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THE SHOCK SOLUTION FOR A CLASS OF NONLINEAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

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Abstract: In this paper, the shock solution for a class of nonlinear singularly perturbed differential equation is considered. Using the method of matched asymptotic expansions, the asymptotic expression of problem is constructed and the uniform validity of asymptotic solution is also proved by the theory of differential inequalities.

Keywords: nonlinear problem; shock wave; boundary layer; matching

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1 Introduction

Singular perturbation theory is a vast and rich ongoing area of exploration for mathematicians,physicists,and other researchers. There are various methods which are used to tackle problems in this field. The more basics of these include the boundary layer method, the methods of matched asymptotic expansion, the method of averaging and multiple scales. During the past decade, many scholars such as O'Malley [1] and Bohé [2], Nayfeh [3] and Howes [4] did a great deal of work. Some domestic scholars such as Jiang [5], Mo [6–11], Ni [12], Tang [13], Han [14], Chen [15] etc. also studied a class of nonlinear boundary value problems for the reaction diffusion equations, a class of activator inhibitor system, the shock wave, the soliton, the laser pulse and the problems of atmospheric physics and so on. The shock wave is an important behavior of solution to singularly perturbed problems. The shock wave of solution implies that the function produces a rapid change and comes into being the shock layer, as an independent variable near the boundary or some interior points of the interval. And the location of shock layer has strong sensitivity with the domain of boundary value. In quantum mechanics, hydrodynamics and electro magnetics, there are many models whose solutions possess the shock behavior. In this paper, we construct asymptotic solution for nonlinear singular perturbed boundary value problems and obtain some expressions of shock solutions, and prove it's uniformly valid.

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We consider the following nonlinear singular perturbed boundary value problems

$$\varepsilon y'' + yy' = f(x, \varepsilon y'), \quad x \in (0,1) \tag{1}$$

$$y(0) = \alpha, \tag{2}$$

$$y(1) = \beta, \tag{3}$$

where ε are positive constants and $0 < \varepsilon \ll 1$,$\alpha < 0 < \beta$. We need the following hypotheses

[H1] $f(x,y) > 0$ is sufficiently smooth with respect to their arguments in corresponding domains and $f_y(x,y) < -\delta < 0$, where δ is a positive constant;

[H2] $0 < \beta^2 - \alpha^2 < 2\int_0^1 f(x,0)dx$.

2 The Outer and Inner Solutions

The reduced equation of (1) is

$$yy' = f(x,0) \tag{4}$$

from the hypotheses, there exists a unique solution to (4),

$$y_0^l(x) = -\sqrt{2\int_0^x f(t,0)dt + \alpha^2} \tag{5}$$

or

$$y_0^r(x) = \sqrt{2\int_x^1 f(t,0)dt + \beta^2}, \tag{6}$$

where $y_0^l(x)$ and $y_0^r(x)$, respectively, satisfy the left and the right boundary value conditions (2), (3). And clearly, because $y_0^l(x) \neq y_0^r(x)$, there may exist a shock solution in interior point of the interval $(0,1)$.

Let the stretched variable

$$\xi = \frac{x - x^*}{\varepsilon^\nu}, \tag{7}$$

where x^* is the shock location, and ν is a positive constant which will be determined below. Substituting (7) into (1), we have

$$\frac{d^2y}{d\xi^2} + \varepsilon^{\nu-1} y \frac{dy}{d\xi} - \varepsilon^{2\nu-1} f\big(\varepsilon^\nu \xi + x^*, \varepsilon^{1-\nu}\frac{dy}{d\xi}\big) = 0,$$

noting that the special limit can be obtained as $\nu = 1$, for inner solution is Y and we have

$$\frac{d^2Y}{d\xi^2} + Y\frac{dY}{d\xi} - \varepsilon f\big(\varepsilon\xi + x^*, \frac{dY}{d\xi}\big) = 0, \tag{8}$$

we assume the inner solution has the form $Y = \sum\limits_{j=0}^{\infty} Y_j(\xi)\varepsilon^j$, so we have the equation satisfied by the first order approximation Y_0,

$$\frac{d^2Y_0}{d\xi^2} + Y_0\frac{dY_0}{d\xi} = 0, \tag{9}$$

which upon integration gives

$$\frac{dY_0}{d\xi} = \frac{1}{2}(c_1 - Y_0^2), \tag{10}$$

where c_1 is a constant of integration. It must be positive, otherwise $\lim_{\xi\to\pm\infty} Y_0 = \mp\infty$ making unmatchable with outer expansions of the solution. Then, separating variables and integrating (10) gives two branches of the first order inner solution Y_0,

$$Y_0 = k\tanh\left(\frac{1}{2}k(\xi + c_2)\right), \ \ Y_0^2 \le k^2 \tag{11}$$

and

$$Y_0 = k\coth\left(\frac{1}{2}k(\xi + c_2)\right), \ \ Y_0^2 \ge k^2, \tag{12}$$

where c_1 is replaced with k^2, c_2 the constant of integration. We note that k maybe taken to be positive because tanh and coth are odd. The constants k and c_2 in either form of the inner expansion need to be determined.

3 Matching the Inner and Outer Solutions

By the matching principle, we can determine k and c_2 from matching the inner and outer solutions. For the shock location x^* in the interval $(0,1)$, combining (5) with (6), we can obtain the zero-th order outer solution to problem (1) with (2), (3)

$$y_0(x) = \begin{cases} y_0^l, & 0 \le x \le x^*, \\ y_0^r, & x^* \le x \le 1. \end{cases} \tag{13}$$

Since the outer solutions must increases from $y_0^l(x^*)$ to $y_0^r(x^*)$, thus from (11) and (12) we can obtain the interior solution must be (11), matching it with the zero-th order outer solution (13). For the left zero-th order outer solution and the interior solution, note that as $\varepsilon \to 0$, $\xi = \frac{x-x^*}{\varepsilon} \to -\infty$. Clearly, the outer limit for left side of the interior solution is

$$(y_0^l)^i = \lim_{x\to x^*}\left(-\sqrt{2\int_0^x f(t,0)dt + \alpha^2}\right) = -\sqrt{2\int_0^{x^*} f(t,0)dt + \alpha^2},$$

the interior limit for the outer solution is

$$(Y_0)^o = \lim_{\xi\to-\infty} Y_0 = \lim_{\xi\to-\infty} k\tanh\left(\frac{1}{2}k(\xi + c_2)\right) = -k.$$

From matching principle, we have

$$k = \sqrt{2\int_0^{x^*} f(t,0)dt + \alpha^2}. \tag{14}$$

Similarly, for the right zero-th order outer solution and the interior solution. Note that as $\varepsilon \to 0$, $\xi = \frac{x-x^*}{\varepsilon} \to +\infty$, the outer limit for right side of the interior solution is

$$(y_0^r)^i = \lim_{x \to x^*} \left(\sqrt{2\int_x^1 f(t,0)dt + \beta^2} \right) = \sqrt{2\int_{x^*}^1 f(t,0)dt + \beta^2},$$

the interior limit for the outer solution is

$$(Y_0)^o = \lim_{\xi \to +\infty} Y_0 = \lim_{\xi \to +\infty} k \tanh\left(\frac{1}{2}k(\xi + c_2)\right) = k.$$

From matching principle, we have

$$k = \sqrt{2\int_{x^*}^1 f(t,0)dt + \beta^2}. \tag{15}$$

Comparing with (14), (15), we have

$$\int_0^{x^*} f(t,0)dt + \int_1^{x^*} f(t,0)dt = \frac{\beta^2 - \alpha^2}{2}. \tag{16}$$

Using the zero theorem and monotonicity, we can prove that (16) has a uniqueness solution x^*, where the shock location x^* can be determined from (16). From the character of the shock location x^*, it is not difficult to see that $c_2 = 0$ and

$$k = \sqrt{2\int_0^{x^*} f(t,0)dt + \alpha^2} > 0.$$

So the zero-th order interior solution is $Y_0 = k\tanh(\frac{k}{2}\xi)$. Then the boundary value problem (1) with (2), (3), exists a solution, and the solution can be asymptotically expanded as

$$y(x) = y_0^l(x) + k\Big(\tanh\Big(\frac{k}{2}(\frac{x-x^*}{\varepsilon})\Big) + 1\Big) + \cdots, 0 < \varepsilon \ll 1, x \in [0, x^*], \tag{17}$$

$$y(x) = y_0^r(x) + k\Big(\tanh\Big(\frac{k}{2}(\frac{x-x^*}{\varepsilon})\Big) - 1\Big) + \cdots, 0 < \varepsilon \ll 1, x \in [x^*, 1]. \tag{18}$$

4 Uniform Validity of the Asymptotic Solution

We have the following theorem.

Theorem Under hypotheses [H1]–[H2], there exists a solution y of the nonlinear singular perturbed boundary value problems (1)–(3), and the solution y can be expanded into the uniformly valid asymptotic expansion

$$y(x) = y_0(x) + Y_0 - (Y_0)^o + O(\varepsilon), 0 < \varepsilon \ll 1, x \in [0, x^*] \cup (x^*, 1],$$

where

$$y_0(x) = \begin{cases} y_0^l, & 0 \le x \le x^*, \\ y_0^r, & x^* \le x \le 1, \end{cases}$$

$$Y_0 = k \tanh\Big(\frac{k}{2}(\frac{x-x^*}{\varepsilon})\Big), \ (Y_0)^o = \begin{cases} -k, & 0 \le x \le x^*, \\ k, & x^* \le x \le 1. \end{cases}$$

Proof The theorem includes two estimates

$$y(x) = y_0^l(x) + k\Big(\tanh\Big(\frac{k}{2}(\frac{x-x^*}{\varepsilon})\Big) + 1\Big) + O(\varepsilon), 0 < \varepsilon \ll 1, x \in [0, x^*], \tag{19}$$

$$y(x) = y_0^r(x) + k\Big(\tanh\Big(\frac{k}{2}(\frac{x-x^*}{\varepsilon})\Big) - 1\Big) + O(\varepsilon), 0 < \varepsilon \ll 1, x \in [x^*, 1]. \tag{20}$$

Now we prove estimate (19). Similarly, we can prove (20). We use the theory of differential inequalities, first we construct the auxiliary functions $\underline{y}(x,\varepsilon)$ and $\overline{y}(x,\varepsilon)$,

$$\underline{y}(x,\varepsilon) = y_0^l(x) + Y_0 - (Y_0)^o - \gamma\varepsilon, \tag{21}$$

$$\overline{y}(x,\varepsilon) = y_0^l(x) + Y_0 - (Y_0)^o + \gamma\varepsilon, \tag{22}$$

where $x \in [0, x^*]$,

$$y_0^l(x) = -\sqrt{2\int_0^x f(t,0)dt + \alpha^2},$$
$$Y_0 = k\tanh\Big(\frac{k}{2}(\frac{x-x^*}{\varepsilon})\Big), (Y_0)^o = -k,$$

γ is a large enough positive constant to be chosen below. Obviously, we have

$$\underline{y}(x,\varepsilon) \le \overline{y}(x,\varepsilon) \tag{23}$$

and

$$\underline{y}(0,\varepsilon) \le \alpha \le \overline{y}(0,\varepsilon), \underline{y}(x^*,\varepsilon) \le 0 \le \overline{y}(x^*,\varepsilon). \tag{24}$$

Now we prove that

$$\varepsilon\underline{y}'' + \underline{y}\ \underline{y}' - f(x, \varepsilon\underline{y}') \ge 0, x \in [0, x^*], \tag{25}$$

$$\varepsilon\overline{y}'' + \overline{y}\ \overline{y}' - f(x, \varepsilon\overline{y}') \le 0, x \in [0, x^*]. \tag{26}$$

From hypotheses [H1], [H2], and considering the character of the tanh, there exists a positive constant M, such that

$$\begin{aligned}
& \varepsilon\overline{y}'' + \overline{y}\ \overline{y}' - f(x, \varepsilon\overline{y}') \\
= \ & \varepsilon\frac{d^2}{dx^2}\big(y_0^l(x) + Y_0 - (Y_0)^o + \gamma\varepsilon\big) + \big(y_0^l(x) + Y_0 - (Y_0)^o + \gamma\varepsilon\big)\frac{d}{dx}\big(y_0^l(x) + Y_0 - (Y_0)^o + \gamma\varepsilon\big) \\
& -f\Big(x, \varepsilon\frac{d}{dx}\big(y_0^l(x) + Y_0 - (Y_0)^o + \gamma\varepsilon\big)\Big) \\
= \ & M\varepsilon + \frac{1}{\varepsilon}\Big(\frac{d^2Y_0}{d\xi^2} + Y_0\frac{dY_0}{d\xi}\Big) + \Big(y_0^l(x)\frac{d(y_0^l(x))}{dx} - f(x,0)\Big) + \big(Y_0 - (Y_0)^o + \gamma\varepsilon\big)\frac{d(y_0^l(x))}{dx} \\
& + \frac{1}{\varepsilon}\big(y_0^l(x) - (Y_0)^o + \gamma\varepsilon\big)\frac{dY_0}{d\xi} - \Big(f\Big(x, \varepsilon\frac{d\big(y_0^l(x) + Y_0 - (Y_0)^o + \gamma\varepsilon\big)}{dx}\Big) - f(x,0)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= M\varepsilon + \big(Y_0 - (Y_0)^o + \gamma\varepsilon\big)\frac{d(y_0^l(x))}{dx} - f_y'(x,\zeta)\Big(\varepsilon\frac{d(y_0^l(x))}{dx} + \varepsilon\frac{dY_0}{dx}\Big)\\
&\quad + \frac{1}{\varepsilon}\big(y_0^l(x) - (Y_0)^o + \gamma\varepsilon\big)\frac{dY_0}{d\xi}\\
&= M\varepsilon + \big(Y_0 - (Y_0)^o + \gamma\varepsilon - \varepsilon f_y'(x,\zeta)\big)\frac{d(y_0^l(x))}{dx}\\
&\quad + \frac{1}{\varepsilon}\big(y_0^l(x) - (Y_0)^o + \gamma\varepsilon - \varepsilon f_y'(x,\zeta)\big)\frac{dY_0}{d\xi},
\end{aligned}
$$

where constant

$$
\zeta \in \left(0, \varepsilon\frac{d\big(y_0^l(x)\big)}{dx} + \varepsilon\frac{dY_0}{dx}\right)
$$

from hypothesis [H1], $f_y'(x,y) < -\delta < 0$ and

$$
y_0^l(x) = -\sqrt{2\int_0^x f(t,0)dt + \alpha^2}, Y_0 = k\tanh\left(\frac{k}{2}\big(\frac{x-x^*}{\varepsilon}\big)\right).
$$

There exist a positive constant ρ,

$$
\frac{d(y_0^l(x))}{dx} \le -\rho < 0,
$$

so selecting $\gamma \ge \frac{M}{\rho} - \delta$, $\varepsilon\overline{y}'' + \overline{y}\ \overline{y}' - f(x,\varepsilon\overline{y}') \le M\varepsilon - \rho(\gamma+\delta)\varepsilon \le 0$.

We prove inequality (26). Similarly, we can prove inequality (25) too. Thus from inequalities (23)–(26), by using the theorem of differential inequalities, there is a solution $y(x)$ of problems (1)–(3), such that

$$
\underline{y}(x,\varepsilon) \le y(x,\varepsilon) \le \overline{y}(x,\varepsilon),\ \ x \in [0,x^*],
$$

then we have equation (19). Similarly, we can prove (20). The proof of the theorem is completed.

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一类非线性奇摄动边值问题的激波解

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摘 要: 本文研究了一类非线性奇摄动微分方程的激波解. 利用匹配渐近展开法, 构造了问题的解的渐近展开式, 并利用微分不等式理论, 证明了解的一致有效性.

关键词: 非线性; 激波; 边值; 匹配

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