Vol. 38 (2018)
No. 4

数 学 杂 志
J. of Math. (PRC)

FROM LEIBNIZ SUPERALGEBRAS TO LIE-YAMAGUTI SUPERALGEBRAS

TANG Xin-xin[1], ZHANG Qing-cheng[1], WANG Chun-yue[2]

(*1.School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China*)

(*2.School of Media and Mathematics and Physics, Jilin Engineering Normal University, Changchun 130052, China*)

Abstract: In this paper, we study the construction of Lie-Yamaguti superalgebras. By using left Leibniz superalgebras, we give the construction of left Leibniz superalgbebras, then give the construction of Lie-Yamaguti superalgebras from left Leibniz superalgebras. So we gain the construction of Lie-Yamaguti superalgebras, which generalizes the construction of Lie-Yamaguti algebras in the situation of superalgebras.

Keywords: Lie-Yamaguiti superlagebras; (left) Leibniz superlagebras; Akivis superalgebras; Lie supertriple systems; construction

2010 MR Subject Classification: 17A30; 17A32; 17D99

Document code: A **Article ID:** 0255-7797(2018)04-0589-13

1 Introduction

Lie algebras were studied for many years in mathematics and physics, such as in quantum field theory. As the noncommutative analogs of Lie algebras, Leibniz algebras were first introduced by Cuvier and Loday in [1] and [2]. Researchers obtained many results about Leibniz algebras and we can find some of them in [3–6]. There are two kinds of Leibniz algebras, left Leibniz algebras and right Leibniz algebras [7]. For a given non-commutative algebra $(A,\cdot)$, if the left multiplication $l_x\cdot y=x\cdot y,\forall x,y\in A$ is a derivation of A, then $(A,\cdot)$ is called a left Leibniz algebra [8]. As non-associative algebras, left Leibniz algebras can construct Akivis algebras [9]. Kinyon and Weinstein found that a left Leibniz algebra has a Lie-Yamaguti algebra structure by using an enveloping Lie algebra of Leibniz algebras.

Recently, Leibniz algebras are generalized to Leibniz superalgebras by Dzhumadil in [10]. Then some important results were obtained such as [11] and [12]. Like left Leibniz algebras and right Leibniz algebras, we can similarly obtain left Leibniz superalgebras and right Leibniz superalgebras. If $(A,\cdot)$ is a left Leibniz superalgebra, we can obtain a right Leibniz superalgebra $(A,\circ)$ by defining $x\circ y=(-1)^{|x||y|}y\cdot x$. In this paper, we study the construction of left Leibniz superalgebras and Lie-Yamaguti superalgebras.

* **Received date:** 2017-03-03 **Accepted date:** 2017-04-26

Foundation item: Supported by the NSFC (11471090).

Biography: Tang Xinxin (1993–), female, Man, born at Changchun, Jilin, master, major in Lie algebra.

This paper is organized as follows. In Section 2, we recall the definition of Leibniz superalgebras and prove that every non-associative superalgebra has an Akivis superalgebra structure. Then we give examples and constructions of left Leibniz superalgebras. In Section 3, we define Lie-Yamaguti superalgebras and prove that every left Leibniz superalgebra has a Lie-Yamaguti superalgebra structure.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero; All vector spaces and algebras are over $\mathbb{K}$; $hg(A)$ denotes the set of homogeneous elements of the superalgebra A.

2 Leibniz Superalgebras

In this section, we introduce the definition of Leibniz superslgebras, and then give the constructions and examples of Leibniz superalgebras.

Definition 2.1 [11] (i) A (left) Leibniz superalgebra is a pair $(A,\cdot)$, in which A is a superspace, $\cdot : A\times A\to A$ an even bilinear map such that

$$\begin{aligned}&|x\cdot y|=|x|+|y|,\\ &x\cdot(y\cdot z)=(x\cdot y)\cdot z+(-1)^{|x||y|}y\cdot(x\cdot z)\end{aligned}\tag{2.1}$$

for all $x,y,z\in hg(A)$.

(ii) A (right) Leibniz superalgebra is a pair $(A,\cdot)$, in which A is a superspace, $\cdot : A\times A\to A$ an even bilinear map such that

$$\begin{aligned}&|x\cdot y|=|x|+|y|,\\ &(x\cdot y)\cdot z=(-1)^{|y||z|}(x\cdot z)\cdot y+x\cdot(y\cdot z)\end{aligned}\tag{2.2}$$

for all $x,y,z\in hg(A)$.

In this paper, “Leibniz superalgebras” means “left Leibniz superalgebras”. A super skew-symmetric Leibniz superalgebra is a Lie superalgebra. In this case, equations (2.1) and (2.2) become the Jacobi super-identity. If $(A,\cdot)$ is a left Leibniz superalgebra, we can obtain a right Leibniz superalgebra $(A,\circ)$ by defining $x\circ y=(-1)^{|x||y|}y\cdot x$.

Definition 2.2 Let $(A,\cdot)$ be a superalgebra.

(i) The super-associator of A is an even trilinear map $\widetilde{as}: A\times A\times A\to A$ defined by

$$\widetilde{as}(x,y,z)=(x\cdot y)\cdot z-x\cdot(y\cdot z)\tag{2.3}$$

for all $x,y,z\in hg(A)$.

If $\widetilde{as}(x,y,z)=0$ for all $x,y,z\in hg(A)$, then $(A,\cdot)$ is said to be associatives.

(ii) The super-Jacobian of A is an even trilinear map $\widetilde{J}: A\times A\times A\to A$ defined by

$$\widetilde{J}(x,y,z)=\circlearrowleft_{x,y,z}(-1)^{|x||z|}(x\cdot y)\cdot z.$$

Remark 2.3 A not necessarily associative superalgebra is called a non-associative superalgebra. That is to say, $\widetilde{as}(x,y,z)\neq 0$ for some $x,y,z\in hg(A)$.

Definition 2.4 [13] An Akivis superalgebra is a triple $(A, \circ, [-,-,-])$, in which A is a superspace, $\circ : A \times A \to A$ an even bilinear map, $[-,-,-] : A \times A \times A \to A$ an even trilinear map such that

$$|x \circ y| = |x| + |y|, \quad |[x, y, z]| = |x| + |y| + |z|,$$
$$x \circ y = -(-1)^{|x|+|y|} y \circ x, \tag{2.4}$$
$$\widetilde{J}(x, y, z) = \circlearrowleft_{x,y,z} (-1)^{|x||z|}[x, y, z] - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}[y, x, z] \tag{2.5}$$

for all $x, y, z \in hg(A)$. Equation (2.5) is called the Akivis super-identity.

Theorem 2.5 Every non-associative superalgebra $(A, \cdot)$ has an Akivis superalgebra $(A, \circ, [-,-,-])$ structure with respect to the operation defined by

$$x \circ y = x \cdot y - (-1)^{|x||y|} y \cdot x, \tag{2.6}$$
$$[x, y, z] = \widetilde{as}(x, y, z) \tag{2.7}$$

for all $x, y, z \in hg(A)$.

Proof First, we proceed to verify that " $\circ$ " is super skew-symmetric.

$$\begin{aligned} -(-1)^{|x||y|} y \circ x &= -(-1)^{|x||y|}(y \cdot x - (-1)^{|x||y|} x \cdot y) \\ &= x \cdot y - (-1)^{|x||y|} y \cdot x = x \circ y. \end{aligned}$$

So we obtain equation (2.4).

Second, consider the Akivis super-identity. On one hand,

$$\begin{aligned} & \circlearrowleft_{x,y,z} (-1)^{|x||z|}[x, y, z] - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}[y, x, z] \\ = \; & \circlearrowleft_{x,y,z} (-1)^{|x||z|}\widetilde{as}(x, y, z) - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}\widetilde{as}(x, y, z) \\ = \; & \circlearrowleft_{x,y,z} (-1)^{|x||z|}((x \cdot y) \cdot z - x \cdot (y \cdot z)) - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}((y \cdot x) \cdot z - y \cdot (x \cdot z)). \end{aligned}$$

On the other hand,

$$\begin{aligned} \widetilde{J}(x, y, z) &= \circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \circ y) \circ z \\ &= \circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \cdot y - (-1)^{|x||y|} y \cdot x) \circ z \\ &= \circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \cdot y) \circ z - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}(y \cdot x) \circ z \\ &= \circlearrowleft_{x,y,z} (-1)^{|x||z|}((x \cdot y) \cdot z - (-1)^{|z|(|x|+|y|)} z \cdot (x \cdot y)) \\ &\quad - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}((y \cdot x) \cdot z - (-1)^{|z|(|x|+|y|)} z \cdot (y \cdot x)) \\ &= \circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \cdot y) \cdot z - \circlearrowleft_{x,y,z} (-1)^{|y||z|} z \cdot (x \cdot y) \\ &\quad - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}(y \cdot x) \cdot z + \circlearrowleft_{x,y,z} (-1)^{|y|(|x|+|z|)} z \cdot (y \cdot x) \\ &= \circlearrowleft_{x,y,z} (-1)^{|x||z|}((x \cdot y) \cdot z - x \cdot (y \cdot z)) \\ &\quad - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}((y \cdot x) \cdot z - y \cdot (x \cdot z)). \end{aligned}$$

That is

$$\widetilde{J}(x, y, z) = \circlearrowleft_{x,y,z} (-1)^{|x||z|}[x, y, z] - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}[y, x, z].$$

So we get equation (2.5).

An Akivis superalgebra derived from a given non-associative superalgebra A by Theorem 2.5 is said associated with A. We are interested in Akivis superalgebras associated with Leibniz superalgebras.

In terms of equation (2.3), equation (2.1) has the form

$$\widetilde{as}(x,y,z) = -(-1)^{|x||y|} y \cdot (x \cdot z). \tag{2.8}$$

Because the operations of the Akivis superalgebra $(A, \circ, [-,-,-])$ defined by the (left) Leibniz superalgebra $(A, \cdot)$ satisfy the super skew-symmetrization and equation (2.4), the Akivis super-identity (2.5) has the form

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \circ y) \circ z = \circlearrowleft_{x,y,z} (-1)^{|x||z|}\widetilde{as}(x,y,z) - \circlearrowleft_{x,y,z} (-1)^{|x|(|y|+|z|)}\widetilde{as}(y,x,z). \tag{2.9}$$

By equations (2.8) and (2.1), we have $-(-1)^{|x||y|}\widetilde{as}(y,x,z) = (x \cdot y) \cdot z - \widetilde{as}(x,y,z)$. So equaiton (2.9) becomes

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \circ y) \circ z = \circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \cdot y) \cdot z. \tag{2.10}$$

Lemma 2.6 Let $(A, \cdot)$ be a Leibniz superalgebra, and consider on $(A, \cdot)$ the operation $[x, y] := x \cdot y - (-1)^{|x||y|} y \cdot x$ for all $x, y \in hg(A)$. Then

(i)

$$(x \cdot y + (-1)^{|x||y|} y \cdot x) \cdot z = 0; \tag{2.11}$$

(ii)

$$x \cdot [y,z] = [x \cdot y, z] + (-1)^{|x||y|}[y, x \cdot z]. \tag{2.12}$$

Proof (i) Equation (2.1) implies that

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) - (-1)^{|x||y|} y \cdot (x \cdot z). \tag{2.13}$$

Likewise, interchanging x and y, we have

$$(y \cdot x) \cdot z = y \cdot (x \cdot z) - (-1)^{|x||y|} x \cdot (y \cdot z). \tag{2.14}$$

Then, consider

$$\begin{aligned} & (x \cdot y + (-1)^{|x||y|} y \cdot x) \cdot z = (x \cdot y) \cdot z + (-1)^{|x||y|}(y \cdot x) \cdot z \\ = \; & x \cdot (y \cdot z) - (-1)^{|x||y|} y \cdot (x \cdot z) + (-1)^{|x||y|} y \cdot (x \cdot z) - x \cdot (y \cdot z) = 0. \end{aligned}$$

(ii) By calculating directly, we have

$$\begin{aligned} & [x \cdot y, z] + (-1)^{|x||y|}[y, x \cdot z] \\ = \; & (x \cdot y) \cdot z - (-1)^{|z|(|x|+|y|)} z \cdot (x \cdot y) + (-1)^{|x||y|} y \cdot (x \cdot z) - (-1)^{|y||z|}(x \cdot z) \cdot y \\ = \; & x \cdot (y \cdot z) - (-1)^{|z|(|x|+|y|)} z \cdot (x \cdot y) - (-1)^{|y||z|}(x \cdot z) \cdot y \\ = \; & x \cdot (y \cdot z) - (-1)^{|z|(|x|+|y|)}(z \cdot x) \cdot y - (-1)^{|y||z|} x \cdot (z \cdot y) - (-1)^{|y||z|}(x \cdot z) \cdot y \\ = \; & x \cdot (y \cdot z) - (-1)^{|y||z|} x \cdot (z \cdot y) = x \cdot [y,z]. \end{aligned}$$

Lemma 2.7 Let $(A, \cdot)$ be a Leibniz superalgebra, $(A, \circ, [-,-,-])$ be an Akivis superalgebra associated with Leibniz superalgebra $(A, \dot{})$. Then

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \circ y) \circ z = \circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \cdot y) \cdot z.$$

Proof We get the result from equation (2.10).

An superalgebra $(A, \cdot)$ is called Lie-super-admissible if its commutator superalgebra $(A, \circ)$ is a Lie superalgebra. We can obtain following lemma immediatly from Lemma 2.7.

Lemma 2.8 A Leibniz superalgebra $(A, \cdot)$ is Lie-super-admissible if and only if

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}(x \cdot y) \cdot z = 0$$

for all $x, y, z \in hg(A)$.

We now give an example of 3-dimensional Leibniz superalgebra and some methods to construct Leibniz superalgebras. We can find following definitions and similar constructions in [14].

Example 2.9 Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a 3-dimensional superspace. $A_{\bar{0}} = \text{span}\{e_1, e_3\}$, $A_{\bar{1}} = \text{span}\{e_2\}$. The nonzero product is given by $e_2 \cdot e_3 = e_2, e_3 \cdot e_1 = e_1, e_2 \cdot e_2 = -e_2$, $e_3 \cdot e_3 = e_1$. Then $(A, \cdot)$ is a left Leibniz superalgebra.

Proposition 2.10 Let $(A, \circ)$ be an associative superalgebra. Consider the linear map $D : A \to A$ which satisfies

$$D(x \circ D(y)) = D(x) \circ D(y) = D(D(x) \circ y), \forall x, y \in A.$$

Define an even bilinear map $[-,-]_D : A \times A \to A$, such that

$$[x, y]_D = [D(x), y] = D(x) \circ y - (-1)^{|x||y|} y \circ D(x).$$

Then $(A, [-,-]_D)$ is a left Leibniz superalgebra.

Proof We only need to verify that $(A, [-,-]_D)$ is a left Leibniz superalgebra. Calculate directly,

$$\begin{aligned}
& [x, [y, z]_D]_D = [D(x), [D(y), z]] \\
= \ & [D(x), D(y) \circ z - (-1)^{|y||z|} z \circ D(y)] \\
= \ & [D(x), D(y) \circ z] - (-1)^{|y||z|}[D(x), z \circ D(y)] \\
= \ & D(x) \circ D(y) \circ z - (-1)^{|x|(|y|+|z|)} D(y) \circ z \circ D(x) \\
& -(-1)^{|y||z|} D(x) \circ z \circ D(y) + (-1)^{|x|(|y|+|z|)+|y||z|} z \circ D(y) \circ D(x).
\end{aligned}$$

and

$$\begin{aligned}
& [[x, y]_D, z]_D + (-1)^{|x||y|}[y, [x, z]_D]_D \\
= \ & [[D(x), y], z]_D + (-1)^{|x||y|}[D(y), [D(x), z]] \\
= \ & [D(x) \circ y - (-1)^{|x||y|} y \circ D(x), z]_D + (-1)^{|x||y|}[D(y), D(x) \circ z - (-1)^{|x||z|} z \circ D(x)] \\
= \ & [D(x) \circ D(y), z] - (-1)^{|x||y|}[D(y) \circ D(x), z] \\
& +(-1)^{|x||y|}[D(y), D(x) \circ z] - (-1)^{|x|(|y|+|z|)}[D(y), z \circ D(x)]
\end{aligned}$$

$$\begin{aligned} = \ & D(x)\circ D(y)\circ z+(-1)^{|z|(|x|+|y|)+|x||y|}z\circ D(y)\circ D(x) \\ & -(-1)^{|y||z|}D(x)\circ z\circ D(y)-(-1)^{|x|(|y|+|z|)}D(y)\circ z\circ D(x). \end{aligned}$$

So we get $[x,[y,z]_D]_D=[[x,y]_D,z]_D+(-1)^{|x||y|}[y,[x,z]_D]_D$. Therefore $(A,[-,-]_D)$ is a left Leibniz superalgebra.

Definition 2.11 [14] A superdialgebra is a triple $(A,\dashv,\vdash)$, in which A is a superspace, $\dashv,\vdash$: $A\times A\to A$ two bilinear maps such that

(1) $x\vdash(y\dashv z)=(x\vdash y)\dashv z$;

(2) $x\dashv(y\dashv z)=(x\dashv y)\dashv z=x\dashv(y\vdash z)$;

(3) $x\vdash(y\vdash z)=(x\vdash y)\vdash z=(x\dashv y)\vdash z$

for all $x,y,z\in hg(A)$.

Proposition 2.12 Let $(A,\dashv,\vdash)$, be a superdialgebra. Define an even bilinear map $[-,-]:A\times A\to A$ such that $[x,y]=(-1)^{|x||y|}y\dashv x-x\vdash y$. Then $(A,[-,-])$ is a left Leibniz superalgebra.

Proof Calculate directly,

$$\begin{aligned} & [x,[y,z]]=[x,(-1)^{|y||z|}z\dashv y-y\vdash z] \\ = \ & (-1)^{|y||z|}[x,z\dashv y]-[x,y\vdash z] \\ = \ & (-1)^{|y||z|}((-1)^{|x|(|y|+|z|)}(z\dashv y)\dashv x-x\vdash(z\dashv y)) \\ & -((-1)^{|x|(|y|+|z|)}(y\vdash z)\dashv x-x\vdash(y\vdash z)) \\ = \ & (-1)^{|y||z|+|x|(|y|+|z|)}(z\dashv y)\dashv x-(-1)^{|y||z|}x\vdash(z\dashv y) \\ & -(-1)^{|x|(|y|+|z|)}(y\vdash z)\dashv x+x\vdash(y\vdash z) \\ = \ & (-1)^{|y||z|+|x|(|y|+|z|)}z\dashv(y\dashv x)-(-1)^{|y||z|}x\vdash(z\dashv y) \\ & -(-1)^{|x|(|y|+|z|)}y\vdash(z\dashv x)+x\vdash(y\vdash z) \end{aligned}$$

and

$$\begin{aligned} & [[x,y],z]+(-1)^{|x||y|}[y,[x,z]] \\ = \ & [(-1)^{|x||y|}y\dashv x-x\vdash y,z]+(-1)^{|x||y|}[y,(-1)^{|x||z|}z\dashv x-x\vdash z] \\ = \ & (-1)^{|x||y|}[y\dashv x,z]-[x\vdash y,z]+(-1)^{|x|(|y|+|z|)}[y,z\dashv x]-(-1)^{|x||y|}[y,x\vdash z] \\ = \ & (-1)^{|x||y|}((-1)^{|z|(|x|+|y|)}z\dashv(y\dashv x)-(y\dashv x)\vdash z) \\ & -((-1)^{|z|(|x|+|y|)}z\dashv(x\vdash y)-(x\vdash y)\vdash z) \\ & +(-1)^{|x|(|y|+|z|)}((-1)^{|y|(|x|+|z|)}(z\dashv x)\dashv y-y\vdash(z\dashv x)) \\ & -(-1)^{|x||y|}((-1)^{|y|(|x|+|z|)}(x\vdash z)\dashv y-y\vdash(x\vdash z)) \\ = \ & (-1)^{|x||y|+|z|(|x|+|y|)}z\dashv(y\dashv x)+x\vdash(y\vdash z) \\ & -(-1)^{|x|(|y|+|z|)}y\vdash(z\dashv x)-(-1)^{|y||z|}x\vdash(z\dashv y). \end{aligned}$$

So we get $[x,[y,z]]=[[x,y],z]+(-1)^{|x||y|}[y,[x,z]]$. Therefore $(A,[-,-])$ is a left Leibniz superalgebra.

Definition 2.13 [14] A dendriform superalgebra is a triple $(A, \prec, \succ)$, in which A is a superspace, $\prec, \succ: A \times A \to A$ two even bilinear maps such that

(1) $(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z)$;

(2) $x \succ (y \succ z) = (x \prec y) \succ z + (x \succ y) \succ z$;

(3) $(x \succ y) \prec z = x \succ (y \prec z)$

for all $x, y, z \in hg(A)$.

Proposition 2.14 Let $(A, \prec, \succ)$ be a dendriform superalgebra. Define two even bilinear maps $*, [-,-] : A \times A \to A$ such that $x * y = x \prec y + y \succ x, [x, y] = (-1)^{|x||y|} y * x - x * y$. Then $(A, [-,-])$ is a left Leibniz superalgebra.

Proof Calculate directly,

$$\begin{aligned}
[x,[y,z]] \;=\; & (-1)^{|y||z|+|x|(|y|+|z|)}(z \prec y) \prec x + (-1)^{|y||z|+|x|(|y|+|z|)}(y \succ z) \prec x \\
& +(-1)^{|y||z|+|x|(|y|+|z|)} x \succ (z \prec y) + (-1)^{|y||z|+|x|(|y|+|z|)} x \succ (y \succ z) \\
& -(-1)^{|y||z|} x \prec (z \prec y) - (-1)^{|y||z|} x \prec (y \succ z) \\
& -(-1)^{|y||z|}(z \prec y) \succ x - (-1)^{|y||z|}(y \succ z) \succ x \\
& -(-1)^{|x|(|y|+|z|)}(y \prec z) \prec x - (-1)^{|x|(|y|+|z|)}(z \succ y) \prec x \\
& -(-1)^{|x|(|y|+|z|)} x \succ (y \prec z) - (-1)^{|x|(|y|+|z|)} x \succ (z \succ y) \\
& +x \prec (y \prec z) + x \prec (z \succ y) + (y \prec z) \succ x + (z \succ y) \succ x
\end{aligned}$$

and

$$\begin{aligned}
[[x,y],z] \;=\; & (-1)^{|x||y|+|z|(|x|+|y|)} z \prec (y \prec x) + (-1)^{|x||y|+|z|(|x|+|y|)} z \prec (x \succ y) \\
& +(-1)^{|x||y|+|z|(|x|+|y|)}(y \prec x) \succ z + (-1)^{|x||y|+|z|(|x|+|y|)}(x \succ y) \succ z \\
& -(-1)^{|x||y|}(y \prec x) \prec z - (-1)^{|x||y|}(x \succ y) \prec z \\
& -(-1)^{|x||y|} z \succ (y \prec x) - (-1)^{|x||y|} z \succ (x \succ y) \\
& -(-1)^{|z|(|x|+|y|)} z \prec (x \prec y) - (-1)^{|z|(|x|+|y|)} z \prec (y \succ x) \\
& -(-1)^{|z|(|x|+|y|)}(x \prec y) \succ z - (-1)^{|z|(|x|+|y|)}(y \succ x) \succ z \\
& +(x \prec y) \prec z + (y \succ x) \prec z + z \succ (x \prec y) + z \succ (y \succ x)
\end{aligned}$$

and

$$\begin{aligned}
(-1)^{|x||y|}[y,[x,z]] \;=\; & (-1)^{|z|(|x|+|y|)}(z \prec x) \prec y + (-1)^{|z|(|x|+|y|)}(x \succ z) \prec y \\
& +(-1)^{|z|(|x|+|y|)} y \succ (z \prec x) + (-1)^{|z|(|x|+|y|)} y \succ (x \succ z) \\
& -(-1)^{|x|(|y|+|z|)} y \prec (z \prec x) - (-1)^{|x|(|y|+|z|)} y \prec (x \succ z) \\
& -(-1)^{|x|(|y|+|z|)}(z \prec x) \succ y - (-1)^{|x|(|y|+|z|)}(x \succ z) \succ y \\
& -(-1)^{|y||z|}(x \prec z) \prec y - (-1)^{|y||z|}(z \succ x) \prec y \\
& -(-1)^{|y||z|} y \succ (x \prec z) - (-1)^{|y||z|} y \succ (z \succ x) \\
& +(-1)^{|x||y|} y \prec (x \prec z) + (-1)^{|x||y|} y \prec (z \succ x) \\
& +(-1)^{|x||y|}(x \prec z) \succ y + (-1)^{|x||y|}(z \succ x) \succ y.
\end{aligned}$$

So we get

$$[x,[y,z]]=[[x,y],z]+(-1)^{|x||y|}[y,[x,z]].$$

Therefore $(A,[-,-])$ is a left Leibniz superalgebra.

Definition 2.15 [14] A Rota-Baxter superalgebra is a triple $(A,\cdot,R)$, in which A is a superspace, $(A,\cdot)$ a superalgebra, $R:A\to A$ an even bilinear map satistying Rota-Baxter super-identity

$$R(x)\cdot R(y)=R(R(x)\cdot y)+R(x\cdot R(y))+\lambda R(x\cdot y),\lambda\in\mathbb{K}$$

for all $x,y\in hg(A)$. $R:A\to A$ is called a Rota-Baxter super-operator of weight λ. If $(A,\cdot)$ is an associative superalgebra, then we call $(A,\cdot,R)$ associative Rota-Baxter superalgebra.

Proposition 2.16 Let $(A,\cdot,R)$ be an associative Rota-Baxter superalgebra with weight 0. Define two even bilinear maps $*,[-,-]:A\times A\to A$ such that

$$x*y=x\cdot R(y)+R(x)\cdot y,[x,y]=(-1)^{|x||y|}y*x-x*y.$$

Then $(A,[-,-])$ is a left Leibniz superalgebra.

Proof Calculate directly,

$$\begin{aligned}[x,[y,z]] &= (-1)^{|y||z|+|x|(|y|+|z|)}y\cdot R(z)\cdot R(x)+(-1)^{|y||z|+|x|(|y|+|z|)}R(y)\cdot z\cdot R(x)\\ &\quad +(-1)^{|y||z|+|x|(|y|+|z|)}R(y\cdot R(z))\cdot x+(-1)^{|y||z|+|x|(|y|+|z|)}R(R(y)\cdot z)\cdot x\\ &\quad -(-1)^{|y||z|}x\cdot R(z\cdot R(y))-(-1)^{|y||z|}x\cdot R(R(z)\cdot y)\\ &\quad -(-1)^{|y||z|}R(x)\cdot z\cdot R(y)-(-1)^{|y||z|}R(x)\cdot R(z)\cdot y\\ &\quad -(-1)^{|x|(|y|+|z|)}y\cdot R(z)\cdot R(x)-(-1)^{|x|(|y|+|z|)}R(y)\cdot z\cdot R(x)\\ &\quad -(-1)^{|x|(|y|+|z|)}R(y\cdot R(z))\cdot x-(-1)^{|x|(|y|+|z|)}R(R(y)\cdot z)\cdot x\\ &\quad +x\cdot R(y\cdot R(x))+x\cdot R(R(y)\cdot z)\\ &\quad +R(x)\cdot y\cdot R(z)+R(x)\cdot R(y)\cdot z\end{aligned}$$

and

$$\begin{aligned}[[x,y],z] &= (-1)^{|x||y|+|z|(|x|+|y|)}z\cdot R(y\cdot R(x))+(-1)^{|x||y|+|z|(|x|+|y|)}z\cdot R(R(y)\cdot x)\\ &\quad +(-1)^{|x||y|+|z|(|x|+|y|)}R(z)\cdot y\cdot R(x)+(-1)^{|x||y|+|z|(|x|+|y|)}R(z)\cdot R(y)\cdot x\\ &\quad -(-1)^{|x||y|}y\cdot R(x)\cdot R(z)-(-1)^{|x||y|}R(y)\cdot x\cdot R(z)\\ &\quad -(-1)^{|x||y|}R(y\cdot R(x))\cdot z-(-1)^{|x||y|}R(R(y)\cdot x)\cdot z\\ &\quad -(-1)^{|z|(|x|+|y|)}z\cdot R(x\cdot R(y))-(-1)^{|z|(|x|+|y|)}z\cdot R(R(x)\cdot y)\\ &\quad -(-1)^{|z|(|x|+|y|)}R(z)\cdot x\cdot R(y)-(-1)^{|z|(|x|+|y|)}R(z)\cdot R(x)\cdot y\\ &\quad +x\cdot R(y)\cdot R(z)+R(x)\cdot y\cdot R(z)\\ &\quad +R(x\cdot R(y))\cdot z+R(R(x)\cdot y)\cdot z\end{aligned}$$

and

$$\begin{aligned}(-1)^{|x||y|}[y,[x,z]] \quad = \quad & (-1)^{|z|(|x|+|y|)}z\cdot R(x)\cdot R(y)+(-1)^{|z|(|x|+|y|)}R(z)\cdot x\cdot R(y)\\ & +(-1)^{|z|(|x|+|y|)}R(z\cdot R(x))\cdot y+(-1)^{|z|(|x|+|y|)}R(R(z)\cdot x)\cdot y\\ & -(-1)^{|x|(|y|+|z|)}y\cdot R(z\cdot R(x))-(-1)^{|x|(|y|+|z|)}y\cdot R(R(z)\cdot x)\\ & -(-1)^{|x|(|y|+|z|)}R(y)\cdot z\cdot R(x)-(-1)^{|x|(|y|+|z|)}R(y)\cdot R(z)\cdot x\\ & -(-1)^{|y||z|}x\cdot R(z)\cdot R(y)-(-1)^{|y||z|}R(x)\cdot z\cdot R(y)\\ & -(-1)^{|y||z|}R(x\cdot R(z))\cdot y-(-1)^{|y||z|}R(R(x)\cdot z)\cdot y\\ & +(-1)^{|x||y|}y\cdot R(x\cdot R(z))+(-1)^{|x||y|}y\cdot R(R(x)\cdot z)\\ & +(-1)^{|x||y|}R(y)\cdot x\cdot R(z)+(-1)^{|x||y|}R(y)\cdot R(x)\cdot z.\end{aligned}$$

By Rota Baxter super-identity, we can get

$$[x,[y,z]]=[[x,y],z]+(-1)^{|x||y|}[y,[x,z]].$$

Therefore $(A,[-,-])$ is a left Leibniz superalgebra.

3 Leibniz Superalgebras, Lie Supertriple Systems, Lie-Yamaguti Superalgebras

Definition 3.1 A Lie-Yamaguti superalgebra (LYSA) is a triple $(A,[-,-],\{-,-,-\})$, in which A is a superspace, $[-,-]:A\times A\to A$ an even bilinear map and $\{-,-,-\}:A\times A\times A\to A$ an even trilinear map such that

(LYS01) $|[x,y]|=|x|+|y|$;

(LYS02) $|\{x,y,z\}|=|x|+|y|+|z|$;

(LYS1) $d[x,y]+(-1)^{|x||y|}[y,x]=0$;

(LYS2) $\{x,y,z\}+(-1)^{|x||y|}\{y,x,z\}=0$;

(LYS3) $\circlearrowleft_{x,y,z}(-1)^{|x||z|}[[x,y],z]+\circlearrowleft_{x,y,z}(-1)^{|x||z|}\{x,y,z\}=0$;

(LYS4) $\circlearrowleft_{x,y,z}(-1)^{|x||z|}\{[x,y],z,u\}=0$;

(LYS5) $\{x,y,[u,v]\}=[\{x,y,u\},v]+(-1)^{|u|(|x|+|y|)}[u,\{x,y,v\}]$;

(LYS6)

$$\begin{aligned}\{x,y,\{u,v,w\}\} \quad = \quad & \{\{x,y,u\},v,w\}+(-1)^{|u|(|x|+|y|)}\{u,\{x,y,v\},w\}\\ & +(-1)^{(|x|+|y|)(|u|+|v|)}\{u,v,\{x,y,w\}\}\end{aligned}$$

for all $x,y,z,u,v,w\in hg(A)$, where $\circlearrowleft_{x,y,z}$ denotes the sum over cyclic permutation of x,y,z.

Definition 3.2 [15] A Lie supertriple system is a pair $(A,\{-,-,-\})$ such that

(1) $\{x,y,z\}=(-1)^{|x||y|}\{y,x,z\}$;

(2) $\circlearrowleft_{x,y,z}(-1)^{|x||z|}\{x,y,z\}=0$;

(3)

$$\begin{aligned}\{x,y,\{u,v,w\}\} \quad = \quad & \{\{x,y,u\},v,w\}+(-1)^{|u|(|x|+|y|)}\{u,\{x,y,v\},w\}\\ & +(-1)^{(|x|+|y|)(|u|+|v|)}\{u,v,\{x,y,w\}\}\end{aligned}$$

for all $x, y, z, u, v, w \in hg(A)$.

If $[x, y] = 0$ for all $x, y \in hg(A)$, then Lie-Yamaguti superalgebras become Lie supertriple systems. So Lie-Yamaguti superalgebras can be seen as general Lie supertriple systems.

Let l_x denote the left multiplication operator on $(A, \cdot)$ which given by $l_x y = x \cdot y$ for all $x, y \in hg(A)$. Then equation (2.1) means that l_x are super-derivations of $(A, \cdot)$. By Lemma 2.6 (ii), we can get following proposition.

Proposition 3.3 Let $(A, \cdot)$ be Leibniz superlagebra, $(A, \circ, [-, -, -])$ be its associate Akivis algebra. Then the operators l_x are derivations of $(A, \circ, [-, -, -])$ for all $x \in A$.

We can obtain a Lie-Yamaguti superalgebra structure from Leibniz superalgebra as following theorem.

Theorem 3.4 Every (left) Leibniz superalgebra $(A, \cdot)$ has a Lie-Yamaguti superalgebra structure $(A, [-, -], \{-, -, -\})$ with respect to the operation defined by

$$\begin{aligned} [x, y] &:= x \cdot y - (-1)^{|x||y|} y \cdot x, && (3.1) \\ \{x, y, z\} &:= (-1)^{|x||y|} \widetilde{as}(y, x, z) - \widetilde{as}(x, y, z). && (3.2) \end{aligned}$$

Proof Equations (3.2), (2.1) and (2.8) imply

$$\{x, y, z\} = -(x \cdot y) \cdot z. \tag{3.3}$$

Moreover, we have

$$[x, y] \cdot z = (x \cdot y - (-1)^{|x||y|} y \cdot x) \cdot z = 2(x \cdot y) \cdot z = -2\{x, y, z\}.$$

So we get

$$\{x, y, z\} = -\frac{1}{2}[x, y] \cdot z. \tag{3.4}$$

Thus equations (3.2), (3.3) and (3.4) are different expressions of the operation "$\{-, -, -\}$". Now we proceed to verify equations (LYS1)–(LYS6). For (LYS1),

$$[x, y] + (-1)^{|x||y|}[y, x] = x \cdot y - (-1)^{|x||y|} y \cdot x + (-1)^{|x||y|}(y \cdot x - (-1)^{|x||y|} x \cdot y) = 0.$$

So we get (LYS1). For (LYS2),

$$\begin{aligned} \{x, y, z\} + (-1)^{|x||y|}\{y, x, z\} &= -\frac{1}{2}[x, y] \cdot z - (-1)^{|x||y|} \frac{1}{2}[y, x] \cdot z \\ &= -\frac{1}{2}[x, y] \cdot z + \frac{1}{2}[x, y] \cdot z = 0. \end{aligned}$$

So we get (LYS2). For (LYS3), (2.10) and (3.3) imply

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}[[x, y], z] = - \circlearrowleft_{x,y,z} (-1)^{|x||z|}\{x, y, z\}.$$

So we get (LYS3) by transposition. For (LYS4),

$$\begin{aligned}
& \circlearrowleft_{x,y,z} (-1)^{|x||z|}\{[x,y],z,u\} = \circlearrowleft_{x,y,z} (-1)^{|x||z|} - ([x,y]\cdot z)\cdot u \\
= \;& 2(\circlearrowleft_{x,y,z} (-1)^{|x||z|}\{x,y,z\})\cdot u \\
= \;& -2((-1)^{|x||z|}(x\cdot y)\cdot z + (-1)^{|x||y|}(y\cdot z)\cdot x + (-1)^{|y||z|}(z\cdot x)\cdot y)\cdot u \\
= \;& -2((-1)^{|x||z|}x\cdot(y\cdot z) - (-1)^{|x|(|y|+|z|)}y\cdot(x\cdot z) \\
& +(-1)^{|x||y|}(y\cdot z)\cdot x + (-1)^{|y||z|}(z\cdot x)\cdot y)\cdot u \\
= \;& -2\cdot(-1)^{|x||z|}(x\cdot(y\cdot z) + (-1)^{|x|(|y|+|z|)}(y\cdot z)\cdot x)\cdot u \\
& -2(-(-1)^{|x|(|y|+|z|)}y\cdot(x\cdot z) + (-1)^{|y||z|}(z\cdot x)\cdot y)\cdot u \\
= \;& -2(-(-1)^{|x|(|y|+|z|)}y\cdot(x\cdot z) + (-1)^{|y||z|}(z\cdot x)\cdot y)\cdot u \\
= \;& -2(-(-1)^{|x|(|y|+|z|)}y\cdot(x\cdot z) + (-1)^{|z|(|x|+|y|)}(x\cdot z)\cdot y)\cdot u \\
= \;& 2\cdot(-1)^{|x|(|y|+|z|)}(y\cdot(x\cdot z) + (-1)^{|y|(|x|+|z|)}(x\cdot z)\cdot y)\cdot u \\
= \;& 0.
\end{aligned}$$

So we get (LYS4). For (LYS5),

$$\begin{aligned}
\{x,y,[u,v]\} &= -(x\cdot y)\cdot[u,v] \\
&= -[(x\cdot y)\cdot u, v] - (-1)^{|u|(|x|+|y|)}[u,(x\cdot y)\cdot v] \\
&= [\{x,y,u\},v] + (-1)^{|u|(|x|+|y|)}[u,\{x,y,v\}].
\end{aligned}$$

So we get (LYS5). For (LYS6),

$$\begin{aligned}
& \{\{x,y,u\},v,w\} + (-1)^{|u|(|x|+|y|)}\{u,\{x,y,v\},w\} \\
& +(-1)^{(|x|+|y|)(|u|+|v|)}\{u,v,\{x,y,w\}\} \\
= \;& \{-(x\cdot y)\cdot u,v,w\} + (-1)^{|u|(|x|+|y|)}\{u,-(x\cdot y)\cdot v,w\} \\
& +(-1)^{(|x|+|y|)(|u|+|v|)}\{u,v,-(x\cdot y)\cdot w\}(By(17)) \\
= \;& -(((-(x\cdot y)\cdot u)\cdot v)\cdot w - (-1)^{|u|(|x|+|y|)}(u\cdot(-(x\cdot y)\cdot v))\cdot w \\
& -(-1)^{(|x|+|y|)(|u|+|v|)}(u\cdot v)\cdot(-(x\cdot y)\cdot w) \\
= \;& ((x\cdot y)\cdot(u\cdot v))\cdot w + (-1)^{(|x|+|y|)(|u|+|v|)}(u\cdot v)\cdot((x\cdot y)\cdot w) \\
= \;& (x\cdot y)\cdot((u\cdot v)\cdot w) = \{x,y,\{u,v,w\}\}.
\end{aligned}$$

So we get (LYS6). Therefore $(A,[-,-],\{-,-,-\})$ is a Lie-Yamaguti superalgebra.

Remark 3.5 By Proposition 2.10, Proposition 2.12, Proposition 2.14 and Proposition 2.16, we can get left Leibniz superalgebras from associative superalgebras, superdialgebras, dendriform superalgebras and associative Rota-Baxter superalgebras. Then using Theorem 3.4, we will obtain corresponding Lie-Yamaguti superalgebras from above superalgebras [16]. Then we give an 3-dimensional example of Lie-Yamaguti superalgebra by Theorem 3.4.

Example 3.6 Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a 3-dimensional superspace. $A_{\bar{0}} = \mathrm{span}\{e_1, e_3\}$, $A_{\bar{1}} = \mathrm{span}\{e_2\}$. The nonzero product is given by $e_2\cdot e_3 = e_2$, $e_3\cdot e_1 = e_1$, $e_2\cdot e_2 = -e_2$,

$e_3 \cdot e_3 = e_1$. Then $(A, \cdot)$ is a left Leibniz superalgebra. By Theorem 2.8, when we define the binary operation and the ternary operation by (3.1) and (3.2), we get a Lie-Yamaguti superalgebra $(A, [-,-], \{-,-,-\})$ with nonzero product

$$[e_2, e_3] = 2e_2 = -[e_3, e_2], \;\; [e_3, e_1] = e_1 = -[e_1, e_3], \;\; \{e_2, e_3, e_3\} = -e_2 = -\{e_3, e_2, e_3\}.$$

The following proposition is a direct conclusion of Theorem 3.4.

Proposition 3.7 Let $(A, \cdot)$ be Leibniz superlagebra, $(A, \circ, [-,-,-])$ be its associate Akivis algebra. Define $\{x, y, z\} = (-1)^{|x||y|}[y, x, z] + [x, y, z]$ for all $x, y, z \in hg(A)$, then $(A, \circ, \{-,-,-\})$ is a Lie-Yamaguti superalgebra.

In Leibniz superlagebra $(A, \cdot)$ and its associate Akivis algebra $(A, \circ, [-,-,-])$, consider even ternaty operation

$$(x, y, z) = (-1)^{|x||y|}[y, x, z] - [x, y, z] + (x \circ y) \circ z \tag{3.5}$$

for all $x, y, z \in hg(L)$. We have

$$(x, y, z) = -(-1)^{|x||y|}(y, x, z) \tag{3.6}$$

and the Akivis super-identity (2.5) is written as

$$\circlearrowleft_{x,y,z} (x, y, z) = 0. \tag{3.7}$$

A superalgebra $(A, (-,-,-))$ called a supertriple system if the even trilinear operation satisfies (3.6) and (3.7). By Theorem 2.5, any non-associative algebra has a supertriple system structure defined by (3.5), and we call it the associate supertriple system.

References

[1] Cuvier C. Homologie de Leibniz et homologie de Hochschild[J]. Comptes rendus de l'Académie des Sciences, Série 1, Mathématique, 1991, 313(9): 569–572.

[2] Loday J L. Une v ersion non commutative des alg ebres de Lie: les alg ebres de Leibniz[J]. Perspectivas Revista Trimestral De Educación Comparada, 1993, 39(1): 269–294.

[3] Loday J L. Cup-product for Leibniz cohomology and dual Leibniz algebras[J]. Math. Scand., 1995, 77(2): 189–196.

[4] Ayupov Sh A, Omirov B A. On 3-dimensional Leibniz algebras[J]. Uzbek. Mat. Zh., 1999, 1: 9–14.

[5] Ayupov Sh A, Omirov B A. On some classes of nilpotent Leibniz algebras[J]. Sibirsk. Math. J., 2001, 42(1): 15–24.

[6] Albeverio S, Ayupov Sh A, Omirov B A. On nilpotent and simple Leibniz algebras[J]. Comm. Alg., 2005, 33(1): 159–172.

[7] Issa A N. Remarks on the construction of Lie-Yamaguti algebras from Leibniz algebras[J]. Int. J. Alg., 2011, 5(14): 667–677.

[8] Fialowski A, Mihálka é Z. Representations of Leibniz algebras[J]. Alg. Repres. The., 2015, 18(2): 477–490.

[9] Hofmann K, Strambach K. Lie's fundamental theorems for local analytical loops[J]. Pacific J. Math., 1986, 123(2): 301–327.

[10] Dzhumadil'daev A S. Cohomologies of colour Leibniz algebras: pre-simplicial approach[J]. Lie The. Appl. Phys., III (Clausthal, 1999), River Edge, NJ: World Sci. Publ., 2000: 124–136.

[11] Liu Dong, Hu Naihong. Leibniz superalgebras and central extensions[J]. J. Alg. Appl., 2006, 5(6): 765–780.

[12] Liu Dong. Steinberg-Leibniz algebras and superalgebras[J]. J. Alg., 2005, 283(1): 199–221.

[13] Albuquerque H, Santana A P. Akivis superalgebras and speciality[C]. Alg., Repres. Appl.: Conference in Honour of Ivan Shestakov's 60th Birthday, August 26-September 1, 2007, Maresias, Brazil, Amer. Math. Soc., 2009, 483: 13.

[14] Zhang Qingcheng, Wang Chunyue. Hom-Leibniz superalgebras and Hom-Leibniz poisson superalgebras[J]. Hacet. J. Math. Stat., 2015, 44(5): 1163–1179.

[15] Muta M, Taniguchi Y, Yamaguti K. A construction of Lie supertriple systems from pairs of Lie triple system and anti-Lie triple system[J]. Bull. College Liberal Arts Kyushu Sangyo Univ., 1994, 30(3):157–162.

[16] Ma Fengmin, Zhang Qingcheng. The derivation algebras of the modular Lie Superalgebras K-type[J]. J. Math., 2000, 20(4):431–435.

从Leibniz超代数到Lie-Yamaguti超代数

唐鑫鑫[1], 张庆成[1], 王春月[2]

(1.东北师范大学数学与统计学院, 吉林 长春 130024)

(2.吉林工程技术师范学院传媒与数理学院, 吉林 长春 130052)

摘要: 本文研究了Lie-Yamaguti超代数的构造. 利用左Leibniz超代数, 先给出左Leibniz超代数的构造方法, 再给出用左Leibniz超代数构造Lie-Yamaguti超代数的方法, 获得了Lie-Yamaguti超代数的构造方法. 将Leibniz代数和Lie-Yamaguti代数的构造推广到超代数的情形.

关键词: Lie-Yamaguiti超代数; (左) Leibniz 超代数; Akivis 超代数; 李超三系; 构造

MR(2010)**主题分类号:** 17A30; 17A32; 17D99 **中图分类号:** O152.5