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OPTIMAL DIVIDENDS WITH EXPONENTIAL AND LINEAR PENALTY PAYMENTS IN A DUAL MODEL

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Abstract: In this paper, we consider the optimal dividend problem with penalty payments in a dual model. We assume that the company doesn't go bankrupt when the surplus becomes negative, but penalty payments occur, and the penalty amounts are dependent on the level of the surplus. By using the stochastic optimal control approach and dynamic programming principles, we obtain the HJB equation and verification theorem for the optimal problem. Finally, when the profits follow an exponential distribution, we obtain the optimal dividend strategies and explicit solutions for exponential and linear penalty payments, respectively.

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1 Introduction

Recently, many papers were published on the dual risk model. The model is suitable for describing companies whose capital reserves involve a constant flow of expenses and occasional profits, such as petroleum, pharmaceutical or commission-based businesses. In actuarial mathematics, the risk of a company is traditionally measured by the probability of ruin, where the time of ruin is defined as the first time when the surplus becomes negative. Classical ruin probability results for the dual model can be found in Grandell [1], Dong and Wang [2], and Zhu and Yang [3].

Another measure considers the expected discounted dividend payments which are paid to the shareholders until ruin. The dividend problem in the dual model was first introduced by Avanzi et al. [4], and the optional dividend strategy in the dual model was a constant barrier strategy. From then, many researchers studied the dual model under different dividend strategies. See Avanzi and Gerber [5], Gerber and Smith [6], Ng [7] and so on. However, the disadvantage of the dividend approach is that, under the optimal strategy, ruin occurs almost surely. Therefore, the idea of capital injections rises. Whenever the surplus becomes

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negative, the shareholders have to inject capital in order to avoid ruin. See, for example, Yao et al. [8, 9], Avanzi et al. [10] and so on.

All of the approaches above have one thing in common: if the surplus becomes negative, the company either has to inject capital or ruin occurs. However, in practice, it can be observed that some companies continue doing business although they had large losses for a long period. The regulator often intervenes in order to avoid that a company goes out of business. Therefore, it is more realistic to allow negative surplus. In the context of negative surplus, Vierkotter and Schmidli [11] considered optimal dividend problem with penalty payments in a diffusion model. Vierkotter and Schmidli assumed that insurer is not ruined when the surplus becomes negative, but penalty payments occur, depending on the level of the surplus.

Motivated by Vierkotter and Schmidli [11], we consider the optimal dividend problem with penalty payments for a dual model. Similarly, we assume that bankruptcy does not occur, but whenever the surplus is negative, penalty payments occur. These payments reflect all costs which are necessary to prevent bankruptcy. For example, penalty payments can occur if the company needs to borrow money, generate additional equity or additional administrative measures have to be taken. These costs may also be extended to positive surplus to penalise small surplus. The penalty payments are rather technical in order to avoid that the surplus becomes small or even negative. Different from Vierkotter and Schmidli [11], we assume that the dividend payments have transaction costs.

The rest of this paper is organized as follows. In Section 2, the model is described and basic concepts are introduced. In Section 3, the HJB equation and verification theorem for the optimization problem are provided. In Section 4, we explicitly derive the optimal value functions and corresponding optimal strategies for exponential and linear penalty payments when the profits follow an exponential distribution.

2 The Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space on which all stochastic processes and random variables introduced in the following are defined. The company's uncontrolled surplus process $R = \{R_t\}_{t\geq 0}$ is the dual model, which is described as

$$R_t = x - ct + S_t, \tag{2.1}$$

where $x \geq 0$ is the initial surplus level; $c > 0$ can be viewed as the rate of expenses; $S_t = \sum\limits_{i=1}^{N(t)} Y_i$ is a compound Poisson process representing the total income amount up to time t, in which $N(t)$ is a Poisson process with a gain arrival intensity λ, and the sequence of gain amounts $\{Y_i\}_{i\geq 1}$ are independent and identically distributed (i.i.d.) positive random variables with mean m_1 and a continuously differentiable distribution function $F(y)$. In addition, since $\mathbb{E}(R_t - x) = t(\lambda m_1 - c)$, we assume that the net profit condition, $\lambda m_1 - c > 0$, is valid.

The manager of the company can control over the dividend payments and the controlled surplus of the company evolves according to

$$R_t^D = x - ct + S_t - D_t, \tag{2.2}$$

where D_t is the cumulative amount of dividend paid out up to time t. The strategy $D = \{D_t\}_{t\geq 0}$ is said to be admissible if D is predictable and non-decreasing cádlág processes with $D_0 = 0$. The set of all admissible strategies is denoted by $\mathcal{D}$. We assume that dividends are paid according to a barrier strategy. Such a strategy has a parameter $b > 0$, the level of the barrier. Whenever the surplus exceeds the barrier, the excess is paid out immediately as a dividend. This means that

$$D_t = (x-b)^+ + \int_0^t I_{\{R_s \geq b\}} dD_s,$$

where I_A represents the indiction function of event A.

The value function of a strategy D is defined by

$$V^D(x) = \mathbb{E}\left[\int_0^\infty \eta e^{-\delta t} dD_t - \int_0^\infty e^{-\delta t}\phi(R_t^D)dt | R_0^D = x\right], \tag{2.3}$$

where $\delta > 0$ denotes the discount rate, $0 < \eta \leq 1$ represents the net proportional of leakages from the surplus received by shareholders after transaction costs are paid. The penalty function ϕ is continuous, decreasing, and convex a function, satisfying $\phi(x) \to 0$ as $x \to \infty$. The manager's objective is to find the optimal strategy $D^* \in \mathcal{D}$ such that

$$V(x) = \sup_{D\in\mathcal{D}} V^D(x) = V^{D^*}(x). \tag{2.4}$$

We have to assume

$$\int_0^\infty e^{-\delta t}\mathbb{E}[\phi(R_t)]dt < \infty.$$

Otherwise, the value function would be minus infinity. Moreover, we assume that

$$\phi(x) - \phi(y) > \delta\eta(y-x) \tag{2.5}$$

for $x < y < x_0$ and some $x_0 \in \mathbb{R}$ in order that it is not optimal to pay an infinite amount of dividends. Since ϕ is assumed to be convex, this means that there is an $x \in \mathbb{R}$ such that $\phi'(x) < -\delta\eta$.

3 The HJB Equation and the Verification Theorem

First, we verify some basic properties of the value function that will help us to prove the following HJB equation.

Lemma 3.1 The function $V(x)$ is concave.

Proof Similarly to Vierkotter and Schmidli [11], let $x, y \in \mathbb{R}$ and $z = kx + (1-k)y$, where $k \in (0,1)$. Applying the strategies D^x and D^y for initial capital x and y, respectively, we define $D_t = kD_t^x + (1-k)D_t^y$ for the initial capital z. Since $-\phi$ is concave and

$$\begin{aligned} R_t^D &= kx + (1-k)y + (k+1-k)(-ct + S_t) - kD_t^x - (1-k)D_t^y \\ &= kR_t^{D^x} + (1-k)R_t^{D^y}, \end{aligned}$$

we obtain

$$\begin{aligned} & V(kx + (1-k)y) \\ &= V(z) \ge V^D(z) = \mathbb{E}\left[\int_0^\infty \eta e^{-\delta t}(kdD_t^x + (1-k)dD_t^y) - \int_0^\infty e^{-\delta t}\phi(R_t^D)dt\right] \\ &\ge k\mathbb{E}\left[\int_0^\infty \eta e^{-\delta t}dD_t^x - \int_0^\infty e^{-\delta t}\phi(R_t^{D^x})dt\right] + (1-k)\mathbb{E}\left[\int_0^\infty \eta e^{-\delta t}dD_t^y - \int_0^\infty e^{-\delta t}\phi(R_t^{D^y})dt\right] \\ &= kV^{D^x}(x) + (1-k)V^{D^y}(y). \end{aligned}$$

Taking the supremum over all strategies D^x and D^y, we get

$$V(kx + (1-k)y) \ge kV(x) + (1-k)V(y).$$

This completes the proof.

Remark 3.1 The concavity implies that V is differentiable from the left and from the right and $V'(x-) \ge V'(x+) \ge V'(y-) \ge V'(y+)$ for $x < y$. In particular, V is differentiable almost everywhere. Moreover, the concavity implies that V is continuous.

Lemma 3.2 $V(x)$ is increasing with $V(y) - V(x) \ge \eta(y-x)$ for $x \le y$ and

$$-\int_0^\infty e^{-\delta t}\mathbb{E}[\phi(R_t)]dt \le V(x) \le \eta x + \frac{\eta\lambda m_1}{\delta}. \tag{3.1}$$

Proof Consider a strategy D with $V^D(x) \ge V(x) - \varepsilon$ for an $\varepsilon > 0$. For $y \ge x$, we define a new strategy as follows: $y - x$ is paid immediately as dividend and then the strategy D with initial capital x is followed. Then for any $\varepsilon > 0$, it holds that

$$V(y) \ge \eta(y-x) + V^D(x) \ge \eta(y-x) + V(x) - \varepsilon.$$

Since ε is arbitrary, we get $V(y) \ge V(x) + \eta(y-x)$. Hence V is increasing.

Let V^0 be the value of the strategy where no dividends are paid. Then, Fubini's theorem implies

$$V(x) \ge V^0(x) = -\int_0^\infty e^{-\delta t}\mathbb{E}[\phi(R_t)]dt.$$

On the other hand, consider another extreme case: there is no operating cost (i.e., $c = 0$), all initial surplus x and profits are paid out immediately as dividends. The company can run smoothly due to no operating costs, there is no penalty payments occur. We find, using the fact that the n-th jump time T_n is Gamma $\Gamma(\lambda, n)$ distributed, the upper bound

$$V(x) \leq \eta x + \eta \mathbb{E}[\sum_{n=1}^{\infty} Y_n e^{-\delta T_n}] = \eta x + \eta m_1 \sum_{n=1}^{\infty} (\frac{\lambda}{\lambda+\delta})^n = \eta x + \frac{\eta \lambda m_1}{\delta}.$$

The proof is completed.

Theorem 3.1 The function $V(x)$ fulfils the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\{(\mathcal{A}-\delta)V(x) - \phi(x),\ \eta - V'(x)\} = 0, \tag{3.2}$$

where operator $\mathcal{A}$ is the infinitesimal generator given by

$$\mathcal{A}V(x) = -cV'(x) - \lambda V(x) + \lambda \int_0^{\infty} V(x+y)dF(y). \tag{3.3}$$

Proof Let $h \geq 0$ and $l \geq 0$. Consider the strategy D^{ε}, such that $V^{D^{\varepsilon}}(x') > V(x') - \varepsilon$ for $x' \in (-\infty, x+(c+l)h]$ and $\varepsilon > 0$. Then, we define the strategy

$$D_t = \begin{cases} lt, & 0 \leq t < T_1 \wedge h, \\ D^{\varepsilon}_{t-T_1\wedge h}, & t \geq T_1 \wedge h. \end{cases}$$

For this strategy, we obtain

$$\begin{aligned} V(x) \geq & V^D(x) \\ \geq & \mathbb{E}\left[\int_0^{T_1\wedge h} e^{-\delta s}(\eta l - \phi(R_s^D))ds + e^{-\delta(T_1\wedge h)} V^{D^{\varepsilon}}(R^D_{T_1\wedge h})\right] \\ > & \mathbb{E}\left[\int_0^{T_1\wedge h} e^{-\delta s}(\eta l - \phi(R_s^D))ds + e^{-\delta(T_1\wedge h)} V(R^D_{T_1\wedge h})\right] - \varepsilon \\ = & \mathbb{E}\left[\left(\int_0^{T_1\wedge h} e^{-\delta s}(\eta l - \phi(R_s^D))ds + e^{-\delta(T_1\wedge h)} V(R^D_{T_1\wedge h})\right)\left(I_{\{T_1>h\}} + I_{\{T_1\leq h\}}\right)\right] - \varepsilon \\ = & e^{-\lambda h}\left(\int_0^h e^{-\delta s}(\eta l - \phi(x-(c+l)s))ds + e^{-\delta h} V(x-(c+l)h)\right) \\ & + \int_0^h \lambda e^{-\lambda t}\left[\int_0^t (\eta l - \phi(x-(c+l)s))ds + e^{-\delta t}\int_0^{\infty} V(x-(c+l)t+y)dF(y)\right]dt \\ & - \varepsilon + V(x-(c+l)h) - V(x-(c+l)h). \end{aligned}$$

Since ε is arbitrary we can let it tend to zero. Then, rearranging the terms and dividing by h implies

$$\begin{aligned} 0 \geq & \frac{V(x-(c+l)h) - V(x)}{h} - \frac{1-e^{-(\lambda+\delta)h}}{h} V(x-(c+l)h) \\ & + \frac{e^{-\lambda h}}{h}\int_0^h e^{-\delta s}(\eta l - \phi(x-(c+l)s))ds \\ & + \frac{1}{h}\int_0^h \lambda e^{-\lambda t}\left[\int_0^t (\eta l - \phi(x-(c+l)s))ds + e^{-\delta t}\int_0^{\infty} V(x-(c+l)t+y)dF(y)\right]dt. \end{aligned}$$

Let $h \to 0$, we get

$$-(c+l)V'(x)-(\lambda+\delta)V(x)+\eta l-\phi(x)+\lambda\int_0^{\infty}V(x+y)dF(y)\leq 0.$$

Since l is arbitrary, we obtain

$$\sup_{l\geq 0}\left\{l(\eta-V'(x))-cV'(x)-(\lambda+\delta)V(x)-\phi(x)+\lambda\int_0^{\infty}V(x+y)dF(y)\right\}\leq 0. \tag{3.4}$$

If $V'(x)<\eta$, then (3.4) would be positive for l large enough. Hence, we can get $V'(x)\geq\eta$. In addition, when $l=0$, we obtain

$$-cV'(x)-(\lambda+\delta)V(x)-\phi(x)+\lambda\int_0^{\infty}V(x+y)dF(y)\leq 0.$$

Thus, we can get

$$\max\left\{-cV'(x)-(\lambda+\delta)V(x)-\phi(x)+\lambda\int_0^{\infty}V(x+y)dF(y),\eta-V'(x)\right\}=0.$$

This completes the proof.

Theorem 3.2 Assume that f is an increasing, concave and twice continuously differentiable solution to (3.2), then $f(x)\geq V(x)$. Moreover, if there exists $b^*\in R^+$ such that

(i) $(\mathcal{A}-\delta)f(x)-\phi(x)=0, f'(x)\geq\eta, \forall x\leq b^*$;

(ii) $(\mathcal{A}-\delta)f(x)-\phi(x)<0, f(x)=f(b^*)+\eta(x-b^*), \forall x>b^*$,

then $f(x)=V(x)$ and b^* is the corresponding optimal dividend barrier.

Proof (i) For an arbitrary constant $n>0$ and admissible strategy $D\in\mathcal{D}$, define the stopping times $\tau_n=\inf\{t\geq 0:|R_t^D|\geq n\}$. Applying generalized Itô's formula yields

$$\begin{aligned}e^{-\delta\tau_n}f(R_{\tau_n}^D)=&f(x)-\delta\int_0^{\tau_n}e^{-\delta t}f(R_t^D)dt-c\int_0^{\tau_n}e^{-\delta t}f'(R_t^D)dt\\&+\int_0^{\tau_n}\lambda\int_0^{\infty}e^{-\delta t}[f(R_{t-}^D+y)-f(R_{t-}^D)]dF(y)dt-\int_0^{\tau_n}e^{-\delta t}f'(R_t^D)dD_t\\&+\sum_{0\leq t\leq\tau_n}e^{-\delta t}[f(R_t^D)-f(R_{R_{t-}}^D)-f'(R_{t-}^D)(R_t^D-R_{t-}^D)].\end{aligned}\tag{3.5}$$

Since f is concave, we have $f(y)\leq f(z)+f'(z)(y-z)$ for all y,z. Thus

$$\sum_{0\leq t\leq\tau_n}e^{-\delta t}[f(R_t^D)-f(R_{R_{t-}}^D)-f'(R_{t-}^D)(R_t^D-R_{t-}^D)]\leq 0.$$

Since f fulfils (3.2) and $f'(x)\geq\eta$, we obtain

$$f(x)\geq\mathbb{E}\left[e^{-\delta\tau_n}f(R_{\tau_n}^D)+\int_0^{\tau_n}\eta e^{-\delta t}dD_t-\int_0^{\tau_n}e^{-\delta t}\phi(R_t^D)dt\right].$$

Let b be the dividend barrier of strategy D, then $R_t^D \leq b$ a.s.. Since f is increasing, we have $e^{-\delta\tau_n} f(R_{\tau_n}^D) \leq e^{-\delta\tau_n} f(b)$. Because of $\lim\limits_{n\to\infty} \mathbb{E}[e^{-\delta\tau_n} f(b)] = 0$, so by bounded convergence theorem, we have $\lim\limits_{n\to\infty} \mathbb{E}[e^{-\delta\tau_n} f(R_{\tau_n}^D)] = 0$. Hence, we get $f(x) \geq V^D(x)$. Since D is arbitrary, we have $f(x) \geq V(x)$.

(ii) With strategy D^*, $f'(R_t^{D^*}) = \eta$ only if $R_t \geq b^*$, and $\{R_t^{D^*}\}$ only jumps downwards when $R_{t-}^{D*} > b^*$. Thus

$$\sum_{0\leq t\leq \tau_n} e^{-\delta t}[f(R_t^{D^*}) - f(R_{R_{t-}}^{D^*}) - f'(R_{t-}^{D^*})(R_t^{D^*} - R_{t-}^{D^*})] = 0.$$

Taking the expectation on both sides of (3.5) and letting $n \to \infty$, we get that

$$f(x) = \mathbb{E}\left[\int_0^\infty \eta e^{-\delta t} dD_t^* - \int_0^\infty e^{-\delta t}\phi(R_t^{D^*})dt\right] = V^{D^*}(x) \leq V(x).$$

This proves the result.

4 Examples

In our examples, we assume that the gain amounts are exponentially distributed. That is $F(y) = 1 - e^{-\gamma y}$, $\gamma > 0$. Then, $m_1 = \frac{1}{\gamma}$. We obtain the explicit expressions of the optimal value functions and corresponding optimal strategies for exponential and linear penalty payments, respectively.

4.1 Exponential Penalty Payments

In this section, we consider the function $\phi(x) = \alpha e^{-\beta x}$ with $\alpha, \beta > 0$. Obviously, (2.5) is fulfilled for $x < y < x_0 = -\beta^{-1}\max\{\ln\delta - \ln(\alpha\beta), 0\}$. Let $M_Y(r) = \mathbb{E}[e^{rY}]$ denote the moment-generating function of the gain amounts. Then

$$\mathbb{E}[e^{-\beta R_t - \delta t}] = \exp[-\beta(x - ct) + \lambda t(M_Y(-\beta) - 1) - \delta t]. \tag{4.1}$$

Putting the distribution function $F(y) = 1 - e^{-\gamma y}$ and penalty function $\phi(x) = \alpha e^{-\beta x}$ into $(\mathcal{A} - \delta)V(x) - \phi(x) = 0$ for $x \leq b^*$, we obtain

$$-cV'(x) + \lambda\gamma\int_0^\infty e^{-\gamma y}V(x+y)dy - (\lambda+\delta)V(x) - \alpha e^{-\beta x} = 0. \tag{4.2}$$

Since $V(x) = V(b^*) + \eta(x - b^*)$ for $x > b^*$ and let $z = x + y$, the above equation can be written as

$$\begin{aligned}&-cV'(x) + \lambda\gamma e^{\gamma x}\int_x^{b^*} e^{-\gamma z}V(z)dz + \lambda e^{(-b^*+x)\gamma}V(b^*)\\ &+\frac{\lambda\eta}{\gamma}e^{(-b^*+x)\gamma} - (\lambda+\delta)V(x) - \alpha e^{-\beta x} = 0.\end{aligned}$$

Applying the operator $(\gamma - \frac{d}{dx})$ to the above equation, yields

$$cV''(x) + (\lambda + \delta - c\gamma)V'(x) - \delta\gamma V(x) - \alpha(\beta+\gamma)e^{-\beta x} = 0. \tag{4.3}$$

This equation is solved by

$$f(x) = C_1 e^{\xi_1 x} + C_2 e^{\xi_2 x} - Ae^{-\beta x}, \tag{4.4}$$

where $\xi_2 < 0 < \xi_1$ are the roots of the equation

$$c\xi^2 + (\lambda + \delta - c\gamma)\xi - \delta\gamma = 0, \tag{4.5}$$

$$A = -\frac{\alpha(\beta + \gamma)}{c\beta^2 - (\lambda + \delta - c\gamma)\beta - \gamma\delta}, \tag{4.6}$$

and C_1, C_2 are constants. Since

$$\begin{aligned} V(x) &\geq -\alpha \int_0^\infty \mathbb{E}[e^{-\beta R_t - \delta t}]dt = -\alpha \int_0^\infty \exp[-\beta(x - ct) + \lambda t(M_Y(-\beta) - 1) - \delta t]dt \\ &= -\alpha \int_0^\infty \exp\{-\beta x + \frac{t(c\beta^2 - (\lambda + \delta - c\gamma)\beta - \gamma\delta)}{\beta + \gamma}\}dt \\ &= -\alpha e^{-\beta x} \int_0^\infty e^{-\frac{\alpha}{A}t}dt, \end{aligned}$$

we see that $V(x) = \infty$ if $-\beta \leq \xi_2$. We therefore assume $-\beta > \xi_2$, this means that $A > 0$, $V(x) \geq -\alpha \int_0^\infty \mathbb{E}[e^{-\beta R_t - \delta t}]dt = -Ae^{-\beta x}$. Now, since $\xi_1 > 0 > -\beta > \xi_2$, we obtain that $V(x)$ is only increasing for x small enough if $C_2 \leq 0$. Furthermore, if $C_2 < 0$, we have $V(x) < -Ae^{-\beta x}$ for x small enough. Thus, it must hold that $C_2 = 0$. Next, we only need to look for b^* and C_1. By $f'(b^*) = \eta$ and $f''(b^*) = 0$, that is,

$$\begin{cases} C_1\xi_1 e^{\xi_1 b^*} + A\beta e^{-\beta b^*} = \eta, \\ C_1\xi_1^2 e^{\xi_1 b^*} - A\beta^2 e^{-\beta b^*} = 0, \end{cases}$$

we get

$$b^* = -\frac{1}{\beta}\ln\frac{\eta\xi_1}{A\beta(\xi_1 + \beta)}$$

and

$$C_1 = \frac{A\beta^2 e^{-(\beta + \xi_1)b^*}}{\xi_1^2}.$$

Our candidate solution becomes now

$$f(x) = \begin{cases} C_1 e^{\xi_1 x} - Ae^{-\beta x}, & x \leq b^*, \\ C_1 e^{\xi_1 b^*} - Ae^{-\beta b^*} + \eta(x - b^*), & x > b^*. \end{cases} \tag{4.7}$$

Let $G(x) = -Ae^{-\beta x}$. For $x \leq b^*$, we have

$$\begin{aligned} f''(x) =& C_1\xi_1^2 e^{\xi_1 x} - A\beta^2 e^{-\beta x} = A\beta^2(e^{-(\beta + \xi_1)b^* + \xi_1 x} - e^{-\beta x}) \\ =& A\beta^2 e^{-\beta x}(e^{(x - b^*)(\beta + \xi_1)} - 1) < 0, \end{aligned}$$

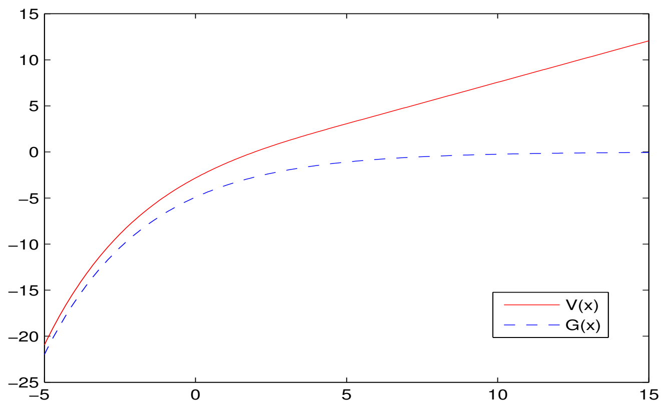

Figure 1: Value function for $\lambda = \gamma = 1, c = 0.8, \delta = 0.05, \alpha = 0.2, \beta = 0.3, \eta = 0.9$

so $f'(x) > f'(b^*) = \eta$. And for $x \geq b^*$, we have $f''(x) = 0$, $f'(x) = \eta$. Therefore, by Theorem 3.2, we can get that $f(x) = V(x)$ and b^* is the optimal dividend barrier.

In Figure 1, the value function is shown for $\lambda = \gamma = 1, c = 0.8, \delta = 0.05, \alpha = 0.2$, $\beta = 0.3$ and $\eta = 0.9$. In this case we have $b^* = 5.47991$. The solid line gives the optimal value, the dotted line gives the value without dividend payments.

4.2 Linear Penalty Payments

In this section, we assume that the penalty payments occur only when the surplus becomes negative. Therefore, we can let $\phi(x) = -\alpha x I_{\{x<0\}}$ with $\alpha > 0$.

Lemma 4.1 (i) If $\alpha < \delta\eta$, an optimal strategy does not exist and $V(x) = \infty$.

(ii) For $\alpha > \delta\eta$ it holds $V(x) \leq \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}$. Moreover, $V(x) \geq \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2} + C$ for some $C < 0$ if $x \leq 0$.

(iii) Let $\alpha = \delta\eta$, then $V(x) = \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}$.

Proof This lemma can be proved analogously as in the proof of [11, Lemma 5.1].

(i) Let $D^0 \in \mathcal{D}$ with the barrier $b = 0$. We define the strategy $D_t^{(0,a)} = D_t^0 + at$ for some $a > 0$. Now, we have $R_t^{(0,a)} \leq 0$ and

$$\mathbb{E}\left[\int_0^\infty \eta e^{-\delta t} dD_t^{(0,a)}\right] = \delta \mathbb{E}\left[\int_0^\infty \eta e^{-\delta t} D_t^{(0,a)} dt\right]. \tag{4.8}$$

Hence, we get

$$\begin{aligned}
V(x) \geq V^{D^{(0,a)}}(x) &= \mathbb{E}\left[\int_0^\infty \eta e^{-\delta t} dD_t^{(0,a)} + \alpha \int_0^\infty e^{-\delta t} R_t^{D^{(0,a)}} dt\right] \\
&= \mathbb{E}\left[(\delta\eta - \alpha) \int_0^\infty e^{-\delta t} D_t^{(0,a)} dt + \alpha \int_0^\infty e^{-\delta t} R_t dt\right] \\
&> \frac{a(\delta\eta - \alpha)}{\delta^2} + \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}.
\end{aligned}$$

If $\alpha < \delta\eta$, let $a \to \infty$, we can get $V(x) = \infty$.

(ii) Let $D \in \mathcal{D}$ be an arbitrary strategy. We assume that $\lim\limits_{t\to\infty} \mathbb{E}[e^{-\delta t} D_t] = 0$. Then

$$\begin{aligned} V^D(x) &\leq \mathbb{E}\left[\int_0^\infty \eta e^{-\delta t} dD_t + \alpha \int_0^\infty e^{-\delta t} R_t^D dt\right] \\ &= \mathbb{E}\left[\int_0^\infty \eta e^{-\delta t} dD_t - \alpha \int_0^\infty e^{-\delta t} D_t dt\right] + \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2} \\ &\leq \mathbb{E}\left[\int_0^\infty \eta e^{-\delta t} dD_t - \delta \int_0^\infty e^{-\delta t} D_t dt\right] + \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2} \\ &= \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}. \end{aligned}$$

Since D is arbitrary, we have $V(x) \leq \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}$. Now, let $x \leq 0$, we have

$$\begin{aligned} V(x) \geq &\alpha\mathbb{E}\left[\int_0^\infty e^{-\delta t} \min(R_t, 0) dt\right] = \alpha\mathbb{E}\left[\int_0^\infty e^{-\delta t} \frac{1}{2}(R_t - |R_t|) dt\right] \\ =&\frac{1}{2}\left(\frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2} - \alpha\int_0^\infty e^{-\delta t}\mathbb{E}[|R_t|] dt\right) \\ \geq&\frac{1}{2}\left(\frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2} + \alpha\int_0^{\frac{-x}{\lambda m_1 - c}} e^{-\delta t}(x - ct + \lambda m_1 t) dt\right. \\ &\left. -\alpha\int_{\frac{-x}{\lambda m_1 - c}}^\infty e^{-\delta t}(x - ct + \lambda m_1 t) dt\right) \\ =&\frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2} - \frac{\lambda m_1 - c}{\delta^2} e^{\frac{\delta x}{\lambda m_1 - c}} \\ =&\frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2} + C \end{aligned}$$

for some $C < 0$.

(iii) Consider the same strategy as in (i). By $\alpha = \delta\eta$, we obtain $V(x) \geq \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}$.

On the other hand, from (ii) we get $V(x) \leq \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}$.

The proof is completed.

From Lemma 4.1, when $\alpha = \delta\eta$, we know that a barrier strategy with a barrier at zero is optimal. So, in the following, we assume that $\alpha > \delta\eta$. This is consistent with our assumption in (2.5). In addition, the dividend barrier must be positive, because it can't be optimal to pay dividends if the surplus is negative.

Next, we use a similar method to obtain the explicit expression of the optimal value functions and corresponding optimal strategy for linear penalty payments.

Now, writting the distribution function $F(y) = 1 - e^{-\gamma y}$ and penalty function $\phi(x) = -\alpha x$ into $(\mathcal{A} - \delta)V(x) - \phi(x) = 0$ for $x \leq b^*$, we obtain

$$-cV'(x) + \lambda\gamma \int_0^\infty e^{-\gamma y} V(x + y) dy - (\lambda + \delta)V(x) + \alpha x = 0. \tag{4.9}$$

Since $V(x) = V(b^*) + \eta(x - b^*)$ for $x > b^*$, and let $z = x + y$, the above equation can be written as

$$-cV'(x) + \lambda\gamma e^{\gamma x}\int_x^{b^*} e^{-\gamma z}V(z)dz + \lambda e^{(-b^*+x)\gamma}V(b^*) + \frac{\lambda\eta}{\gamma}e^{(-b^*+x)\gamma} - (\lambda+\delta)V(x) + \alpha x = 0.$$

Applying the operator $(\gamma - \frac{d}{dx})$ to the above equation, yields

$$cV''(x) + (\lambda + \delta - c\gamma)V'(x) - \delta\gamma V(x) + \alpha(\gamma x - 1) = 0. \tag{4.10}$$

This equation is solved by

$$f_1(x) = B_1 e^{\xi_1 x} + B_2 e^{\xi_2 x} + \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}, \tag{4.11}$$

where B_1, B_2 are constants and ξ_1, ξ_2 are the roots of (4.5). Let $G(x) = \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2}$.

When $x \in (0, b^*]$, we have $\phi(x)$=0. Putting the distribution function $F(y) = 1 - e^{-\gamma y}$ and penalty function $\phi(x) = 0$ into $(\mathcal{A} - \delta)V(x) - \phi(x) = 0$, similarly, we obtain

$$cV''(x) + (\lambda + \delta - c\gamma)V'(x) - \delta\gamma V(x) = 0. \tag{4.12}$$

This equation is solved by

$$f_2(x) = B_3 e^{\xi_1 x} + B_4 e^{\xi_2 x}, \tag{4.13}$$

where B_3, B_4 are constants and ξ_1, ξ_2 as above.

Now, $f_1(x)$ is increasing for x small enough only if $B_2 \le 0$. Furthermore, if $B_2 < 0$, we have $f_1(x) < \frac{\alpha(\delta x - c + \lambda m_1)}{\delta^2} + C$ for x small enough. Thus, we let $B_2 = 0$. Note that the continuity of ϕ in $x = 0$ together with $f_1(0) = f_2(0)$ and $f_1'(0) = f_2'(0)$ implies $f_1''(0) = f_2''(0)$. At the dividend barrier, we have $f_2'(b^*) = \eta$ and $f_2''(b^*) = 0$. By these bounded conditions, we obtain

$$B_3 = -\frac{\eta\xi_2 e^{-\xi_1 b^*}}{\xi_1(\xi_1 - \xi_2)}, \qquad B_4 = \frac{\eta\xi_1 e^{-\xi_2 b^*}}{\xi_2(\xi_1 - \xi_2)},$$
$$B_1 = -\frac{-\eta\xi_2 e^{-\xi_1 b^*} + \eta\xi_1 e^{-\xi_2 b^*}}{\xi_1(\xi_1 - \xi_2)} - \frac{\alpha}{\delta\xi_1},$$

and

$$b^* = -\frac{1}{\xi_2}\ln\frac{\alpha}{\delta\eta} > 0.$$

Thus, we can get that $B_3 > 0$, $B_4 < 0$ and

$$B_1 = \frac{-\eta\xi_2(\alpha/\delta\eta)^{\xi_1/\xi_2} + \eta\xi_1(\alpha/\delta\eta)}{\xi_1(\xi_1 - \xi_2)} - \frac{\alpha}{\delta\xi_1} = \frac{\alpha\xi_2(1 - (\alpha/\delta\eta)^{-1+\xi_1/\xi_2})}{\delta\xi_1(\xi_1 - \xi_2)} < 0.$$

From the above, our candidate solution becomes now

$$f(x) = \begin{cases} f_1(x), & x \le 0, \\ f_2(x), & 0 < x \le b^*, \\ f_2(b^*) + \eta(x - b^*), & x > b^*. \end{cases} \tag{4.14}$$

By construction, we know that $f(x)$ is twice continuously differentiable. Now,

$$f_2'''(x) = \xi_1^3 B_3 e^{\xi_1 x} + \xi_2^3 B_4 e^{\xi_2 x} > 0.$$

Then, if $0 < x \le b^*$, $f_2''(x) \le f_2''(b^*) = 0$ and $f_2'(x) \ge f_2'(b^*) = \eta$. Furthermore, $f_1''(x) = \xi_1^2 B_1 e^{\xi_1 x} < 0$. Therefore, $f_1'(x) \ge f_1'(0) = f_2'(0) > f_2'(b^*) = \eta$ for $x \le 0$. So f is concave and $f'(x) \ge \eta$ for all $x \le b^*$.

In conclusion, $f(x)$ satisfies all of conditions of Theorem 3.2. Therefore, we can get that $f(x) = V(x)$ and b^* is the optimal dividend barrier.

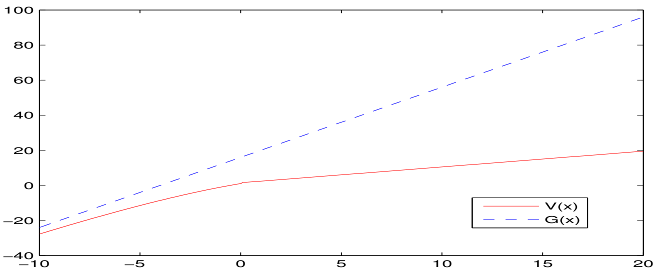

Figure 2: Value function for $\lambda = \gamma = 1, c = 0.8, \delta = 0.05, \alpha = 0.2, \eta = 0.9$

Figure 2 illustrates the value function for $\lambda = \gamma = 1, c = 0.8, \delta = 0.05$ and $\alpha = 0.2$. The optimal dividend barrier is given by $b^* = 3.03070$.

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对偶模型中带指数或线性罚函数的最优分红问题

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摘要: 本文研究了带罚函数的对偶模型的最优分红问题. 假设当公司的盈余资金为负值时, 公司不会发生破产, 但是会进行相应的惩罚, 惩罚金额取决于公司的余额水平. 利用随机最优控制方法和动态规划原则, 得到了最优化问题的HJB方程及其验证定理. 最后, 当收益服从指数分布时, 得到了带指数罚函数和带线性罚函数两种情形各自的最优分红策略及最优值函数的解析式.

关键词: 对偶风险模型; 分红; 罚金; HJB方程

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