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CHARACTERIZATIONS OF SOBOLEV CLASSES OF BANACH SPACE-VALUED FUNCTIONS ON METRIC MEASURE SPACE

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Abstract: In the paper, we investigate the Sobolev function classes on Euclidean space when the index is infinity and the ones of Banach space-valued functions on metric measure space when the index is constant. By using the method of Banach space and potential theory, we give various characterizations of Sobolev classes of Banach space-valued functions on metric measure space when the index is infinity. Moreover, we compare the Sobolev classes with the corresponding Lipschitz and Hajlasz-Sobolev classes, which generalizes the related ones for Sobolev classes of Banach space-valued functions on metric measure space as well as Euclidean setting.

Keywords: Sobolev class; Banach space-valued function; Lipschitz function; Poincaré inequality; metric measure space

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1 Introduction

The complete metric measure space $X = (X, d, \mu)$ is equipped with a metric d and a Borel regular outer measure μ such that $0 < \mu(B(x,r)) < \infty$ for all balls $B(x,r) = \{y \in X : d(x,y) < r\}$. The measure μ is doubling, if there exists a doubling constant C_μ such that for all balls $B \subset X$,

$$\mu(2B) \leq C_\mu \mu(B).$$

Let (X, d) be a metric space and V an arbitrary Banach space of positive dimension. We call that a measurable map $F : X \to V$ belongs to the Sobolev class $N^{1,\infty}(X : V)$ with the norm

$$\| F \|_{N^{1,\infty}(X:V)} = \| F \|_{L^\infty(X:V)} + \inf_\rho \| \rho \|_{L^\infty(X)}$$

if $F \in L^\infty(X : V)$ and if there exists a Borel function $\rho : X \to [0, \infty]$ so that $\rho \in L^\infty(X)$ and that

$$\| F(\gamma(a)) - F(\gamma(b)) \| \leq \int_\gamma \rho \mathrm{d}s \tag{1.1}$$

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for ∞-a.e rectifiable curve $\gamma : [a, b] \to X$, where each function ρ is called an ∞-weak V-upper gradient of F and the norm $\| \cdot \|_{N^{1,\infty}(X:V)}$ satisfies the equivalence relation by $F_1 = F_2$ μ-a.e. if and only if $\| F_1 - F_2 \|_{N^{1,\infty}(X:V)} = 0$. If ρ satisfies (1.1) for all rectifiable curves γ, then ρ is called a V-upper gradient of F.

The map $F : X \to V$ is C-Lipschitz if there exists a constant $C > 0$ so that

$$\| F(\gamma(x)) - F(\gamma(y)) \| \leq Cd(x, y) \tag{1.2}$$

holds for $x, y \in X$, here $\| \cdot \|$ is the norm of element in V. Given a map F, the pointwise Lipschitz constant of F at non-isolated point $x \in X$ is defined as follows

$$\mathrm{Lip}F(x) = \limsup_{y \to x, y \neq x} \frac{\| F(\gamma(x)) - F(\gamma(y)) \|}{d(x, y)}. \tag{1.3}$$

If x is an isolated point, we define $\mathrm{Lip}F(x) = 0$. $\mathrm{LIP}^\infty(X : V)$ is the space of bounded Lischitz maps F from X to V with the norm

$$\| F \|_{\mathrm{LIP}^\infty(X:V)} = \| F \|_{L^\infty(X:V)} + \mathrm{LIP}(F),$$

where

$$\mathrm{LIP}(F) = \sup_{x,y \in X; x \neq y} \frac{\| F(\gamma(x)) - F(\gamma(y)) \|}{d(x, y)}. \tag{1.4}$$

Let $\Lambda = \Lambda(X)$ denote the family of all nonconstant rectifiable curves γ and Γ_E the family of all rectifiable curves γ such that $\gamma \cap E \neq \phi$. For $\Gamma \subset \Lambda$, define the ∞-modulus of Γ by

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \{\| \rho \|_{L^\infty(X)}\} \in [0, \infty], \tag{1.5}$$

here $\mathcal{F}(\Gamma)$ is the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho ds \geq 1 \text{ for all } \gamma \in \Gamma.$$

If some property holds for all $\gamma \in \Lambda \setminus \Gamma$ with $\mathrm{Mod}_\infty(\Gamma) = 0$, then we call that the property holds for ∞-a.e. curve.

For each 1-Lipschitz function $\varphi : V \to \mathbb{R}$, the map $\varphi \circ F : X \to \mathbb{R}$ belongs to $N^{1,\infty}(X)$, and there exists $\rho \in L^\infty(X)$ that is an upper gradient of $\varphi \circ F$ for all such φ. The ∞-capacity of a set $E \subset X$ with respect to the space $N^{1,\infty}(X : V)$ is defined by

$$\mathrm{Cap}_\infty(E : V) = \inf_F \| F \|_{N^{1,\infty}(X:V)},$$

where the infimum is taken over all maps F in $N^{1,\infty}(X : V)$ such that $\varphi \circ F \mid_E \geq 1$ for all 1-Lipschitz functions φ. We remark that if $V = \mathbb{R}$ and $\varphi = 1$, the definition is the classical case for metric measure space (due to Cartagena [1]). Clearly, for $E_1 \subset E_2$, it satisfies that $\mathrm{Cap}_\infty(E_1 : V) \leq \mathrm{Cap}_\infty(E_2 : V)$.

In the paper, the aim is to consider the properties related to Sobolev classes of Banach space-valued functions on metric measure space. In the classical Euclidean setting the generalizations of the theory of Sobolev spaces were based on the distributional the derivatives, we may refer to the books [2] and [3]. The definitions of Sobolev classes of real-valued functions on metric measure spaces were given by Cheeger [4], Hajlasz [5, 6], Koskela [7], Romanovskiǐ [8], Shanmugalingam [9], etc. There were many applications in areas of analysis, particularly the first order differential calculus, for example, nonlinear potential theory [10–12], quasiconformal and quasiregular theories [13, 14], Carnot groups [15] and fractal analysis [16]. To overcome the difficulties of derivatives and extra hypothesis on metric measure spaces, Shanmugalingam [9] applied a nonnegative Borel function to take the place of distributional derivatives and defined the Newtonian space $N^{1,p}(X)$ for $1 \leq p < \infty$. In addition, Hajlasz [5] ever considered a integrable function named by Hajlasz gradient to play a role as the same as upper gradient and introduced Hajlasz-Sobolev space $M^{1,p}(X)$ for $1 \leq p < \infty$. Under the suitable conditions, Cartagena [1] pointed out that the all approaches to Sobolev type spaces, even the spaces which support Poincaré inequality, turn to be equivalent. Thus Cartagena et al. [1, 17] studied the Newtonian space $N^{1,\infty}(X)$ and Hajlasz-Sobolev space $M^{1,\infty}(X)$ as well as the cases that Poincaré inequality holds. The works of Ambrosio [18], Korevaar-Schoen [19] and Reshetnyak [20] etc., were about the Sobolev mappings from the domains in Euclidean or Riemannian space into a complete metric space. Since every metric space may be isometrically embedded in the Banach space $\ell^\infty(\cdot)$ of bounded functions, many mathematicians focused on the case when the target is an arbitrary Banach space, refer to Cheeger and Kleiner [21], Heinonen et al. [22], Järvenpää et al. [23], Wildrick and Zürcher [24], and the references therein. Motivated by the studies of Cartagena and Heinonen et al., we prove the characterizations of $N^{1,\infty}(X:V)$ and its comparisons with $\mathrm{LIP}^\infty(X:V)$ and $M^{1,\infty}(X:V)$ (refer to the definition in Section 2).

The remainder of the paper is organized as follows: in Section 2 we will establish our main theorems; in Section 3, some preliminary lemmas will be given; in Section 4, we will prove the main results.

2 Statements of Main Results

Assume that Γ_E^+ is the family of all paths γ in Γ such that Lebeguse measure $\mathcal{L}^1(\gamma^{-1}(\gamma \cap E))$ is positive. Denote by V^* the dual space of V, which is endowed with the norm

$$\| v^* \| = \sup\{ | \langle v^*, v \rangle | : v \in V, \| v \| \leq 1\}.$$

At first, we consider Newtion-Sobolev classes $N^{1,\infty}(X:V)$ of Banach space-valued functions on metric measure space X.

Theorem 2.1 Let $X = (X, d, \mu)$ be a metric measure space of finite total measure and V a Banach space. For each map $F \in L^\infty(X:V)$, there exists four equivalent results as follows:

(I) $F \in N^{1,\infty}(X:V)$;

(II) for each 1-Lipschitz function $\varphi : V \to \mathbb{R}$, the map $\varphi \circ F : X \to \mathbb{R}$ belongs to $N^{1,\infty}(X)$, and there exists $\rho \in L^\infty(X)$ that is an ∞-weak upper gradient of $\varphi \circ F$ for all such φ;

(III) for each $v^* \in V^*$ with $\| v^* \| \leq 1$, the map $\langle v^*, F\rangle : X \to \mathbb{R}$ belongs to $N^{1,\infty}(X : V)$, and there exists $\rho \in L^\infty(X)$ that is an ∞-weak upper gradient of $\langle v^*, F\rangle$ for all such v^*;

(IV) for each $z \in F(X)$, the map $d_zF : X \to \mathbb{R}$ defined by $d_zF(x) = \| F(x) - z \|$ belongs to $N^{1,\infty}(X : V)$, and there exists $\rho \in L^\infty(X)$ that is an ∞-weak upper gradient of d_zF for all such z.

It is well known that the space with the doubling measure can be isometrically embedded into a separable Banach space. If X is equipped with the doubling measure, then we may obtain three results being equivalent to the four ones above.

(V) for each 1-Lipschitz function $\varphi : V \to \mathbb{R}$, the map $\varphi \circ F : X \to \mathbb{R}$ belongs to $N^{1,\infty}(X)$, and there exists $\widetilde{\rho} \in L^\infty(X)$ such that $\rho_{\varphi\circ F} \leq \widetilde{\rho}$ for all such φ;

(VI) for each $v^* \in V^*$ with $\| v^* \| \leq 1$, the map $\langle v^*, F\rangle : X \to \mathbb{R}$ belongs to $N^{1,\infty}(X : V)$, and there exists $\rho \in L^\infty(X)$ such that $\rho_{\varphi\circ F} \leq \widetilde{\rho}$ for all such v^*;

(VII) for each $z \in F(X)$, the map $d_zF : X \to \mathbb{R}$ defined by $d_zF(x) = \| F(x) - z \|$ belongs to $N^{1,\infty}(X : V)$, and there exists $\rho \in L^\infty(X)$ such that $\rho_{\varphi\circ F} \leq \widetilde{\rho}$ for all such z.

Following the ways of Cartagena (see [1]) and Shanmugalingam (see [9]), by some extra techniques we may establish the next theorems on $F \in N^{1,\infty}(X : V)$.

Theorem 2.2 Let $F_i \in N^{1,\infty}(X : V)$ and $\rho_i \in L^\infty(X)$ be an ∞-weak V-upper gradient of F_i for $i \in \mathbb{N}$. Suppose that there exist $F \in L^\infty(X : V)$ and $\rho \in L^\infty(X)$ so that the sequences $\{F_i\}$ and $\{\rho_i\}$ converge to F in $L^\infty(X : V)$ and ρ in $L^\infty(X)$, respectively. Then there exists a map $\widetilde{F} = F$ μ-a.e. such that ρ is an ∞-weak V-upper gradient of $\widetilde{F}$. Moreover, $\widetilde{F} \in N^{1,\infty}(X : V)$.

Theorem 2.3 $N^{1,\infty}(X : V)$ is a Banach space. Moreover, every map $F \in N^{1,\infty}(X : V)$ has a minimal ∞-weak V-upper gradient ρ_F in $L^\infty(X)$.

For $1 \leq p \leq \infty$, the Hajlasz-Sobolev space $M^{1,p}(X : V))$ is the set of all maps $F \in L^p(X : V)$ with the norm

$$\| F \|_{M^{1,P}(X:V)} = \| F \|_{L^P(X:V)} + \inf_g \| g \|_{L^p(X)}$$

for which there exists a nonnegative function $g \in L^p(X)$ such that

$$\| F(\gamma(x)) - F(\gamma(y)) \| \leq d(x, y)(g(x) + g(y)) \ \ \mu\text{-a.e.}, \tag{2.1}$$

here $\| \cdot \|_{M^{1,p}(X:V)}$ also satisfies the equivalence relation by $F_1 = F_2$ μ-a.e. if and only if $\| F_1 - F_2 \|_{M^{1,p}(X:V)} = 0$.

For $F \in L^1(X : V)$ and $E \subset X$ with $\mu(E) > 0$, define the mean value of F over the set E by the vector

$$F_E = \frac{1}{\mu(E)} \int_E F(x) d\mu(x). \tag{2.2}$$

For $\lambda \geq 1$ and an open ball $B(x,r)$ in X, let $F \in L^1(\lambda B(x,r):V)$ and $\rho : \lambda B(x,r) \to [0,\infty]$ be a Borel measurable function. If there exists a constant C_p for $1 \leq p < \infty$ so that

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} \| F - F_{B(x,r)} \| d\mu \leq C_p r \left(\frac{1}{\mu(\lambda B(x,r))}\int_{\lambda B(x,r)} \rho^p d\mu\right)^{\frac{1}{p}} \tag{2.3}$$

holds, then we call that the function pair (F,ρ) satisfies the weak p-Poincaré inequality in $\lambda B(x,r)$. Next it shows the relations among $\mathrm{LIP}^{\infty}(X:V)$, $N^{1,\infty}(X:V)$ and $M^{1,\infty}(X:V)$ in the sense of equivalent norm.

Theorem 2.4 Suppose that X supports a weak p-Poincaré inequality for $1 \leq p < \infty$ and the doubling Borel measure μ. Then $\mathrm{LIP}^{\infty}(X:V) = N^{1,\infty}(X:V) = M^{1,\infty}(X:V)$ with equivalent norms.

3 Some Preliminary Lemmas

Before we continue to deal with our arguments, we will recall and prove the following lemmas.

Lemma 3.1 (see [1], Lemma III.2.9 for $p = \infty$ and Fuglede [25] for $1 \leq p < \infty$) Let $\rho_k : X \to [-\infty, +\infty]$ be a sequence of Borel functions which converge to a Borel representative ρ in $L^p(X)$. Then there exists a subsequence $\{\rho_{k_n}\}$ of Borel functions such that

$$\int_{\gamma} | \rho_{k_n} - \rho | ds \to 0 \text{ as } n \to \infty$$

for p-a.e. curve $\gamma \in \Lambda$, where $1 \leq p \leq \infty$.

Lemma 3.2 (see [1], Lemma III.2.6) If $\mu(E) = 0$ for $E \subset X$, then $\mathrm{Mod}_{\infty}(\Gamma_E^+) = 0$.

Lemma 3.3 If $\{E_k\}$ is a sequence in X, then

$$\mathrm{Cap}_{\infty}\left(\bigcup_{k=1}^{\infty} E_k : V\right) \leq \sum_{k=1}^{\infty} \mathrm{Cap}_{\infty}(E_k : V).$$

Proof Obviously, we only need to consider the case for $\sum\limits_{k=1}^{\infty} \mathrm{Cap}_{\infty}(E_k : V) < \infty$. For $\epsilon > 0$ and positive integer k, we may choose a sequence of maps $U_k \in N^{1,\infty}(X:V)$ with $\varphi \circ U_k |_{E_k} \geq 1$ for all 1-Lipschitz functions φ and ∞-weak V-upper gradient ϱ_k of $\varphi \circ U_k$ such that

$$\| U_k \|_{L^{\infty}(X:V)} + \| \varrho_k \|_{L^{\infty}(X)} \leq \mathrm{Cap}_{\infty}(E_k : V) + \frac{\varepsilon}{2^k}.$$

Set

$$F_n = \sum_{k=1}^{n} \| U_k \|_{L^{\infty}(X:V)} \text{ and } \rho_n = \sum_{k=1}^{n} \varrho_k,$$

where ρ_n is an ∞-weak V-upper gradient of $\varphi \circ F_n$. Then $\sum\limits_{k=1}^{\infty} \| U_k \|_{L^{\infty}(X:V)}$ and $\sum\limits_{k=1}^{\infty} \| \varrho_k \|_{L^{\infty}(X)}$ are finite. It implies that

$$\| F_n - F_m \|_{L^{\infty}(X:V)} \leq \sum_{k=m+1}^{n} \| U_k \|_{L^{\infty}(X:V)} \to 0 \text{ as } m \to \infty.$$

Therefore, $\{F_n\}$ is a Cauchy sequence which converges to $F = \sum\limits_{k=1}^{\infty} \| U_k \|_{L^\infty(X:V)}$ in $L^\infty(X : V)$. Similarly, $\{\rho_n\}$ converges to $\rho = \sum\limits_{k=1}^{\infty} \varrho_k$ in $L^\infty(X)$. According to Theorem 2.2 there exists a map $\widetilde{F} = F$ μ-a.e. such that ρ is an ∞-weak V-upper gradient of $\widetilde{F}$. Moreover, $\widetilde{F} \in N^{1,\infty}(X : V)$. Since $\varphi \circ F \geq 1$ in $\bigcup\limits_{k=1}^{\infty} E_k$, we obtain that

$$\begin{aligned} &\mathrm{Cap}_\infty\left(\bigcup_{k=1}^{\infty} E_k : V\right) \leq \| \widetilde{F} \|_{N^{1,\infty}(X:V)} \\ \leq &\sum_{k=1}^{\infty} \big(\| U_k \|_{L^\infty(X:V)} + \| \varrho_k \|_{L^\infty(X)}\big) \leq \sum_{k=1}^{\infty} \mathrm{Cap}_\infty(E_k : V) + \varepsilon, \end{aligned}$$

which is the desired result and so Lemma 3.3 holds.

Lemma 3.4 If $\mathrm{Cap}_\infty(E : V) = 0$ for $E \subset X$, then $\mathrm{Mod}_\infty(\Gamma_E) = 0$.

Proof For $\epsilon > 0$ and positive integer k, we may choose a sequence of maps $U_k \in N^{1,\infty}(X : V)$ with $\varphi \circ U_k \mid_E \geq 1$ for all 1-Lipschitz functions φ and ∞-weak V-upper gradient ϱ_k of $\varphi \circ U_k$ such that

$$\| U_k \|_{L^\infty(X:V)} + \| \varrho_k \|_{L^\infty(X)} \leq \frac{\varepsilon}{2^k}.$$

Put

$$F_n = \sum_{k=1}^{n} \| U_k \|_{L^\infty(X:V)} \quad \text{and} \quad \rho_n = \sum_{k=1}^{n} \varrho_k,$$

where ρ_n is an ∞-weak V-upper gradient of $\varphi \circ F_n$. Following the procedure of proof in Lemma 3.3 there exists a map $\widetilde{F} = F$ μ-a.e. such that ρ is an ∞-weak V-upper gradient of $\widetilde{F}$. Moreover, $\widetilde{F} \in N^{1,\infty}(X : V)$. By Theorem 2.2, we infer that

$$F(x) = \lim_{k\to\infty} F_k(x)$$

outside a set G satisfying $\mathrm{Mod}_\infty(\Gamma_G) = 0$. Since $E \subset G$, that is to say $\Gamma_E \subset \Gamma_G$, we have $\mathrm{Mod}_\infty(\Gamma_E) \leq \mathrm{Mod}_\infty(\Gamma_G)$ and it follows $\mathrm{Mod}_\infty(\Gamma_E) = 0$.

Lemma 3.5 (see Cartagena [1], Lemma III.2.5) For $\Gamma \in \Lambda$, the following results are equivalent:

(I) $\mathrm{Mod}_\infty(\Gamma_E) = 0$;

(II) there exists a nonnegative Borel function $\rho \in L^\infty(X)$ such that $\int_\gamma \rho ds = +\infty$ for each $\gamma \in \Gamma$;

(III) there exists a nonnegative Borel function $\rho \in L^\infty(X)$ such that $\int_\gamma \rho ds = +\infty$ for each $\gamma \in \Gamma$ and $\| \rho \|_{L^\infty(X)} = 0$.

Lemma 3.6 (see Cartagena [1], Theorem III.3.3) Suppose that X supports a weak p-Poincaré inequality for $1 \leq p < \infty$ and the doubling Borel measure μ which is nontrivial and finite on balls. For nonnegative $\rho \in L^\infty(X)$, there exists a set $E \subset X$ of measure zero and positive constant K depending only on X such that for all $x, y \in X \setminus E$ there exists a rectifiable curve γ connecting x and y so that $\int_\gamma \rho ds < +\infty$ and $\hbar(\gamma) \leq K d(x, y)$.

4 Proofs of Main Theorems

Proof of Theorem 2.1 Assume that result (I) is true. Let $\rho \in L^\infty(X)$ be an ∞-weak V-upper gradient of F and $\varphi : V \to \mathbb{R}$ be 1-Lipschitz function. If γ is a rectifiable curve in X with the ends x and y, then

$$| \varphi \circ F(x) - \varphi \circ F(y) | \leq \| F(x) - F(y) \| \leq \int_\gamma \rho ds.$$

Since X has finite mass, we obtain that

$$\| \varphi \circ F \|_{L^\infty(X)} | \leq \| F \|_{L^\infty(X:V)} + | \varphi(0) | \leq \| F \|_{N^{1,\infty}(X:V)} + | \varphi(0) | < \infty.$$

Therefore, $\varphi \circ F$ is in $N^{1,\infty}(X : V)$ and ρ is an ∞-weak upper gradient of $\varphi \circ F$ which is independent of φ. Hence, it follows result (II).

Since both the mappings $\langle v^*, \cdot \rangle : V \to \mathbb{R}$ with $\| v^* \| \leq 1$ for $v^* \in V^*$ and the mappings $d_z : V \to \mathbb{R}$ with $d_z(v) = \| v - z \|$ for $z \in V$ are 1-Lipschitz, by (II), results (III) and (IV) clearly hold.

Suppose that the map $F \in L^\infty(X : V)$ and $\rho \in L^\infty(X)$ satisfy result (III). Let γ be a rectifiable curve in X with the ends x and y. If $F(x) = F(y)$, then the result is trivial. Otherwise, we choose $v^* \in V^*$ satisfying

$$\langle v^*, F(x) - F(y) \rangle = \| F(x) - F(y) \|$$

with $\| v^* \| \leq 1$. Since ρ is an ∞-weak upper gradient of $\langle v^*, F \rangle$ for all such v^*, we know that

$$\| F(x) - F(y) \| = \langle v^*, F(x) - F(y) \rangle \leq \int_\gamma \rho ds.$$

Hence, ρ is an ∞-weak V-upper gradient of F and so $F \in N^{1,\infty}(X : V)$. Similarly, from result (IV) it infers (I). When X is equipped with the doubling measure, we may assume that $F(X)$ is separable. For the equalities of (I), (V), (VI) and (VII), we only need repeat the procedures of Heinonen et al. [22].

Proof of Theorem 2.2 Set

$$\widetilde{F}(x) = \frac{1}{2} \left(\limsup_{k\to\infty} F_k(x) + \liminf_{k\to\infty} F_k(x) \right).$$

Since $F_i \to F$ in $L^\infty(X : V)$, clearly it converges μ-a.e. Hence $\widetilde{F} = F$ μ-a.e. and $\widetilde{F} \in L^\infty(X : V)$. Therefore, the map $\widetilde{F}$ is well defined outside the zero-measurable set

$$E = \{x : \lim_{k\to\infty} | \varphi \circ F_k(x) | = \infty\},$$

where $\varphi : V \to \mathbb{R}$ is 1-Lipschtiz function. Let Γ be the collection of paths $\gamma \in \Lambda$ such that either $\int_\gamma \rho ds = \infty$ or

$$\lim_{k\to\infty} \int_\gamma \rho_k ds \neq \int_\gamma \rho ds.$$

From Lemma 3.1, it easily infers that $\mathrm{Mod}_\infty(\Gamma) = 0$. Since $\mu(E) = 0$, by Lemma 3.2 we know that $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. For any nonconstant path $\gamma \in \Lambda \setminus (\Gamma \cup \Gamma_E^+)$, we may fix a point $y \in\mid \gamma \mid \setminus E$, here $\mid \gamma \mid$ is the image of γ. Since ρ_k is an ∞-weak V-upper gradient of F_k, we know that for all points $x \in\mid \gamma \mid$,

$$\mid \varphi \circ F_k(x) \mid - \mid \varphi \circ F_k(y) \mid \leq \mid \varphi \circ F_k(x) - \varphi \circ F_k(y) \mid \leq \parallel F_k(x) - F_k(y) \parallel \leq \int_\gamma \rho_k ds.$$

Hence,

$$\mid \varphi \circ F_k(x) \mid \leq \mid \varphi \circ F_k(y) \mid + \int_\gamma \rho_k ds.$$

Because $\gamma \in \Lambda \setminus (\Gamma \cup \Gamma_E^+)$, we obtain that

$$\lim_{k\to\infty} \mid \varphi \circ F_k(x) \mid \leq \lim_{k\to\infty} \mid \varphi \circ F_k(y) \mid + \int_\gamma \varrho ds < \infty,$$

and so $x \in X \setminus E$. That is to say, $\gamma \in \Gamma_E$ fails and so $\Gamma_E \subset \Gamma \cup \Gamma_E^+$. Further $\mathrm{Mod}_\infty(\Gamma_E) = 0$.

For $\gamma \in \Lambda \setminus \Gamma$, let x and y be the end of points of its images. Clearly, x, y don't belong to E, and so we know that

$$\begin{aligned}
\parallel \widetilde{F}(x) - \widetilde{F}(y) \parallel &= < v^*, \widetilde{F}(x) - \widetilde{F}(y) > \\
&= \frac{1}{2} < v^*, \limsup_{k\to\infty}(F_k(x) - F_k(y)) > + \frac{1}{2} < v^*, \liminf_{k\to\infty}(F_k(x) - F_k(y)) > \\
&\leq \frac{1}{2} \limsup_{k\to\infty} \parallel F_k(x) - F_k(y) \parallel + \frac{1}{2} \liminf_{k\to\infty} \parallel F_k(x) - F_k(y) \parallel \\
&\leq \frac{1}{2} \limsup_{k\to\infty} \int_\gamma \rho_k ds + \frac{1}{2} \liminf_{k\to\infty} \int_\gamma \rho_k ds = \int_\gamma \rho ds,
\end{aligned}$$

where the map $\langle v^*, \cdot \rangle : V \to \mathbb{R}$ for v^* with $\parallel v^* \parallel \leq 1$ is 1-lipschitz. Therefore, ρ is an ∞-weak V-upper gradient of $\widetilde{F}$. Further, $\widetilde{F} \in N^{1,\infty}(X : V)$.

Proof of Theorem 2.3 For a Cauchy sequence $\{F_k\}$ in $N^{1,\infty}(X : V)$, it can be assumed that

$$\parallel F_{k+1} - F_k \parallel_{N^{1,\infty}(X:V)} < 4^{-k}$$

and $\parallel \rho_{k+1,k} \parallel_{L^\infty(X:V)} < 2^{-k}$, where $\rho_{k,\ell}$ is an ∞-weak V-upper gradient of $F_k - F_\ell$.

Let

$$E_k = \{x \in X : \parallel F_{k+1}(x) - F_k(x) \parallel \geq 2^{-k}\}$$

and

$$G_\ell = \bigcup_{k=\ell}^\infty E_k \text{ and } G = \bigcap_{\ell=1}^\infty G_\ell.$$

If $x \in G$ fails, then exists ℓ satisfying

$$\| F_{k+1}(x) - F_k(x) \| < 2^{-k} \text{ for all } k \geq \ell,$$

and thus the Cauchy sequence $\{F_k(x)\}$ converges in X. Here we put $F(x) = \lim\limits_{k\to\infty} F_k(x)$. Now we claim that the set G is of ∞-capacity zero. Since $2^k \| F_{k+1}(x) - F_k(x) \| \geq 1$ holds on E_k, we know that

$$\mathrm{Cap}_\infty(E_k : V) \leq 2^k \| F_{k+1} - F_k \|_{N^{1,\infty}(X:V)} \leq 2^{-k}.$$

From the countably subadditivity of $\mathrm{Cap}_\infty(\cdot : V)$ in Lemma 3.3, we obtain that

$$\mathrm{Cap}_\infty(G_\ell : V) \leq \sum_{k=\ell}^{\infty} \mathrm{Cap}_\infty(E_k : V) \leq \sum_{k=\ell}^{\infty} 2^k = 2^{1-\ell}.$$

Hence $\mathrm{Cap}_\infty(G : V) = 0$.

When $x \in X \setminus G$, the sequence $\{F_k(x)\}$ is convergent. Therefore, we know that

$$F(x) = \lim_{\ell\to\infty} F_\ell(x) = F_k(x) + \sum_{\ell=k}^{\infty}(F_{\ell+1}(x) - F_\ell(x)).$$

By Lemma 3.4, we get that $\mathrm{Mod}_\infty(\Gamma_G) = 0$. For $\gamma \in \Lambda \setminus \Gamma_G$ being connected x and y, we have that

$$\begin{aligned}\| (F - F_k)(x) - (F - F_k)(y) \| &\leq \sum_{\ell=k}^{\infty} \| (F_{\ell+1} - F_\ell)(x) - (F_{\ell+1} - F_\ell)(y) \| \\ &\leq \sum_{\ell=k}^{\infty} \int_\gamma g_{\ell+1,\ell} ds = \int_\gamma \sum_{\ell=k}^{\infty} g_{\ell+1,\ell} ds.\end{aligned}$$

Therefore, $\sum\limits_{\ell=k}^{\infty} g_{\ell+1,\ell}$ is an ∞-weak V-upper gradient of $F - F_k$.

$$\begin{aligned}\| F - F_k \|_{N^{1,\infty}(X:V)} &\leq \| F - F_k \|_{L^\infty(X:V)} + \sum_{\ell=k}^{\infty} \| g_{\ell+1,\ell} \|_{L^\infty(X)} \\ &\leq \| F - F_k \|_{L^\infty(X:V)} + \sum_{\ell=k}^{\infty} 2^\ell \\ &\leq \| F - F_k \|_{L^\infty(X:V)} + 2^{1-k} \text{ as } k \to \infty.\end{aligned}$$

Hence, $\{F_k\}$ converges into F in $N^{1,\infty}(X : V)$. That is to say, $N^{1,\infty}(X : V)$ is a Banach space.

On the other hand, from Theorem 2.2, we may choose a sequence $\{\rho_k\}$ satisfying $\rho_k \to \rho = \inf\{\varrho_k\}$ as $k \to \infty$ in $L^\infty(X)$ so that there exists a map $\widetilde{F} = F$ μ-a.e. such that ρ is an ∞-weak V-upper gradient of $\widetilde{F}$, that is to say, ρ is an ∞-weak V-upper gradient of F. Thus Theorem 2.3 follows.

Proof of Theorem 2.4 According to the definitions of $\mathrm{LIP}^\infty(X:V)$ and $M^{1,\infty}(X:V)$, we easily see that $\frac{1}{2}\parallel\cdot\parallel_{\mathrm{LIP}^\infty(X:V)}\leq\parallel\cdot\parallel_{M^{1,\infty}(X:V)}\leq\parallel\cdot\parallel_{\mathrm{LIP}^{1,\infty}(X:V)}$, and so $\mathrm{LIP}^\infty(X:V)=M^{1,\infty}(X:V)$. Assume that $F\in N^{1,\infty}(X:V)$. Then there exists an ∞-weak V-upper gradient $\rho\in L^\infty(X)$ of F. Now we denote by Γ_N the family of curves for which ρ is not a V-upper gradient of F, and so $\mathrm{Mod}_\infty(\Gamma_N)=0$. From Lemma 3.5, we know that there exists a nonnegative Borel function $\widetilde{\rho}\in L^\infty(X)$ such that $\int_\gamma\widetilde{\rho}ds=+\infty$ for each $\gamma\in\Gamma_N$ and $\parallel\widetilde{\rho}\parallel_{L^\infty(X)}=0$. Hence, $\rho+\widetilde{\rho}\in L^\infty(X)$ which is a V-upper gradient of F and satisfies $\parallel\rho+\widetilde{\rho}\parallel_{L^\infty(X)}=\parallel\rho\parallel_{L^\infty(X)}$. By Lemma 3.5 we remark that $\Gamma=\{\gamma\in\Lambda:\int_\gamma\rho ds=+\infty\}$ has ∞-modulus zero. Therefore, if $\int_\gamma\rho ds<+\infty$, then $\int_\gamma\rho ds\leq\parallel\rho\parallel_{L^\infty(X)}\hbar(\gamma)$. From Lemma 3.6, there exists a set $E\subset X$ of measure zero and positive constant K depending only on X such that for all $x,y\in X\setminus E$ there exists a rectifiable curve γ connecting x and y so that $\int_\gamma\rho ds<+\infty$ and $\hbar(\gamma)\leq Kd(x,y)$. Hence,

$$\parallel F(x)-F(y)\parallel\leq\int_\gamma\rho ds\leq\parallel\rho\parallel_{L^\infty(X)}\hbar(\gamma)\leq K\parallel\rho\parallel_{L^\infty(X)}d(x,y).$$

Then F is $K\parallel\rho\parallel_{L^\infty(X)}$-Lischitz μ-a.e. That is to say, there exits a $\widetilde{F}\in\mathrm{LIP}^\infty(X:V)$ so that $F=\widetilde{F}$ holds μ-a.e. Therefore, $\mathrm{LIP}^\infty(X:V)=N^{1,\infty}(X:V)$.

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度量测度空间中Sobolev类Banach 空间值函数的刻画

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摘要: 本文研究了在指标是无穷大时欧式空间情形下Sobolev函数类理论和指标是有限常数时度量空间下Sobolev类Banach空间值函数理论. 利用Banach空间理论和位势理论的方法, 得到了在指标是无穷大时度量测度空间中Sobolev类Banach空间值函数的各种刻画, 进而比较了该Sobolev类与对应的Lipschitz 类和Hajlasz-Sobolev 类. 所获结果推广了欧式空间和度量测度空间中Sobolev函数类相应的结论.

关键词: Sobolev类; Banach 空间值函数; Lipschitz 函数; Poincaré 不等式; 度量测度空间

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