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WEIGHTED NORM INEQUALITIES FOR ANISOTROPIC FRACTIONAL INTEGRAL OPERATORS

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Abstract: Let A be an expansive dilation, $\alpha \in (0, 1)$, $p := 1/\alpha$ and function v satisfy the anisotropic Muckenhoupt condition $\mathbb{A}_{p,\infty}(A)$. In this paper, we study the boundedness of anisotropic fractional integral operators. By $L(p, \infty)$ Hölder's inequality and the σ-subaddictive property of $\|\cdot\|_{p',1}$, we obtain some weighted norm inequalities for anisotropic fractional integral operators associated with the weight v^p, which are anisotropic extension of Muckenhoupt and Wheeden [6].

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1 Introduction

Anisotropy is a common attribute of nature, which shows different characterizations in different directions of all or part of the physical or chemical properties of an object. For example, the elastic modulus, hardness or fracture strength of a crystal is different in different directions, which shows the anisotropic property of the crystal. The anisotropic property, in mathematics, can be expressed by a general discrete group of dilations $\{A^k : k \in \mathbb{Z}\}$, where A is a real $n \times n$ matrix with all its eigenvalues λ satisfying $|\lambda| > 1$, which was first introduced by Bownik [1] and who further introduced the anisotropic Hardy spaces [2]. We point out that such spaces include the classical isotropic Hardy spaces of Fefferman-Stein [3], the parabolic Hardy spaces of Calderón-Torchinsky [4, 5], and still maintain the main properties of the corresponding classical Hardy spaces.

Fractional integrals played an important role in harmonic analysis and other fields, such as PDE (see [6, 7]). Many scholars devoted to research the properties of fractional integrals (for example, see [8–10]). The celebrated result of fractional integrals is the Hardy-Littlewood-Sobolev inequality (see [11]). Hardy and Littlewood [12] proved that when $n = 1$,

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fractional integral operator is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$, where $\alpha \in (0, 1)$, $p \in (1, \infty)$ and $q := (1/p - \alpha)^{-1}$, and Sobolev [13] obtained that for general n this result also holds true. The weighted $(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$-boundedness of fractional integral operator was established by Muckenhoupt and Wheeden [6], where $\alpha \in (0, n)$, $p \in (1, n/\alpha)$ and $q := (1/p - \alpha/n)^{-1}$. Ding and Lan [14] introduced the anisotropic fractional integral operator and Lan et al. [15] generalized the result of Muckenhoupt and Wheeden [6] to the anisotropic settings except the case $q = \infty$.

Motivated by [1, 6, 15], we generalize the result of Muckenhoupt and Wheeden when $q = \infty$ [6, Theorems 7 and 8] (see Theorem 2.8 below). It is worth pointing out that any Schwartz function is an anisotropic fractional integral kernel (see Remark 2.6 (ii) (b) below). Moreover, we also obtain that if v^{-1} is locally bounded, then T_α is bounded from $L^{p,\infty}_{v^p}(\mathbb{R}^n)$ to the anisotropic bounded mean oscillation function space $\mathrm{BMO}(A)$ (see Theorem 2.10 below), which is probably new even for classical fractional integral operator of Sobolev [13].

This article is organized as follows.

In Section 2, we recall some notations and definitions concerning expansive dilations, anisotropic Muckenhoupt condition $\mathbb{A}_{p,\,q}(A)$, anisotropic fractional integral operator and $\mathrm{BMO}(A)$ space and state the main results, the proofs of which are given in Section 3.

Finally, we make some conventions on notations. Let $\mathbb{Z}_+ := \{1, 2, \cdots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $D \lesssim F$ means that $D \le CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is simply denoted by $\mathcal{X}$. For example, $L^p(\mathbb{R}^n)$ is simply denoted by L^p. For sets $E, F \subset \mathbb{R}^n$, we use $E^\complement$ to denote the set $\mathbb{R}^n \setminus E$, χ_E its characteristic function and $E + F$ the algebraic sum $\{x + y : \ x \in E, y \in F\}$. For any index $q \in [1, \infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$.

2 Notion and Main Results

First we recall the notion of expansive dilations on $\mathbb{R}^n$, see [1, p. 5]. A real $n \times n$ matrix A is called an expansive dilation, shortly a dilation, if $\min\limits_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all eigenvalues of A. Let λ_- and λ_+ be two positive numbers such that

$$1 < \lambda_- < \min\{|\lambda| : \ \lambda \in \sigma(A)\} \le \max\{|\lambda| : \ \lambda \in \sigma(A)\} < \lambda_+.$$

In the case when A is diagonalizable over $\mathbb{C}$, we can even take

$$\lambda_- := \min\{|\lambda| : \ \lambda \in \sigma(A)\} \ \text{and} \ \lambda_+ := \max\{|\lambda| : \ \lambda \in \sigma(A)\}.$$

Otherwise, we need to choose them sufficiently close to these equalities according to what we need in our arguments.

It was proved in [1, p. 5, Lemma 2.2] that, for a given dilation A, there exist a number $r \in (1, \infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : \ |Px| < 1\}$, where P is some non-degenerate $n \times n$

matrix, such that $\Delta \subset r\Delta \subset A\Delta$, and by a scailing, one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-dimensional Lebesgue measure of the set Δ. Let $B_k := A^k\Delta$ for $k \in \mathbb{Z}$. Then B_k is open, convex, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$, here and hereafter, $b := |\det A|$. Throughout the whole paper, let σ be the minimum positive integer such that $2B_0 \subset A^\sigma B_0 = B_\sigma$. Then, for all $k,\, j \in \mathbb{Z}$ with $k \le j$, it holds true that

$$B_k + B_j \subset B_{j+\sigma}, \tag{2.1}$$

$$B_k + (B_{j+\sigma})^{\complement} \subset (B_j)^{\complement}. \tag{2.2}$$

Definition 2.1 A quasi-norm, associated with an expansive matrix A, is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$, for simplicity, denoted by ρ, satisfying

(i) $\rho(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$, here and hereafter, $\mathbf{0}_n := (0, \cdots, 0)$;

(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$, where, as above, $b := |\det A|$;

(iii) $\rho(x + y) \le H[\rho(x) + \rho(y)]$ for all $x,\, y \in \mathbb{R}^n$, where $H \in [1, \infty)$ is a constant independent of x and y.

In the standard dyadic case $A := 2\mathrm{I}_{n\times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$ is an example of quasi-norms associated with A, here and hereafter, $|\cdot|$ always denotes the Euclidean norm in $\mathbb{R}^n$.

It was proved in [1, p. 6, Lemma 2.4] that all quasi-norms associated with a given dilation A are equivalent. Therefore, for a given expansive dilation A, in what follows, for simplicity, we always use the step quasi-norm ρ defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{k\in\mathbb{Z}} b^k \chi_{B_{k+1}\setminus B_k}(x) \text{ if } x \ne \mathbf{0}_n, \quad \text{or else} \quad \rho(\mathbf{0}_n) := 0.$$

By (2.1) and (2.2), we know that, for all $x,\, y \in \mathbb{R}^n$,

$$\rho(x + y) \le b^\sigma \left(\max\{\rho(x),\, \rho(y)\}\right) \le b^\sigma[\rho(x) + \rho(y)].$$

Moreover, $(\mathbb{R}^n,\, \rho,\, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [16, 17], where dx denotes the n-dimensional Lebesgue measure.

In what follows, for convenience, we always define

$$\mathcal{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}.$$

Definition 2.2 Let

$$\alpha \in (0,\, 1),\;\; p \in (1,\, 1/\alpha),\;\; p' := p/(p-1),\;\; q := (1/p - \alpha)^{-1}.$$

A function $v : \mathbb{R}^n \to [0,\, \infty)$ is said to satisfy the anisotropic Muckenhoupt condition $\mathbb{A}_{p,\,q}(A)$, denoted by $v \in \mathbb{A}_{p,\,q}(A)$, if there exists a positive constant C such that, for any $B \in \mathcal{B}$,

$$\left\{\frac{1}{|B|}\int_B [v(x)]^q\,dx\right\}^{1/q}\left\{\frac{1}{|B|}\int_B [v(x)]^{-p'}\,dx\right\}^{1/p'} \le C,$$

and when $q = \infty$,

$$\left[\operatorname*{ess\,sup}_{x\in B} v(x)\right]\left\{\frac{1}{|B|}\int_B [v(x)]^{-p'}\,dx\right\}^{1/p'} \le C.$$

Definition 2.3 Let $p \in (0, \infty]$ and $\omega : \mathbb{R}^n \to [0, \infty)$ be a measurable function. The weighted Lebesgue space L^p_ω is defined to be the space of all measurable functions f such that

$$\|f\|_{L^p_\omega} := \left[\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx\right]^{1/p} < \infty.$$

The weighted weak Lebesgue space $L^{p,\infty}_\omega$ is defined to be the space of all measurable functions f such that

$$\|f\|_{L^{p,\infty}_\omega} := \sup_{\lambda>0} \lambda\left[\omega\left(\{x \in \mathbb{R}^n :\ |f(x)| > \lambda\}\right)\right]^{1/p} < \infty,$$

here and here after, $\omega(E) := \int_E \omega(x)\,dx$ for any subset $E \subset \mathbb{R}^n$.

Denote the space of all Schwartz functions on $\mathbb{R}^n$ by $\mathcal{S}$, namely, the set of all C^∞ functions ϕ satisfying that, for any $\alpha \in \mathbb{N}^n$ and any $\ell \in \mathbb{N}$,

$$\|\phi\|_{\alpha,\,\ell} := \sup_{x\in\mathbb{R}^n} |\partial^\alpha \phi(x)|[1 + \rho(x)]^\ell < \infty.$$

The dual space of $\mathcal{S}$, namely, the space of all tempered distributions, equipped with the weak-$*$ topology, is denoted by $\mathcal{S}'$.

Remark 2.4 By [1, p. 11, Lemma 3.2], we know that the Schwartz function space $\mathcal{S}$, equipped with the pseudo-norms $\{\|\cdot\|_{\alpha,\,\ell}\}_{\alpha\in\mathbb{N}^n,\,\ell\in\mathbb{N}}$, is equivalent to the classical Schwartz function space, equipped with the pseudo-norms $\{\|\cdot\|^*_{\alpha,\,\ell}\}_{\alpha\in\mathbb{N}^n,\,\ell\in\mathbb{N}}$, where, for any $\alpha \in \mathbb{N}^n$, $\ell \in \mathbb{N}$ and any $\phi \in \mathcal{S}$, $\|\phi\|^*_{\alpha,\,\ell} := \sup_{x\in\mathbb{R}^n} |\partial^\alpha \phi(x)|\left[1 + |x|^2\right]^{\ell/2}$.

Now we recall the definition of anisotropic fractional integral operators associated with a quasi-norm ρ, which comes from [15].

Definition 2.5 Let $\alpha \in (0, 1)$. A locally integrable function K on $\Omega := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \ne y\}$ is said to be an anisotropic fractional integrable kernel (with respect to a dilation A and a quasi-norm ρ) if there exist positive constants C and γ such that

(i) for all $(x, y) \in \Omega$, $|K(x, y)| \le \frac{C}{[\rho(x-y)]^{1-\alpha}}$;

(ii) if $(x, y) \in \Omega$ and $x' \in \mathbb{R}^n$ with $\rho(x - y) \ge b^{2\sigma}\rho(x' - x)$, then

$$|K(x', y) - K(x, y)| \le C\frac{[\rho(x' - x)]^\gamma}{[\rho(x - y)]^{1-\alpha+\gamma}}; \tag{2.3}$$

(iii) if $(x, y) \in \Omega$ and $y' \in \mathbb{R}^n$ with $\rho(x - y) \ge b^{2\sigma}\rho(y' - y)$, then

$$|K(x, y') - K(x, y)| \le C\frac{[\rho(y' - y)]^\gamma}{[\rho(x - y)]^{1-\alpha+\gamma}}.$$

We call that T_α is an anisotropic fractional integrable operator if T_α is a continuous linear operator maps $\mathcal{S}$ into $\mathcal{S}'$ and there exists an anisotropic fractional integrable kernel K such that, for all $f \in C_c^\infty$ and $x \notin \operatorname{supp} f$,

$$T_\alpha f(x) := \int_{\mathbb{R}^n} K(x,y)f(y)\,dy.$$

In particular, if $K(x,\, y) := [\rho(x-y)]^{-1+\alpha}$ with $(x,y) \in \Omega$ and $\alpha \in (0,\, 1)$, we denote the corresponding fractional integral operator by T_α^ρ.

Remark 2.6 (i) If $K(x,\, y) = 1/|x-y|^{n-\alpha}$ with $(x,\, y) \in \Omega$ and $\alpha \in (0,\, n)$, then T_α is reduced to the classical fractional integral operator [12] for the one dimensional case and [13] for the n dimensional case.

(ii) If T_α is a fractional integral operator with convolutional kernel $K(x)$ on $\mathbb{R}^n \setminus \{\mathbf{0}_n\}$, then the above conditions (ii) and (iii) are reduced to

$$|K(x-y) - K(x)| \le C\frac{[\rho(y)]^\gamma}{[\rho(x)]^{1-\alpha+\gamma}} \quad \text{when} \quad \rho(x) \ge b^{2\sigma}\rho(y). \tag{2.4}$$

We give three examples for this case.

(a) if $\alpha \in (0,\, n)$, $\rho(x) := |x|^n$ and $K(x) := 1/|x|^{n-\alpha}$ with $x \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$, then $K(x)$ is a convolutional kernel. Moreover, Lan [15, p. 4] pointed out that, for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$ and $\alpha \in (0,\, 1)$, $K(x) := [\rho(x)]^{-1+\alpha}$ is also a convolutional kernel, or see [18, Lemma 2.3] for more details.

(b) if $K \in \mathcal{S}$, then the operator $T_\alpha : \mathcal{S} \to \mathcal{S}'$ with such convolutional kernel K on $\mathbb{R}^n \setminus \{\mathbf{0}_n\}$ is a fractional integral operator with $\alpha \in (0,\, 1)$ and $\gamma \in (0,\, \log_b(\lambda_-)]$. In fact, Definition 2.5(i) is obvious, we just need to prove (2.4) with $\alpha \in (0,\, 1)$ and $\gamma \in (0,\, \log_b(\lambda_-)]$. There exist some $l \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that $\rho(y) = b^l$ and $\rho(x) = b^{l+m+2\sigma}$. By the mean value theorem, we have

$$|K(x-y) - K(x)| \le |y| \sup_{\theta\in(0,\,1)} |\nabla K(x-\theta y)| \lesssim \sup_{\theta\in(0,\,1)} \frac{|y|}{[1+\rho(x-\theta y)]^M},$$

where $M \in (0,\, \infty)$ to be fixed later. If $l \le 0$, for $x \in (B_{l+m+2\sigma})^{\complement}$ and $\theta y \in B_{l+1}$ (since $y \in B_{l+1}$ and B_{l+1} is convex), by $\sigma > 1$, $m \ge 0$ and (2.2), we obtain

$$x - \theta y \in (B_{l+m+2\sigma})^{\complement} - B_{l+1} \subset (B_{l+m+\sigma})^{\complement},$$

which implies that

$$\rho(x-\theta y) \ge b^{l+m+\sigma}. \tag{2.5}$$

From this, $|y| \lesssim [\rho(y)]^{\log_b(\lambda_-)}$ when $\rho(y) \le 1$ (see [1, p. 11, Lemma 3.2]) and $m \in \mathbb{N}$, it follows that, for any $\gamma \in (0,\, \log_b(\lambda_-)]$ and by choosing $M = 1 - \alpha + \log_b(\lambda_-)$,

$$\frac{|y|}{[1+\rho(x-\theta y)]^M} \lesssim \frac{b^{l\log_b(\lambda_-)}}{b^{M(l+m+\sigma)}} \lesssim \frac{1}{b^{(l+m)(1-\alpha)}}\left(\frac{b^l}{b^{l+m}}\right)^{\log_b(\lambda_-)} \lesssim \frac{[\rho(y)]^\gamma}{[\rho(x)]^{1-\alpha+\gamma}}.$$

If $l > 0$, by (2.5), $|y| \lesssim [\rho(y)]^{\log_b(\lambda_+)}$ when $\rho(y) > 1$ (see [1, p. 11. Lemma 3.2]), $m \in \mathbb{N}$ and $\log_b(\lambda_+) \geq \log_b(\lambda_-)$, we see that, for any $\gamma \in (0, \log_b(\lambda_-)]$ and by choosing $M = 1 - \alpha + \log_b(\lambda_+)$,

$$\frac{|y|}{[1+\rho(x-\theta y)]^M} \lesssim \frac{b^{l\log_b(\lambda_+)}}{b^{M(l+m+\sigma)}} \lesssim \frac{1}{b^{(l+m)(1-\alpha)}}\left(\frac{b^l}{b^{l+m}}\right)^{\log_b(\lambda_-)} \lesssim \frac{[\rho(y)]^\gamma}{[\rho(x)]^{1-\alpha+\gamma}}.$$

Definition 2.7 A locally integrable function b is said to be in anisotropic bounded mean oscillation function space $\mathrm{BMO}(A)$ if

$$\|b\|_{\mathrm{BMO}(A)} := \sup_{B\in\mathcal{B}} \frac{1}{|B|}\int_B |b(x) - b_B|\, dx < \infty,$$

where $b_B := \frac{1}{|B|}\int_B b(x)\, dx$.

The main results of this article are the following two theorems, the proofs of which are given in Section 3.

Theorem 2.8 Let $\alpha \in (0, 1)$, $p := 1/\alpha$ and $v(x)$ be a nonnegative function on $\mathbb{R}^n$. If $v \in \mathbb{A}_{p,\infty}(A)$, then there exists a positive constant C such that, for every $B \in \mathcal{B}$,

$$\left[\operatorname*{ess\,sup}_{x\in B} v(x)\right] \frac{1}{|B|}\int_B |T_\alpha f(x) - (T_\alpha f)_B|\, dx \leq C\|f\|_{L^{p,\infty}_{v^p}}. \tag{2.6}$$

In particular, when $T_\alpha = T^\rho_\alpha$ satisfies (2.6) if and only if $v \in \mathbb{A}_{p,\infty}(A)$.

Remark 2.9 Lan [15, Theorems 1.3 and 1.4] obtained the weighted boundedness of anisotropic fractional singular integral operator associated with $\mathbb{A}_{p,\,q}(A)$, where $q < \infty$. The above Theorem 2.8 considers the case $q = \infty$, which is also an anisotropic extension of Muckenhoupt and Wheeden [6, Theorems 7 and 8].

Theorem 2.10 Let $\alpha \in (0, 1)$, $p := 1/\alpha$ and $v(x)$ be a nonnegative function on $\mathbb{R}^n$. If v^{-1} is locally bounded, then T_α is bounded from $L^{p,\infty}_{v^p}$ to $\mathrm{BMO}(A)$.

3 Proofs of Theorems 2.8 and 2.10

Proof of Theorem 2.8 Let $\alpha \in (0, 1)$. Fix a dilated ball $B := x_0 + B_j \in \mathcal{B}$ with some $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, and $B^* := x_0 + B_{j+4\sigma}$. Then

$$T_\alpha f(x) = \int_{B^*} f(y)K(x, y)\, dy + \int_{(B^*)^\complement} f(y)K(x, y)\, dy =: \mathrm{I}_1 + \mathrm{I}_2$$

and

$$\begin{aligned}
&\left[\operatorname*{ess\,sup}_{x\in B} v(x)\right] \frac{1}{|B|}\int_B |T_\alpha f(x) - (T_\alpha f)_B|\, dx \\
&\leq \left[\operatorname*{ess\,sup}_{x\in B} v(x)\right] \frac{1}{|B|}\int_B \left|\mathrm{I}_1 - (\mathrm{I}_1)_B\right|\, dx + \left[\operatorname*{ess\,sup}_{x\in B} v(x)\right] \frac{1}{|B|}\int_B \left|\mathrm{I}_2 - (\mathrm{I}_2)_B\right|\, dx \\
&= : \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}$$

To prove (2.6) holds true, it is sufficient to prove $\mathrm{II}_1 \lesssim \|f\|_{L^{p,\infty}_{v^p}}$ and $\mathrm{II}_2 \lesssim \|f\|_{L^{p,\infty}_{v^p}}$.

Let $E := \operatorname*{ess\,sup}_{x\in B} v(x)$. By (2.1) with $x \in B$ and $y \in B^*$, we have $x-y \in B-B^* \subset B_{j+5\sigma}$. From this and Definition 2.5 (i), we deduce that

$$\begin{aligned}
\mathrm{II}_1 &\le \frac{E}{|B|}\int_B\int_{B^*}|f(y)K(x,\,y)|\,dy\,dx + \frac{E}{|B|}\int_B \frac{1}{|B|}\int_B\int_{B^*}|f(y)K(z,\,y)|\,dy\,dz\,dx\\
&\lesssim \frac{E}{|B|}\int_{B^*}|f(y)|\int_B |K(x,\,y)|\,dx\,dy \lesssim \frac{E}{|B|}\int_{B^*}|f(y)|\int_{B_{j+5\sigma}}\frac{1}{[\rho(z)]^{1-\alpha}}\,dz\,dy\\
&\sim \frac{E}{|B|}\int_{B^*}|f(y)|\sum_{i=0}^{\infty}\int_{B_{j+5\sigma-i}\setminus B_{j+5\sigma-i-1}}\frac{1}{[\rho(z)]^{1-\alpha}}\,dz\,dy\\
&\lesssim b^{j\alpha}\sum_{i=0}^{\infty}b^{-i\alpha}\frac{E}{|B|}\int_{B^*}|f(y)|\,dy \lesssim E|B|^{-1+\alpha}\int_{B^*}|f(y)|\,dy.
\end{aligned}$$

By $p = 1/\alpha$, $1/p' = 1-\alpha$, $L(p,\,\infty)$ Hölder's inequality (see [19, p. 262]), $\|\cdot\|_{p',1} \sim \|\cdot\|^*_{p',1}$ (see [19, (2.2)]), $\|\chi_{B^*}/v\|_{p',1} \lesssim \{\int_{B^*}[v(x)]^{-p'}dx\}^{1/p'}$, the proof of which is identity to that of [6, Lemma 2], and $v \in \mathbb{A}_{p,\infty}(A)$, we see that

$$\begin{aligned}
E|B|^{-1+\alpha}&\int_{B^*}|f(y)|dy \lesssim \frac{E}{|B|^{1/p'}}\left\|\frac{\chi_{B^*}}{v}\right\|^*_{p',1}\|f\|_{L^{p,\infty}_{v^p}}\\
&\lesssim \frac{E}{|B|^{1/p'}}\left\|\frac{\chi_{B^*}}{v}\right\|_{p',1}\|f\|_{L^{p,\infty}_{v^p}} \lesssim \frac{E}{|B|^{1/p'}}\left\{\int_{B^*}[v(x)]^{-p'}dx\right\}^{1/p'}\|f\|_{L^{p,\infty}_{v^p}}\\
&\lesssim \left[\sup_{B^*\in\mathcal{B}} v(x)\right]\left\{\frac{1}{|B^*|}\int_{B^*}[v(x)]^{-p'}dx\right\}^{1/p'}\|f\|_{L^{p,\infty}_{v^p}} \lesssim \|f\|_{L^{p,\infty}_{v^p}},
\end{aligned}\tag{3.1}$$

which implies that $\mathrm{II}_1 \lesssim \|f\|_{L^{p,\infty}_{v^p}}$.

By (2.1) and (2.2) with $x,\,z \in B$ and $y \in (B^*)^{\complement}$, we see that $x-z \in B_{j+\sigma}$ and $x-y \in (B_{j+3\sigma})^{\complement}$, which implies that $\rho(x-z) < b^{j+\sigma}$ and $\rho(x-y) \ge b^{j+3\sigma}$ and hence $\rho(x-y) \ge b^{2\sigma}\rho(x-z)$. From this, (2.3) and $\rho(x-z) \le b^{j+\sigma}$, we deduce that

$$\begin{aligned}
\mathrm{II}_2 &= \frac{E}{|B|}\int_B\left|\frac{1}{|B|}\int_B\left\{\int_{(B^*)^{\complement}} f(y)[K(x,\,y)-K(z,\,y)]\,dy\right\}dz\right|dx\\
&\lesssim \frac{E}{|B|}\int_B\frac{1}{|B|}\int_B\int_{(B^*)^{\complement}}|f(y)|\frac{[\rho(x-z)]^{\gamma}}{[\rho(x-y)]^{1-\alpha+\gamma}}\,dy\,dz\,dx\\
&\lesssim \frac{Eb^{\gamma j}}{|B|}\int_B\int_{(B^*)^{\complement}}\frac{|f(y)|}{[\rho(x-y)]^{1-\alpha+\gamma}}\,dy\,dx.
\end{aligned}$$

By $\rho(x-y) \ge b^{j+3\sigma}$ and $\rho(x_0-x) < b^j$, we obtain $\rho(x-y) > b^{3\sigma}\rho(x-x_0)$. Thus we have

$$\rho(x_0-y) \le b^{\sigma}[\rho(x_0-x)+\rho(x-y)] \lesssim \rho(x-y).$$

From this, $L(p,\,\infty)$ Hölder's inequality (see [19, p. 262]) and the fact $\|\cdot\|_{p',1} \sim \|\cdot\|^*_{p',1}$ (see

[19, (2.2)]), it follows that

$$\begin{aligned}&\frac{Eb^{\gamma j}}{|B|}\int_B\int_{(B^*)^\complement}\frac{|f(y)|}{[\rho(x-y)]^{1-\alpha+\gamma}}\,dy\,dx\lesssim Eb^{\gamma j}\int_{(B^*)^\complement}\frac{|f(y)|}{[\rho(x_0-y)]^{1-\alpha+\gamma}}\,dy\\ \lesssim&Eb^{\gamma j}\|f\|_{L^{p,\infty}_{vp}}\left\|\frac{\chi_{(B^*)^\complement}}{v[\rho(x_0-\cdot)]^{1-\alpha-\gamma}}\right\|^*_{p',1}\lesssim Eb^{\gamma j}\|f\|_{L^{p,\infty}_{vp}}\left\|\frac{\chi_{(B^*)^\complement}}{v[\rho(x_0-\cdot)]^{1-\alpha-\gamma}}\right\|_{p',1}.\end{aligned}\tag{3.2}$$

Then in order to obtain $\mathrm{II}_2\lesssim\|f\|_{L^{p,\infty}_{vp}}$, it suffices to show that

$$Eb^{\gamma j}\left\|\frac{\chi_{(B^*)^\complement}}{v[\rho(x_0-\cdot)]^{1-\alpha-\gamma}}\right\|_{p',1}\lesssim 1,$$

where the implicit constant is independent of B.

Let $k\in\mathbb{N}$ and $B_k^*:=x_0+B_{j+4\sigma+k}$. Since $\|\cdot\|_{p',1}$ satisfies the σ-subaddictive property, by $\|\chi_{B_k^*}/v\|_{p',1}\lesssim\{\int_{B_k^*}[v(x)]^{-p'}dx\}^{1/p'}$, the proof of which is identity to that of [6, Lemma 2], and $v\in\mathbb{A}_{p,\infty}(A)$, we see that

$$\begin{aligned}&Eb^{\gamma j}\left\|\frac{\chi_{(B^*)^\complement}}{v[\rho(x_0-\cdot)]^{1-\alpha-\gamma}}\right\|_{p',1}\lesssim Eb^{\gamma j}\sum_{k=1}^\infty\left\|\frac{\chi_{B_k^*\setminus B_{k-1}^*}}{vb^{(j+\sigma+k)(1-\alpha+\gamma)}}\right\|_{p',1}\\ \lesssim&E\sum_{k=1}^\infty b^{-k\gamma}|B_k^*|^{-1/p'}\left\|\frac{\chi_{B_k^*}}{v}\right\|_{p',1}\lesssim\sum_{k=1}^\infty b^{-k\gamma}\left[\operatorname*{ess\,sup}_{x\in B_k^*}v(x)\right]\left\{\frac{1}{|B_k^*|}\int_{B_k^*}[v(x)]^{-p'}\right\}^{1/p'}\lesssim 1,\end{aligned}\tag{3.3}$$

which implies that $\mathrm{II}_2\lesssim\|f\|_{L^{p,\infty}_{vp}}$.

Finally, it remains to prove that if $T_\alpha=T_\alpha^\rho$ as in Definition 2.5 satisfies (2.6), then $v\in\mathbb{A}_{p,\infty}(A)$. To prove this, we first show that there exists a constant $k_0\in\mathbb{Z}_+$ such that, for every $B:=x_0+B_j\in\mathcal{B}$ and every $y\in B$,

$$\int_{\widetilde{B}_{k_0}}\frac{1}{[\rho(x-y)]^{1-\alpha}}\,dx\le\frac{b^{k_0}}{2}\int_B\frac{1}{[\rho(x-y)]^{1-\alpha}}\,dx,\tag{3.4}$$

where $\widetilde{B}_{k_0}:=x_0+B_{j+k_0}$.

By (2.1) with $x\in\widetilde{B}_{k_0}$ and $y\in B$, we have $x-y\in\widetilde{B}_{k_0}-B\subset B_{j+k_0+\sigma}$. From this, it follows that

$$\begin{aligned}\int_{\widetilde{B}_{k_0}}\frac{1}{[\rho(x-y)]^{1-\alpha}}\,dx&\le\int_{B_{j+k_0+\sigma}}\frac{1}{[\rho(z)]^{1-\alpha}}\,dz=\sum_{i=0}^\infty\int_{B_{j+k_0+\sigma-i}\setminus B_{j+k_0+\sigma-i-1}}\frac{1}{[\rho(z)]^{1-\alpha}}\,dz\\&\le\ b^{\alpha(j+k_0+\sigma)-\alpha+1}\sum_{i=0}^\infty b^{-i\alpha}=\frac{b^{\alpha(j+k_0+\sigma)+1}}{b^\alpha-1}.\end{aligned}\tag{3.5}$$

By (2.1) with $x,\,y\in B$, we have $x-y\in B_{j+\sigma}$, which implies that

$$\rho(x-y)<b^{j+\sigma}.\tag{3.6}$$

Then we have

$$\int_B \frac{1}{[\rho(x-y)]^{1-\alpha}}\,dx > b^{(j+\sigma)(\alpha-1)+j} = b^{\alpha(j+\sigma)-\sigma}.$$

Therefore, from this and (3.5), it follows that

$$\int_{\widetilde{B}_{k_0}} \frac{1}{[\rho(x-y)]^{1-\alpha}}\,dx \le \frac{b^{1+\sigma+\alpha k_0}}{b^\alpha-1}\int_B \frac{1}{[\rho(x-y)]^{1-\alpha}}\,dx.$$

Since $\alpha\in(0,\,1)$, by choosing k_0 sufficiently large, we see that

$$\frac{b^{1+\sigma+\alpha k_0}}{(b^\alpha-1)} \le \frac{1}{2}b^{k_0}$$

and hence (3.4) holds true.

Now fix a ball $B\in\mathcal{B}$ and let f be a nonnegative integrable function with $\operatorname{supp} f\subset B$ and k_0 being as in (3.4). Then, by Fubini's theorem and (3.4), we have

$$\begin{aligned}
&(T^\rho_\alpha f)_B-(T^\rho_\alpha f)_{\widetilde{B}_{k_0}}\\
=&\frac{1}{|B|}\int_B\int_B \frac{f(y)}{[\rho(x-y)]^{1-\alpha}}\,dy\,dx-\frac{b^{-k_0}}{|B|}\int_{\widetilde{B}_{k_0}}\int_B \frac{f(y)}{[\rho(x-y)]^{1-\alpha}}\,dy\,dx\\
=&\frac{1}{|B|}\int_B f(y)\left\{\int_B \frac{1}{[\rho(x-y)]^{1-\alpha}}\,dx-b^{-k_0}\int_{\widetilde{B}_{k_0}} \frac{1}{[\rho(x-y)]^{1-\alpha}}dx\right\}dy\\
\ge&\frac{1}{2|B|}\int_B f(y)\int_B \frac{1}{[\rho(x-y)]^{1-\alpha}}\,dx\,dy=\frac{1}{2}(T^\rho_\alpha f)_B.
\end{aligned}$$

Therefore, by Minkowski's inequality, (2.6) and $\operatorname{supp} f\subset B$, we see that

$$\begin{aligned}
(T^\rho_\alpha f)_B &\le 2\left[(T^\rho_\alpha f)_B-(T^\rho_\alpha f)_{\widetilde{B}_{k_0}}\right]\\
&\lesssim \frac{1}{|B|}\int_B |(T^\rho_\alpha f)_B-T^\rho_\alpha f(x)|\,dx+\frac{1}{|B|}\int_B \left|T^\rho_\alpha f(x)-(T^\rho_\alpha f)_{\widetilde{B}_{k_0}}\right|\,dx\\
&\lesssim \frac{1}{|B|}\int_B |(T^\rho_\alpha f)_B-T^\rho_\alpha f(x)|\,dx+\frac{1}{|\widetilde{B}_{k_0}|}\int_{\widetilde{B}_{k_0}} \left|T^\rho_\alpha f(x)-(T^\rho_\alpha f)_{\widetilde{B}_{k_0}}\right|\,dx\\
&\lesssim \left[\operatorname*{ess\,sup}_{x\in B} v(x)\right]^{-1}\left\{\int_B |f(x)|^p[v(x)]^p\,dx\right\}^{1/p},
\end{aligned}$$

namely,

$$(T^\rho_\alpha f)_B \lesssim \left[\operatorname*{ess\,sup}_{x\in B} v(x)\right]^{-1}\left\{\int_B |f(x)|^p[v(x)]^p\,dx\right\}^{1/p}. \tag{3.7}$$

Now, if $\int_B [v(x)]^{-p'}dx=0$, $v\in\mathbb{A}_{p,\infty}(A)$ is immediate since $0\cdot\infty=0$.

If $0<\int_B [v(x)]^{-p'}dx<\infty$, by (3.6), we obtain

$$\begin{aligned}&\frac{1}{|B|^{1-\alpha}}\int_B [v(y)]^{-p'}\,dy=\frac{1}{|B|}\int_B\int_B \frac{[v(y)]^{-p'}}{|B|^{1-\alpha}}\,dy\,dx\\ \lesssim&\frac{1}{|B|}\int_B\int_B \frac{[v(y)]^{-p'}}{[\rho(x-y)]^{1-\alpha}}\,dy\,dx\sim\left(T_\alpha^\rho(v^{-p'}\chi_B)\right)_B.\end{aligned}\tag{3.8}$$

Then combining (3.8) and (3.7) with $f(x)=[v(x)]^{-p'}\chi_B(x)$, we have

$$\frac{1}{|B|^{1-\alpha}}\int_B [v(y)]^{-p'}\,dy\lesssim\left[\operatorname*{ess\,sup}_{x\in B} v(x)\right]^{-1}\left\{\int_B [v(x)]^{p(1-p')}\,dx\right\}^{1/p},$$

which, together with $p(p'-1)=p'$, implies that

$$\frac{1}{|B|^{1-\alpha}}\left\{\int_B [v(y)]^{-p'}\,dy\right\}^{1/p'}\lesssim\left[\operatorname*{ess\,sup}_{x\in B} v(x)\right]^{-1}$$

and hence $v\in\mathbb{A}_{p,\infty}(A)$.

If $\int_B [v(x)]^{-p'}dx=\infty$, then $v^{-1}\notin L^{p'}(B)$, so there exists a negative function $g\in L^p(B)$ such that $\int_B g(x)/v(x)dx=\infty$. In fact, suppose on the contrary that, for any $g\in L^p(B)$, $\int_B g(x)/v(x)dx<\infty$, then $v^{-1}\in(L^p(B))^*=L^{p'}(B)$, which is contradict with $\int_B [v(x)]^{-p'}dx=\infty$, which is desired. Let $f:=(g/v)\chi_B$. Then, for any $x\in B$, we claim that $(T_\alpha^\rho f)_B=\infty$. In fact, for any $x,y\in B$, by (2.1), we have $x-y\in B_{j+\sigma}$. From this and $\int_B g(y)/v(y)dy=\infty$, we obtain

$$(T_\alpha^\rho f)_B=\frac{1}{|B|}\int_B\int_B\frac{g(y)/v(y)}{[\rho(x-y)]^{1-\alpha}}\,dy\,dx>b^{(j+\sigma)(\alpha-1)}\int_B g(y)/v(y)\,dy=\infty,$$

which is desired. Then

$$\begin{aligned}&\left\{\frac{1}{|B|}\int_B [v(x)]^{-p'}\,dx\right\}^{1/p'}=(T_\alpha^\rho f)_B\\ \lesssim&\left[\operatorname*{ess\,sup}_{x\in B} v(x)\right]^{-1}\left\{\int_B |f(x)|^p[v(x)]^p\,dx\right\}^{1/p}\sim\left[\operatorname*{ess\,sup}_{x\in B} v(x)\right]^{-1}\|g\|_{L^p(B)}\end{aligned}$$

and hence

$$\left[\operatorname*{ess\,sup}_{x\in B} v(x)\right]\left\{\frac{1}{|B|}\int_B [v(x)]^{-p'}\,dx\right\}^{1/p'}\lesssim\|g\|_{L^p(B)},$$

which implies that $v\in\mathbb{A}_{p,\infty}(A)$. This finishes the proof of Theorem 2.8.

Proof of Theorem 2.10 The proof of Theorem 2.10 is similar to that of Theorem 2.8. So we use the same notations as in the proof of Theorem 2.8. Since

$$\begin{aligned}&\frac{1}{|B|}\int_B |T_\alpha f(x)-(T_\alpha f)_B|\ dx\\ \le&\frac{1}{|B|}\int_B |\mathrm{I}_1-(\mathrm{I}_1)_B|\ dx+\frac{1}{|B|}\int_B |\mathrm{I}_2-(\mathrm{I}_2)_B|\ dx=:\mathrm{I}+\mathrm{II}.\end{aligned}$$

Then, when v^{-1} is locally bounded, we only need to prove that, for any $B\in\mathcal{B}$, $\mathrm{I}\lesssim\|f\|_{L^{p,\infty}_{vp}}$ and $\mathrm{II}\lesssim\|f\|_{L^{p,\infty}_{vp}}$.

For I, by (3.1) and $v^{-1}\in L^\infty_{\mathrm{loc}}$, we see that

$$\mathrm{I}\lesssim\|f\|_{L^{p,\infty}_{vp}}\left\{\frac{1}{|B^*|}\int_{B^*}[v(x)]^{-p'}\,dx\right\}^{1/p'}\lesssim\|f\|_{L^{p,\infty}_{vp}}. \tag{3.9}$$

For II, by (3.2), (3.3), $\gamma\in(0,\infty)$ and $v^{-1}\in L^\infty_{\mathrm{loc}}$, we see that

$$\mathrm{II}\lesssim\|f\|_{L^{p,\infty}_{vp}}\sum_{k=1}^\infty b^{-k\gamma}\left\{\frac{1}{|B_k^*|}\int_{B_k^*}[v(x)]^{-p'}\,dx\right\}^{1/p'}\lesssim\|f\|_{L^{p,\infty}_{vp}},$$

which, together with (3.9), completes the proof of Theorem 2.10.

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各向异性分数次积分算子的加权范数不等式

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摘 要: 设 A 是一个扩张矩阵, $\alpha \in [0,\, 1)$, $p := 1/\alpha$ 且函数v满足各向异性Muckenhoupt $\mathbb{A}_{p,\,\infty}(A)$ 权条件. 本文研究了各向异性分数次积分算子的有界性的问题. 利用$L(p,\, \infty)$ 空间的 Hölder 不等式和范数 $\|\cdot\|_{p',\,1}$ 的 σ-次可加性得到了各向异性分数次积分算子关于权v^p的一些加权范数不等式. 这些结果是 Muckenhoupt 和 Wheeden 的结果[6] 在各向异性情形下的推广.

关键词: 各向异性; Muckenhoupt 权; 分数次积分算子; BMO 空间

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