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WEIGHTED MIXED INEQUALITIES ON PRODUCT SPACES WITH MUCKENHOUPT BASES

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Abstract: We study weighted mixed inequalities on product spaces with different Muckenhoupt bases. Our approaches are mainly based on the abstract formalism of families of extrapolation pairs and Minkowski's integral inequality. Moreover, we can deduce the general case of our results by induction.

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1 Introduction and Statements of Main Results

In this paper, we shall deal with spaces of measurable functions defined in the following way. Let (X_i, S_i, μ_i), for $1 \le i \le n$, be totally σ-finite measure spaces and $P = (p_1, p_2, \cdots, p_n)$ a given n-tuple with $1 \le p_i \le \infty$. We always suppose that none of the spaces (X_i, S_i, μ_i) admits as the only measurable functions the constant ones. A function $f(x_1, x_2, \cdots, x_n)$ measurable in the product space $(X, S, \mu) = (\prod_{i=1}^n X_i, \prod_{i=1}^n S_i, \prod_{i=1}^n \mu_i)$, is said to belong to $L^P(X)$ if the number obtained after taking successively the p_1 norm in x_1, the p_2 norm in x_2, $\cdots$, the p_n norm in x_n and in that order, is finite. The number so obtained, finite or not, will be denoted by $\|f\|_P$, $\|f\|_{(p_1,\cdots,p_n)}$ or $\|f\|_{p_1,\cdots,p_n}$. When for every i, $p_i < \infty$, we have in particular

$$\|f\|_P = (\int_{X_n} \cdots (\int_{X_2} (\int_{X_1} |f(x_1, x_2, \cdots, x_n)|^{p_1} d\mu_1)^{\frac{p_2}{p_1}} d\mu_2)^{\frac{p_3}{p_2}} \cdots d\mu_n)^{\frac{1}{p_n}}.$$

If further, each p_i is equal to p :

$$\|f\|_P = \|f\|_{(p,\cdots,p)} = (\int_X |f(x_1, x_2, \cdots, x_n)|^p d\mu)^{\frac{1}{p}} = \|f\|_p$$

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and $L^P(X) = L^p(X)$ (see [1] for more information). In this paper, we only deal with the case when every component space (X_i, S_i, μ_i) is a d_i-dimensional Euclidean space $\mathbb{R}^{d_i}$ with Lebesgue's measure. Then, the product space (X, S, μ) is a d-dimensional Euclidean space with Lebesgue's measure, where $d = d_1 + d_2 + \cdots + d_n$.

In the following, we consider weighted inequalities on these spaces. For simplicity of notations, we only consider the case of $n = 2$ and our results can be extended to the general case by induction.

Throughout, ω will denote a weight, i.e., a nonnegative, locally integrable function. All cubes in $\mathbb{R}^{d_i}$ will be half open with sides parallel to the axes. Given a set $E \subset \mathbb{R}^{d_i}$, $|E|$ will denote the Lebesgue's measure of E, $\omega(E) = \int_E \omega(x)dx$ the weighted measure of E, and $⨍_E \omega dx = |E|^{-1} \int_E \omega(x)dx = \frac{\omega(E)}{|E|}$ the average of ω over E.

To define the classes of weights which we will consider, we first introduce the concept of basis $\mathscr{B}$ and the maximal operator $M_{\mathscr{B}}$ defined with respect to $\mathscr{B}$ (see [2, 3] for more information). A basis $\mathscr{B}$ is a collection of open sets $B \subset \mathbb{R}^d$. A weight ω is associated with the basis $\mathscr{B}$, if $\omega(B) < \infty$ for every $B \in \mathscr{B}$. Given a basis $\mathscr{B}$, the corresponding maximal operator is defined by

$$M_{\mathscr{B}} f(x) = \begin{cases} \sup\limits_{B \ni x} ⨍_B |f(y)|dy, \text{ if } x \in \bigcup\limits_{B \in \mathscr{B}} B, \\ 0, \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

A weight ω associated with $\mathscr{B}$ is in the Muckenhoupt class $A_{p,\mathscr{B}}(\mathbb{R}^d)$, $1 < p < \infty$, if there exists a constant C such that for every $B \in \mathscr{B}$,

$$(⨍_B \omega(x)dx)(⨍_B \omega(x)^{-\frac{1}{p-1}}dx)^{p-1} < C.$$

When $p = 1$, ω belongs to $A_{1,\mathscr{B}}(\mathbb{R}^d)$ if $M_{\mathscr{B}}\omega(x) \le C\omega(x)$ for almost every $x \in \mathbb{R}^d$. The infimum of all such C, denoted by $[\omega]_{A_{p,\mathscr{B}}(\mathbb{R}^d)}$. For simplicity, $A_{p,\mathscr{B}}(\mathbb{R}^d)$ is denoted by $A_{p,\mathscr{B}}$, if no confusion can arise. Clearly, if $1 \le q \le p$, then $A_{q,\mathscr{B}} \subseteq A_{p,\mathscr{B}}$. Further, from the definitions we get the following factorization property: if ω_1, $\omega_2 \in A_{1,\mathscr{B}}$, then $\omega_1\omega_2^{1-p} \in A_{p,\mathscr{B}}$. Finally, we let $A_{\infty,\mathscr{B}} = \bigcup\limits_{p \ge 1} A_{p,\mathscr{B}}$.

We are going to restrict our attention to the following class of bases. A basis $\mathscr{B}$ is a Muckenhoupt basis if for each p, $1 < p < \infty$, and for every $\omega \in A_{p,\mathscr{B}}$, the maximal operator $M_{\mathscr{B}}$ is bounded on $L^p(\omega)$, that is,

$$\int_{\mathbb{R}^d} M_{\mathscr{B}} f(x)^p \omega(x)dx \le C \int_{\mathbb{R}^d} |f(x)|^p \omega(x)dx. \tag{1.1}$$

Let $\mathscr{B}$ be a Muckenhoupt basis. Let $1 < p < \infty$ and ω be a weight. If there exists a constant C such that

$$\int_{\mathbb{R}^d} M_{\mathscr{B}} f(x)^p \omega(x)dx \le C \int_{\mathbb{R}^d} |f(x)|^p \omega(x)dx \quad \text{for all } f \in L^p(\omega), \tag{1.2}$$

then standard arguments give $\omega \in A_{p,\mathscr{B}}$.

Muckenhoupt bases were introduced and characterized in [3, Theorem 2.1]. Three immediate examples of Muckenhoupt bases are $\mathscr{D}$, the set of dyadic cubes in $\mathbb{R}^d$; $\mathscr{C}$, the set of all cubes in $\mathbb{R}^d$ whose sides are parallel to the coordinate axes, and $\mathscr{R}$, the set of all rectangles (i.e., parallelepipeds) in $\mathbb{R}^d$ whose sides are parallel to the coordinate axes (see [4, Theorem 7.14]). One advantage of these bases is that by using them we avoid any direct appeal to the underlying geometry: the relevant properties are derived from (1.1), and we do not use covering lemmas of any sort.

In this paper, we will use the Rubio de Francia extrapolation as our main tool to deal with our inequalities. As is well known, the extrapolation theorem of Rubio de Francia is one of the deepest results in the study of weighted norm inequalities in harmonic analysis [5]. Recently, an approach to extrapolation is based on the abstract formalism of families of extrapolation pairs and summarized by Cruz-Uribe [6]. This approach was introduced in [7] and first fully developed in [8] (see [9] for more information). It was implicit from the beginning that in extrapolating from an inequality of the form

$$\int_{\mathbb{R}^d} |Tf|^p w\,dx \le C \int_{\mathbb{R}^d} |f|^p w\,dx,$$

the operator T and its properties (positive, linear, etc.) played no role in the proof. Instead, all that mattered was that there existed a pair of non-negative functions $(|Tf|, |f|)$ that satisfied a given collection of norm inequalities. Therefore, the proof goes through working with any pair (f, g) of non-negative functions.

Hereafter, we will adopt the following conventions. A family of extrapolation pairs $\mathcal{F}$ will consist of pairs of non-negative, measurable functions (f, g) that are not equal to 0. Given such a family $\mathcal{F}$, $0 < p < \infty$, $1 \le q < \infty$ and $\omega \in A_{q,\mathscr{B}}(\mathbb{R}^d)$, if we say

$$\int_{\mathbb{R}^d} f^p w\,dx \le C \int_{\mathbb{R}^d} g^p w\,dx, \qquad (f, g) \in \mathcal{F},$$

we mean that this inequality holds for all pairs $(f, g) \in \mathcal{F}$ such that $\|f\|_{L^p(w)} < \infty$, i.e., that the left-hand side of the inequality is finite and the constant C depends only upon p, q, d, and the $[w]_{A_q}$ constant of w. Moreover, given a family $\mathcal{F}$, $0 < p < \infty$ and $w \in A_{\infty,\mathscr{B}}(\mathbb{R}^d)$, we always say

$$\int_{\mathbb{R}^d} f^p w\,dx \le C \int_{\mathbb{R}^d} g^p w\,dx, \qquad (f, g) \in \mathcal{F}. \tag{1.3}$$

Since $A_{\infty,\mathscr{B}} = \bigcup_{q \ge 1} A_{q,\mathscr{B}}$, there is $q \ge 1$ such that $\omega \in A_{q,\mathscr{B}}(\mathbb{R}^d)$. In (1.3), we mean that the constant C depends only upon p, d, q and the $[w]_{A_q}$ constant of w.

The key to the new approach is the family of extrapolation pairs $\mathcal{F}$. If the family of extrapolation pairs $\mathcal{F}$ seems abstract and mysterious, it may help to think of the particular family

$$\mathcal{F} = \big\{(|Tf|, |Sf|), f \in \mathcal{N}\big\},$$

where T and S are some operators that we are interested in and $\mathcal{N}$ is some "nice" family of functions: L_c^∞, C_c^∞, etc. We refer to Cruz-Uribe [6, Sections 5 and 6] and Cruz-Uribe, Martell and Pérez [9, Section 3.8] for more information.

Using the above conventions, we can give the following main results related with strong maximal operator, Riesz potential and multiparameter fractional integral operators.

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be Muckenhoupt bases in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. We consider the space $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ which identify with $\mathbb{R}^{d_1+d_2}=\mathbb{R}^d$ and the product basis $\mathscr{B}\triangleq\mathscr{B}_1\times\mathscr{B}_2=\{Q_{d_1}\times Q_{d_2}:Q_{d_i}\in\mathscr{B}_i\}$. The corresponding maximal operator is called strong maximal operator and is denoted by $\mathcal{M}_s$.

Let $1<p<\infty$, $\omega_i(x_i)$ be a weight in $\mathbb{R}^{d_i}$ and $\omega_i(x_i)\in A_{p,\mathscr{B}_i}(\mathbb{R}^{d_i})$. Then there is a constant C such that

$$\int_{\mathbb{R}^d}\mathcal{M}_s f(x)^p\omega(x)dx\le C\int_{\mathbb{R}^d}|f(x)|^p\omega(x)dx,$$

where $\omega(x_1,x_2)=\omega_1(x_1)\omega_2(x_2)$. Moreover, let $\mathscr{B}\triangleq\mathscr{C}_1\times\mathscr{C}_2$, it follows by Fubini's theorem that $\mathscr{B}$ is a Muckenhoupt basis (see [8, Page 424]).

Then we have the following Theorem 1.1, which is a weighted version of [10, Theroerm 4.1].

Theorem 1.1 Let $1<p_i<\infty$ and $\omega(x_1,x_2)=\omega_1(x_1)\omega_2(x_2)$. Let $\omega_i(x_i)$ be a weight in $\mathbb{R}^{d_i}$, then the following statements are equivalent.

(1) There is a constant C, independing of f such that

$$\begin{aligned}&\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_1}}\mathcal{M}_s f(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\ \le\ & C\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_1}}|f(x_1,x_2)|^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2\Big)^{\frac{1}{p_2}}.\end{aligned}\tag{1.4}$$

(2) $\omega_1^{p_1}\in A_{p_1,\mathscr{B}_1}(\mathbb{R}^{d_1})$ and $\omega_2^{p_2}\in A_{p_2,\mathscr{B}_2}(\mathbb{R}^{d_2})$.

Let $\mathbb{R}^d$ be the d-dimensional Euc1idean space. The Riesz potential of order α, $0<\alpha<d$, of a function f is defined by

$$Rf(x)=\int_{\mathbb{R}^d}\frac{f(\bar{x})}{|x-\bar{x}|^{d-\alpha}}d\bar{x}.\tag{1.5}$$

We also define the fractional maximal operator $M^{(\alpha)}f(x)$ by

$$M^{(\alpha)}f(x)=\sup_{Q_d\ni x}\frac{1}{|Q_d|^{1-\frac{\alpha}{d}}}\int_{Q_d}|f(\bar{x})|d\bar{x},$$

where the supremum is over all cubes Q_d with sides parallel to the axes and containing x.

We extend [11, Theorem 1] to the case on mixed norm Lebesgue spaces. Let us consider the d_i-dimensional Euclidean space $\mathbb{R}^{d_i}$ and the basis $\mathscr{C}_i$ in it, where $i=1,2$. For $\mathbb{R}^d=\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$, we consider two bases $\mathscr{C}$ and $\mathscr{B}=\mathscr{C}_1\times\mathscr{C}_2$ in it. It is clear that $\mathscr{C}\subseteq\mathscr{C}_1\times\mathscr{C}_2$. For $\mathscr{C}$, Sjödin [12] obtained the following Theorem 1.2, which gave the weighted norm inequalities for Riesz potentials and fractional maximal functions in mixed norm Lebesgue spaces. We reprove the theorem by the abstract formalism of families of extrapolation pairs as following.

Theorem 1.2 Let $0 < p_i < \infty$ and $\omega(x_1, x_2) = \omega_1(x_1)\omega_2(x_2)$. Let $\omega_i(x_i)$ be a weight in $\mathbb{R}^{d_i}$ and $\omega_i(x_i)^{p_i} \in A_{\infty,\mathscr{C}_i}(\mathbb{R}^{d_i})$. Then there is a constant C such that

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_1}} Rf(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\
\le\ & C\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_1}} M^{(\alpha)}f(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2\Big)^{\frac{1}{p_2}}. \qquad (1.6)
\end{aligned}$$

Also we can consider the fractional integral operators in our case. We define a multi-parameter version of the fractional integral operator of order 1 (see, e.g. [8, Page 423]): for $(x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, let

$$Tf(x_1,x_2) = \int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} \frac{f(\bar{x}_1,\bar{x}_2)}{|x_1-\bar{x}_1|^{d_1-1}|x_2-\bar{x}_2|^{d_2-1}}d\bar{x}_1 d\bar{x}_2. \qquad (1.7)$$

Given $(x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and a function $f \in L^1_{loc}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, define the multi-parameter fractional maximal operators

$$\begin{aligned}
M_1^{(1)}f(x_1,x_2) &= \sup_{Q_{d_1}\ni x_1} \frac{1}{|Q_{d_1}|^{1-\frac{1}{d_1}}}\int_{Q_{d_1}} |f(\bar{x}_1,x_2)|d\bar{x}_1,\\
M_2^{(1)}f(x_1,x_2) &= \sup_{Q_{d_2}\ni x_2} \frac{1}{|Q_{d_2}|^{1-\frac{1}{d_2}}}\int_{Q_{d_2}} |f(x_1,\bar{x}_2)|d\bar{x}_2.
\end{aligned}$$

A simple estimate shows that $M_1^{(1)} \circ M_2^{(1)} f(x_1, x_2) \le C \cdot Tf(x_1, x_2)$ and similarly with the order of composition reversed. As in the one-variable case, the reverse inequality does not hold pointwise, but does hold in the sense of weighted L^p norms. For the product $\mathscr{B}$, we have the following theorem, which is an extension of [8, Proposition 3.5].

Theorem 1.3 Let $1 \le p_1 < \infty$, $0 < p_2 < \infty$ and $\omega(x_1, x_2) = \omega_1(x_1)\omega_2(x_2)$. Let $\omega_i(x_i)$ be a weight in $\mathbb{R}^{d_i}$ and $\omega_i(x_i)^{p_i} \in A_{\infty,\mathscr{C}_i}(\mathbb{R}^{d_i})$. Then there is a constant C such that

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_1}} |Tf(x_1,x_2)|^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2\Big)^{\frac{1}{p_2}} \qquad (1.8)\\
\le\ & C\Big(\int_{\mathbb{R}^{d_2}} M_2^{(1)}\Big(\Big(\int_{\mathbb{R}^{d_1}} M_1^{(1)}f(x_1,\cdot)^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{1}{p_1}}\Big)(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2\Big)^{\frac{1}{p_2}}.
\end{aligned}$$

Throughout this paper, C denote a constant not necessarily the same at each occurrence.

2 Proof of Main Results

In this section, we give the proofs of Theorems 1.1–1.3.

To prove Theorem 1.1, we need the following theorem which was proved by Cruz-Urible, Martell and Pérez in [9].

Theorem 2.1 (see [6, Theorem 3.9]) Given a family of extrapolation pairs $\mathcal{F}$, let $\mathscr{B}$ be a Muckenhoupt basis. Suppose that for some p_0, $1 \le p_0 < \infty$, and every $w_0 \in A_{p_0,\mathscr{B}}$,

$$\int_{\mathbb{R}^d} f(x)^{p_0} w_0(x)dx \le C \int_{\mathbb{R}^d} g(x)^{p_0} w_0(x)dx, \qquad (f,g) \in \mathcal{F}.$$

Then for every p, $1 < p < \infty$, and every $w \in A_{p,\mathscr{B}}$,

$$\int_{\mathbb{R}^d} f(x)^{p} w(x)dx \le C \int_{\mathbb{R}^d} g(x)^{p} w(x)dx, \qquad (f,g) \in \mathcal{F}.$$

Then, we can give the proof of Theorem 1.1 as follows.

Proof of Theorem 1.1 Suppose that (1.4) is valid. We prove $\omega_1^{p_1} \in A_{p_1,\mathscr{B}_1}(\mathbb{R}^{d_1})$ and $\omega_2^{p_2} \in A_{p_2,\mathscr{B}_2}(\mathbb{R}^{d_2})$. For any $h(x_1) \in L^{p_1}(\omega_1^{p_1})$ and $Q_{d_2} \in \mathscr{B}_2$, let

$$f(x_1,x_2) = h(x_1)\chi_{Q_{d_2}}(x_2).$$

We have $\mathcal{M}_s f(x_1,x_2) = M_{\mathscr{B}_1} h(x_1)\chi_{Q_{d_2}}(x_2)$. Then, we rewrite (1.4) as

$$\Big(\int_{\mathbb{R}^{d_1}} M_{\mathscr{B}_1} h(x_1)^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{1}{p_1}} \le C\Big(\int_{\mathbb{R}^{d_1}} |h(x_1)|^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{1}{p_1}}. \tag{2.1}$$

It follows from (1.2) that $\omega_1(x_1)^{p_1} \in A_{p_1,\mathscr{B}_1}(\mathbb{R}^{d_1})$. Similarly, we have $\omega_2(x_2)^{p_2} \in A_{p_2,\mathscr{B}_2}(\mathbb{R}^{d_2})$.

Conversely, to prove (1.4). Fix $p_1 > 1$ and $\omega_1^{p_1} \in A_{p_1,\mathscr{B}_1}(\mathbb{R}^{d_1})$, let

$$\begin{aligned} \mathcal{S}_2 \ &= \ \Big\{(f_2,g_2):\ f_2(x_2) = \Big(\int_{\mathbb{R}^{d_1}} \mathcal{M}_s f(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{1}{p_1}}, \\ &\qquad g_2(x_2) = \Big(\int_{\mathbb{R}^{d_1}} |f(x_1,x_2)|^{p_1}\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{1}{p_1}}\Big\}. \end{aligned}$$

If $p_2 = p_1$, then, for all $\omega_2(x_2)^{p_2} \in A_{p_2,\mathscr{B}_2}(\mathbb{R}^{d_2})$, we have

$$\omega(x_1,x_2)^{p_1} \triangleq \omega_1(x_1)^{p_1}\omega_2(x_2)^{p_2} \in A_{p_1,\mathscr{B}}(\mathbb{R}^d).$$

It follows that

$$\begin{aligned} &\Big(\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} \mathcal{M}_s f(x_1,x_2)^{p_1}\omega(x_1,x_2)^{p_1}dx_1dx_2\Big)^{\frac{1}{p_1}} \\ \le\ & C\Big(\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} |f(x_1,x_2)|^{p_1}\omega(x_1,x_2)^{p_1}dx_1dx_2\Big)^{\frac{1}{p_1}}. \end{aligned}$$

Then, we have

$$\int_{\mathbb{R}^{d_2}} f_2(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2 \le C\int_{\mathbb{R}^{d_2}} g_2(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2,\ (f_2,g_2)\in\mathcal{S}_2.$$

Thus, we check (2.1) of Theorem 2.1 with $\mathcal{F} \triangleq \mathcal{S}_2$ and $p_0 \triangleq p_2$. Using Theorem 2.1, we get, for every $1 < p_2 < \infty$ and every $v_2(x_2) \in A_{p_2,\mathscr{B}_2}(\mathbb{R}^{d_2})$,

$$\int_{\mathbb{R}^{d_2}} f_2(x_2)^{p_2}v_2(x_2)dx_2 \le C\int_{\mathbb{R}^{d_2}} g_2(x_2)^{p_2}v_2(x_2)dx_2,\ (f_2,g_2)\in\mathcal{S}_2.$$

Note that $\omega_2(x_2)^{p_2} \in A_{p_2,\mathscr{B}_2}(\mathbb{R}^{d_2})$. It follows from the above inequality that

$$\int_{\mathbb{R}^{d_2}} f_2(x_2)^{p_2}\omega_2^{p_2}(x_2)dx_2 \le C\int_{\mathbb{R}^{d_2}} g_2(x_2)^{p_2}\omega_2^{p_2}(x_2)dx_2,\ (f_2,g_2)\in\mathcal{S}_2.$$

Thus

$$\begin{aligned}&(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}\mathcal{M}_s f(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}\\ \le\ & C(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}|f(x_1,x_2)|^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}.\end{aligned}$$

Kurtz [13] obtained Theorem 1.1 in the space $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ involving $\mathscr{C}_1\times\mathscr{C}_2$ and Theorem 1.1 is a general case of [13, Theorem 1]. Here, it is natural to expect a more general result.

Let $\mathscr{B}_i$ be a Muckenhoupt basis in $\mathbb{R}^{d_i}$, $i=1,2,\cdots,n$. Then $\mathscr{B}=\{\prod_{i=1}^{n}Q_{d_i}:Q_{d_i}\in\mathscr{B}_i\}$ is a product basis in the space $\mathbb{R}^d=\prod_{i=1}^{n}\mathbb{R}^{d_i}$. If $\mathscr{B}_i\times\mathscr{B}_{i+1}$ is a Muckenhoupt basis in $\mathbb{R}^{d_i}\times\mathbb{R}^{d_{i+1}}$, $i=1,2,\cdots,n-1$, using our approach to Theorem 1.1, we have the following Corollary 2.2 by induction.

Corollary 2.2 Let $1<p_i<\infty$ and $\omega(x_1,x_2,\cdots,x_n)=\prod_{i=1}^{n}\omega_i(x_i)$. Let $\omega_i(x_i)$ be a weight in $\mathbb{R}^{d_i}$, then the following statements are equivalent:

(1) There is a constant C, independing of f such that

$$\begin{aligned}&(\int_{\mathbb{R}^{dn}}\cdots(\int_{\mathbb{R}^{d_1}}\mathcal{M}_s f(x_1,\cdots,x_n)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\omega_n(x_n)^{p_n}dx_n)^{\frac{1}{pn}}\\ \le\ & C(\int_{\mathbb{R}^{dn}}\cdots(\int_{\mathbb{R}^{d_1}}|f(x_1,\cdots,x_n)|^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\omega_n(x_n)^{p_n}dx_n)^{\frac{1}{pn}}.\end{aligned}$$

(2) $\omega_i^{p_i}\in A_{p_i,\mathscr{B}_i}(\mathbb{R}^{d_i})$, $i=1,2\cdots,n$.

Proof We only prove (2) $\Rightarrow$ (1) and this is done by induction beginning with the case $n=2$. For $n=2$, it is valid because of Theorem 1.1.

Assuming that the inequality is valid for $n-1$, we set

$$\begin{aligned}\mathcal{S}_n=&\Big\{(f_n,g_n):\\ &f_n(x_n)=(\int_{\mathbb{R}^{d_{n-1}}}\cdots(\int_{\mathbb{R}^{d_1}}\mathcal{M}_s f(x_1,x_2,\cdots,x_n)^{p_1}\omega(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\omega(x_{n-1})^{p_{n-1}}dx_{n-1})^{\frac{1}{p_{n-1}}},\\ &g_n(x_n)=(\int_{\mathbb{R}^{d_{n-1}}}\cdots(\int_{\mathbb{R}^{d_1}}|f(x_1,x_2,\cdots,x_n)|^{p_1}\omega(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\omega(x_{n-1})^{p_{n-1}}dx_{n-1})^{\frac{1}{p_{n-1}}}\Big\}.\end{aligned}$$

If $p_n=p_{n-1}$, then, for all $\omega_n(x_n)^{p_n}\in A_{p_n,\mathscr{B}_n}(\mathbb{R}^{d_n})$, we have

$$\nu_{n-1}(\bar{x}_{n-1})^{p_{n-1}}\triangleq\omega_{n-1}(x_{n-1})^{p_{n-1}}\omega_n(x_n)^{p_n}\in A_{p_{n-1},\mathscr{B}_{n-1}\times\mathscr{B}_n}(\mathbb{R}^{d_{n-1}+d_n}).$$

It follows from the induction hypothesis that

$$\begin{aligned}
&(\int_{\mathbb{R}^{d_n}}\cdots(\int_{\mathbb{R}^{d_1}}\mathcal{M}_s f(x_1,\cdots,x_n)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\omega_n(x_n)^{p_n}dx_n)^{\frac{1}{p_n}}\\
= &(\int_{\mathbb{R}^{d_{n-1}+d_n}}\cdots(\int_{\mathbb{R}^{d_1}}\mathcal{M}_s f(x_1,\cdots,\bar{x}_{n-1})^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\nu_{n-1}(\bar{x}_{n-1})^{p_{n-1}}d\bar{x}_{n-1})^{\frac{1}{p_{n-1}}}\\
\leq &C(\int_{\mathbb{R}^{d_{n-1}+d_n}}\cdots(\int_{\mathbb{R}^{d_1}}|f(x_1,\cdots,\bar{x}_{n-1})|^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\nu_{n-1}(\bar{x}_{n-1})^{p_{n-1}}d\bar{x}_{n-1})^{\frac{1}{p_{n-1}}}.
\end{aligned}$$

Then, we have

$$\int_{\mathbb{R}^{d_n}} f_n(x_n)^{p_n}\omega_n(x_n)^{p_n}dx_n\leq C\int_{\mathbb{R}^{d_n}} g_n(x_n)^{p_n}\omega_n(x_n)^{p_n}dx_n,\ (f_n,g_n)\in\mathcal{S}_n.$$

Using Theorem 2.1, we get, for every $1<p_n<\infty$ and every $v_n(x_n)\in A_{p_n,\mathscr{B}_n}(\mathbb{R}^{d_n})$,

$$\int_{\mathbb{R}^{d_n}} f_n(x_n)^{p_n}v_n(x_n)dx_n\leq C\int_{\mathbb{R}^{d_n}} g_n(x_n)^{p_n}v_n(x_n)dx_n,\ (f_n,g_n)\in\mathcal{S}_n.$$

Note that $\omega_n(x_n)^{p_n}\in A_{p_n,\mathscr{B}_n}(\mathbb{R}^{d_n})$. It follows from the above inequality that

$$\int_{\mathbb{R}^{d_n}} f_n(x_n)^{p_n}\omega_n(x_n)^{p_n}dx_n\leq C\int_{\mathbb{R}^{d_n}} g_n(x_n)^{p_n}\omega_n(x_n)^{p_n}dx_n,\ (f_n,g_n)\in\mathcal{S}_n.$$

Thus

$$\begin{aligned}
&(\int_{\mathbb{R}^{d_n}}\cdots(\int_{\mathbb{R}^{d_1}}\mathcal{M}_s f(x_1,\cdots,x_n)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\omega_n(x_n)^{p_n}dx_n)^{\frac{1}{p_n}}\\
\leq &C(\int_{\mathbb{R}^{d_n}}\cdots(\int_{\mathbb{R}^{d_1}}|f(x_1,\cdots,x_n)|^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\cdots\omega_n(x_n)^{p_n}dx_n)^{\frac{1}{p_n}}.
\end{aligned}$$

This completes the induction step.

And therefore, we have the following remark.

Remark 2.3 Let $\mathscr{B}_i=\mathscr{C}_i$, $i=1,2,\cdots,n$, in the assumption of Corollary 2.2, then $\mathscr{C}_i\times\mathscr{C}_{i+1}$ is a Muckenhoupt basis in $\mathbb{R}^{d_i}\times\mathbb{R}^{d_{i+1}}$, $i=1,2,\cdots,n-1$.

In order to prove Theorem 1.2, we need make some preparations.

It was well known that Muckenhoupt and Wheeden [11, Theorem 1] proved the following lemma.

Lemma 2.4 For every weight $\omega\in A_{\infty,\mathscr{C}}(\mathbb{R}^d)$ and $0<q<\infty$,

$$\int_{\mathbb{R}^d}|Rf(x)|^q\omega(x)dx\leq C\int_{\mathbb{R}^d}M^{(\alpha)}f(x)^q\omega(x)dx.$$

Also, we need the following theorem proved in [9].

Theorem 2.5 Given a family of extrapolation pairs $\mathcal{F}$, let $\mathscr{B}$ be a Muckenhoupt basis. Suppose that for some p_0, $0<p_0<\infty$, and every $w_0\in A_{\infty,\mathscr{B}}$,

$$\int_{\mathbb{R}^d} f(x)^{p_0}w_0(x)dx\leq C\int_{\mathbb{R}^d} g(x)^{p_0}w_0(x)dx,\qquad (f,g)\in\mathcal{F}. \tag{2.2}$$

Then for every p, $0 < p < \infty$, and every $w \in A_{\infty,\mathscr{B}}$,

$$\int_{\mathbb{R}^d} f(x)^p w(x)dx \leq C \int_{\mathbb{R}^d} g(x)^p w(x)dx, \qquad (f,g) \in \mathcal{F}.$$

Then, we can prove Theorem 1.2 in the following.

Proof of Theorem 1.2 Fix $0 < p_1 < \infty$ and $\omega_1(x_1)^{p_1} \in A_{\infty,\mathscr{C}_1}(\mathbb{R}^{d_1})$, let

$$\begin{aligned} \mathcal{S}_2 = \Big\{&(f_2,g_2):\ f_2(x_2) = (\int_{\mathbb{R}^{d_1}} Rf(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{1}{p_1}},\\ &g_2(x_2) = (\int_{\mathbb{R}^{d_1}} M^{(\alpha)}f(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{1}{p_1}}\Big\}. \end{aligned}$$

If $p_2 = p_1$, then, for all $\omega_2(x_2)^{p_2} \in A_{\infty,\mathscr{C}_2}(\mathbb{R}^{d_2})$, we have

$$\omega(x_1,x_2)^{p_1} \triangleq \omega_1(x_1)^{p_1}\omega_2(x_2)^{p_2} \in A_{\infty,\mathscr{C}}(\mathbb{R}^d).$$

It follows from Lemma 2.4 that

$$\begin{aligned} &(\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} Rf(x_1,x_2)^{p_1}\omega(x_1,x_2)^{p_1}dx_1dx_2)^{\frac{1}{p_1}}\\ \leq\ & C(\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} M^{(\alpha)}f(x_1,x_2)^{p_1}\omega(x_1,x_2)^{p_1}dx_1dx_2)^{\frac{1}{p_1}}. \end{aligned}$$

Then, we have

$$\int_{\mathbb{R}^{d_2}} f_2(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2 \leq C\int_{\mathbb{R}^{d_2}} g_2(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2,\ (f_2,g_2) \in \mathcal{S}_2.$$

Thus, we check (2.2) of Theorem 2.5 with $\mathcal{F} \triangleq \mathcal{S}_2$ and $p_0 \triangleq p_2$. Using Theorem 2.5, we get, for every $0 < p_2 < \infty$ and every $v_2(x_2) \in A_{\infty,\mathscr{C}_2}(\mathbb{R}^{d_2})$,

$$\int_{\mathbb{R}^{d_2}} f_2(x_2)^{p_2}v_2(x_2)dx_2 \leq C\int_{\mathbb{R}^{d_2}} g_2(x_2)^{p_2}v_2(x_2)dx_2,\ (f_2,g_2) \in \mathcal{S}_2.$$

Note that $\omega_2(x_2)^{p_2} \in A_{\infty,\mathscr{C}_2}(\mathbb{R}^{d_2})$. It follows from the above inequality that

$$\int_{\mathbb{R}^{d_2}} f_2(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2 \leq C\int_{\mathbb{R}^{d_2}} g_2(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2,\ (f_2,g_2) \in \mathcal{S}_2.$$

Thus

$$\begin{aligned} &(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}} Rf(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}\\ \leq\ & C(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}} M^{(\alpha)}f(x_1,x_2)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{1}{p_1}}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}. \end{aligned}$$

At last, we give the proof of Theorem 1.3. First, we should remind that Cruz-Uribe, Martell and Pérez proved the following proposition in [8, Proposition 3.5], which is a special case of Theorem 1.3.

Proposition 2.6 [8, Proposition 3.5] For every weight $\omega \in A_{\infty,\mathscr{C}_1\times\mathscr{C}_2}(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$,

$$\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}}|Tf(x_1,x_2)|\omega(x_1,x_2)dx_1dx_2 \le C\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} M_1^{(1)}\circ M_2^{(1)}f(x_1,x_2)\omega(x_1,x_2)dx_1dx_2.$$

Now, we give the proof of Theorem 1.3 as follows.

Proof of Theorem 1.3 In view of the definition, we have

$$\begin{aligned}
Tf(x_1,x_2) &= \int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}\frac{f(\bar{x}_1,\bar{x}_2)}{|x_1-\bar{x}_1|^{d_1-1}|x_2-\bar{x}_2|^{d_2-1}}d\bar{x}_1)d\bar{x}_2\\
&= \int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}f(\bar{x}_1,\bar{x}_2)\frac{d\bar{x}_1}{|x_1-\bar{x}_1|^{d_1-1}})\frac{d\bar{x}_2}{|x_2-\bar{x}_2|^{d_2-1}}\\
&\le \int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}|f(\bar{x}_1,\bar{x}_2)|\frac{d\bar{x}_1}{|x_1-\bar{x}_1|^{d_1-1}})\frac{d\bar{x}_2}{|x_2-\bar{x}_2|^{d_2-1}}.
\end{aligned}$$

Using Minkowski's integral inequality, we obtain that

$$\begin{aligned}
&(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}|Tf(x_1,x_2)|^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}\\
\le&\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_1}}\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_1}}|f(\bar{x}_1,\bar{x}_2)|\frac{d\bar{x}_1}{|x_1-\bar{x}_1|^{d_1-1}}\Big)\frac{d\bar{x}_2}{|x_2-\bar{x}_2|^{d_2-1}}\Big)^{p_1}\\
&\times\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2\Big)^{\frac{1}{p_2}}\\
\le&\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_2}}\Big(\int_{\mathbb{R}^{d_1}}\Big(\int_{\mathbb{R}^{d_1}}|f(\bar{x}_1,\bar{x}_2)|\frac{d\bar{x}_1}{|x_1-\bar{x}_1|^{d_1-1}}\Big)^{p_1}\\
&\times\omega_1(x_1)^{p_1}dx_1\Big)^{\frac{1}{p_1}}\frac{d\bar{x}_2}{|x_2-\bar{x}_2|^{d_2-1}}\Big)^{p_2}\omega_2(x_2)^{p_2}dx_2\Big)^{\frac{1}{p_2}}.
\end{aligned}$$

Combining this estimate with Lemma 2.4, we have that

$$\begin{aligned}
&(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}|Tf(x_1,x_2)|^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}\\
\le&C(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}M_1^{(1)}f(x_1,\bar{x}_2)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{1}{p_1}}\frac{d\bar{x}_2}{|x_2-\bar{x}_2|^{d_2-1}})^{p_2}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}.
\end{aligned}$$

Using Lemma 2.4 again, we prove that

$$\begin{aligned}
&(\int_{\mathbb{R}^{d_2}}(\int_{\mathbb{R}^{d_1}}|Tf(x_1,x_2)|^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{p_2}{p_1}}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}\\
\le&C(\int_{\mathbb{R}^{d_2}}(M_2^{(1)}(\int_{\mathbb{R}^{d_1}}M_1^{(1)}f(x_1,\bar{x}_2)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{1}{p_1}})(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}\\
=&C(\int_{\mathbb{R}^{d_2}}(M_2^{(1)}(\int_{\mathbb{R}^{d_1}}M_1^{(1)}f(x_1,\cdot)^{p_1}\omega_1(x_1)^{p_1}dx_1)^{\frac{1}{p_1}})(x_2)^{p_2}\omega_2(x_2)^{p_2}dx_2)^{\frac{1}{p_2}}.
\end{aligned}$$

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具有Muckenhoupt基的乘积空间上的混合加权不等式

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摘要: 本文研究了在具有Muckenhoupt基的乘积空间上混合加权问题. 利用外插函数族和Minkowski不等式获得了混合加权不等式. 根据数学归纳法, 可以得到混合加权不等式的一般形式.

关键词: 混合范数Lebesgue空间; Muckenhoupt基; 外插

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