta\right) - \langle f,f\rangle - \delta.
\end{aligned}$$

By Corollary 2.2 in [1],

$$\begin{aligned}&\left(\log h_n^{-1}\right)^{-1}\log P\left(\|X(\cdot)\|\le\sqrt{2}(\log h_n^{-1})^{-\alpha}(1+\varepsilon)\beta\right)-\langle f,f\rangle-\delta\\ \ge&\left(\log h_n^{-1}\right)^{-1}\left(-2^{-\frac{1}{2\alpha}}C(1+\varepsilon)^{-1/\alpha}\beta^{-1/\alpha}\log h_n^{-1}\right)-\langle f,f\rangle-\delta.\end{aligned}$$

Thus

$$I_1:=P\left\{\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}}\|(2\log h_n^{-1})^{-1/2}X(\cdot)-f\|\le(1+\varepsilon)\beta\right\}\ge\left(h_n\right)^{\eta'}.$$

Similar to the proof of (3.15), we have the following estimate for I_2,

$$\begin{aligned}I_2:=&P\left\{\left(\log h_n^{-1}\right)^{\alpha+\frac{1}{2}}\|(2(\log h_n^{-1})^{-1/2}\tilde{X}_n(\cdot)\|\ge\varepsilon\beta\right\}\\ =&P\left\{\|\tilde{X}_n(\cdot)\|\ge\sqrt{2}(\log h_n^{-1})^{-\alpha}\beta\varepsilon\right\}\\ =&P\left\{\sup_{0\le t\le 1}|Y_n(s_n,t)|\ge\sqrt{2}(C'^{1/2}s_n^{\alpha}n^{-\delta/2})C'^{-1/2}(\log h_n^{-1})^{-\alpha}n^{\delta/2}\beta\varepsilon\right\}\\ \le&C_1\exp\left(-\frac{C''n^{\delta}(2^{-1/2}C^{\alpha}(1-\langle f,f\rangle)^{-\alpha})^2\varepsilon^2}{2+\varepsilon}\left(\log h_n^{-1}\right)^{-2\alpha}\right).\end{aligned}$$

Thus $\sum\limits_{n=1}^{\infty}I_2<\infty$. The proof of (3.14) is completed.

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分数Brown运动Lévy连续模的一个Liminf结果

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摘 要: 本文研究了分数Brown运动Lévy连续模的泛函极限. 利用分数Brown运动的大偏差与小偏差, 得到了分数Brown运动Lévy连续模的一个Liminf. 推广了Brown运动的相应结果.

关 键 词: 分数Brown运动; Lévy连续模; liminf结果

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