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NABLA-HUKUHARA DERIVATIVE OF FUZZY-VALUED FUNCTIONS ON TIME SCALES

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Abstract: In this paper, we introduce and investigate the concept of Nabla-Hukuhara derivative of fuzzy-valued functions on time scales. By using the theory of time scales, we show some basic properties of the Nabla-Hukuhara derivative, which extend and improve the related ones in [12].

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1 Introduction

The Hukuhara derivative was the starting point for the topic of set differential equations and later also for fuzzy differential equations. But the Hukuhara derivative in FDE suffers certain disadvantages (see [3]) related to the properties of the space $\mathcal{K}^n$ of all nonempty compact sets of $\mathbb{R}^n$ and in particular to the fact that Minkowski addition does not possess an inverse subtraction. To overcome this obstacle, several generalized fuzzy derivative concepts were studied from different viewpoints by some authors [4–8].

Recently, the authors in [9] introduced the concept of fuzzy derivative for fuzzy-valued functions on time scales, which provides a natural extension of the Hukuhara derivative. In this paper, we define the Nabla-Hukuhara derivative of fuzzy-valued functions on time scales, which gives another type of generalization of the Hukuhara derivative. We also show some basic properties of the Nabla-Hukuhara derivative. Results obtained in this paper extend and improve the related ones in [12].

2 Preliminaries

In this section, we recall some basic definitions, notation, properties and results on fuzzy sets and the time scale calculus, which are used throughout the paper. Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the reals $u : \mathbb{R} \to [0,1]$, satisfying the following properties

(1) u is normal, i.e., there exists $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;

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(2) u is a convex fuzzy set, i.e., for all $x_1, x_2 \in \mathbb{R}$, $\lambda \in (0,1)$, we have

$$u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\};$$

(3) u is upper semi-continuous;

(4) $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of the set A.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$. From conditions (1) to (4), it follows that the α-level set $[u]^\alpha$ is a nonempty compact interval for all $\alpha \in [0,1]$. We write $[u]^\alpha = [\underline{u^\alpha}, \overline{u^\alpha}]$ and denote the lower and upper branches of u by $\underline{u}$ and $\overline{u}$, respectively. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, the sum $u \oplus v$ and the product between crisp numbers and fuzzy numbers, $\lambda \odot u$, is defined by $[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda \odot u]^\alpha = \lambda [u]^\alpha$ respectively for all $\alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ is the Minkowski addition of sets and $\lambda[u]^\alpha$ is the product between real numbers and intervals of $\mathbb{R}$.

As a distance between fuzzy numbers, we use the Hausdorff metric defined by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u^\alpha} - \underline{v^\alpha}|, |\overline{u^\alpha} - \overline{v^\alpha}|\}$$

for $u, v \in \mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ with the subspace topology inherited from the standard topology on $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t)$ by $\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$ where $\inf \emptyset = \sup\{\mathbb{T}\}$, while the backward jump operator $\rho(t)$ is defined by $\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$ where $\sup \emptyset = \inf\{\mathbb{T}\}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. A point $t \in \mathbb{T}$ is dense if it is both right and left dense; isolated if it is both right and left scattered. The forward graininess function μ and the backward graininess function η are defined by $\mu(t) = \sigma(t) - t$, $\eta(t) = t - \rho(t)$ for all $t \in \mathbb{T}$, respectively. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^k := \mathbb{T} \setminus \sup \mathbb{T}$, otherwise $\mathbb{T}^k := \mathbb{T}$; if $\inf \mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_k := \mathbb{T} \setminus \inf \mathbb{T}$, otherwise $\mathbb{T}_k := \mathbb{T}$.

Definition 2.1 [7] A function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is said to be left differentiable at t if there exist A and $\delta > 0$, such that

(1) $f(t) \ominus f(t-h)$ exists for $0 < h < \delta$ and $\lim_{h \to 0^+} \frac{1}{h} \odot (f(t) \ominus f(t-h)) = A$; or

(2) $f(t-h) \ominus f(t)$ exists for $0 < h < \delta$ and $\lim_{h \to 0^+} \frac{1}{-h} \odot (f(t-h) \ominus f(t)) = A$.

The element A is said to be the left derivative of f at t, noted as $f'_-(t)$.

Definition 2.2 [7] A function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is said to be right differentiable at t if there exist A and $\delta > 0$, such that

(1) $f(t+h) \ominus f(t)$ exists for $0 < h < \delta$ and $\lim_{h \to 0^+} \frac{1}{h} \odot (f(t+h) \ominus f(t)) = A$; or

(2) $f(t) \ominus f(t+h)$ exists for $0 < h < \delta$ and $\lim_{h \to 0^+} \frac{1}{-h} \odot (f(t) \ominus f(t+h)) = A$.

The element A is said to be the right derivative of f at t, noted as $f'_+(t)$.

Definition 2.3 [7] A function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is said to be differentiable at t if f is both left and right differentiable at t, and $f'_-(t) = f'_+(t)$. The element $f'_-(t)$ or $f'_+(t)$ is said to be the derivative of f at t, denoted as $f'(t)$.

3 Nabla-Hukuhara Derivative

Definition 3.1 Assume that $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ is a fuzzy function and let $t \in \mathbb{T}_k$. Then we define $f^{\nabla_H}(t)$ to be the number (provided that it exists) with the property that given any $\epsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\begin{cases} D[f(t+h) \ominus_g f^{\rho}(t), f^{\nabla_H}(t)(h+\eta(t))] \leq \epsilon|h+\eta(t)|, \\ D[f^{\rho}(t) \ominus_g f(t-h), f^{\nabla_H}(t)(h-\eta(t))] \leq \epsilon|h-\eta(t)| \end{cases}$$

for all $t-h, t+h \in U$ with $0 \leq h < \delta$.

We call $f^{\nabla_H}(t)$ the Nabla-Hukuhara derivative (∇_H-derivative for short) at t. Moreover, we say that f is ∇_H-differentiable on $\mathbb{T}_k$ provided that $f^{\nabla_H}(t)$ exists for all $t \in \mathbb{T}_k$. The fuzzy function $f^{\nabla_H}(t) : \mathbb{T}_k \to \mathbb{R}_{\mathcal{F}}$ is then called the ∇_H-derivative on $\mathbb{T}_k$.

Some useful properties of the ∇_H-derivative of f are given in the next theorem.

Theorem 3.2 Let $f : \mathbb{T} \to \mathbb{R}_{\mathcal{F}}$ and $t \in \mathbb{T}_k$. Then we have the following.

(i) Function f has at most one ∇_H-derivative at t.

(ii) If f is ∇_H-differentiable at t, then f is continuous at t.

(iii) If f is continuous at t and t is left-scattered, then f is ∇_H-differentiable at t with

$$f^{\nabla_H}(t) = \frac{f(t) \ominus_g f^{\rho}(t)}{\eta(t)}.$$

(iv) If t is left-dense, then f is ∇_H-differentiable at t if and only if the limits $\lim\limits_{h\to 0^+} \frac{f(t+h)\ominus_g f(t)}{h}$ and $\lim\limits_{h\to 0^+} \frac{f(t)\ominus_g f(t-h)}{h}$ exist as a finite number and are equal. In this case,

$$f^{\nabla_H}(t) = \lim_{h\to 0^+} \frac{f(t+h) \ominus_g f(t)}{h} = \lim_{h\to 0^+} \frac{f(t) \ominus_g f(t-h)}{h}.$$

(v) If f is ∇_H-differentiable, then $f(t) = f^{\rho}(t) \oplus f^{\nabla_H}(t)\eta(t)$.

Proof (i) The proof is easy and will be omitted.

(ii) Assume that f is ∇_H-differentiable at t. Let $\epsilon \in (0,1)$. Denote

$$\epsilon^* = \frac{\epsilon}{3(h + \|f^{\nabla_H}(t)\|_{\mathcal{F}})}.$$

Then $\epsilon^* \in (0,1)$, here we have for $u \in \mathbb{R}_{\mathcal{F}}$, $D[u, \tilde{0}] = \|u\|_{\mathcal{F}}$ with $\tilde{0}$, a zero element of $\mathbb{R}_{\mathcal{F}}$. By Definition 3.1 there exists a neighborhood U of t such that

$$\begin{cases} D[f(t+h) \ominus_g f^{\rho}(t), f^{\nabla_H}(t)(h+\eta(t))] \leq \epsilon^*|h+\eta(t)|, \\ D[f^{\rho}(t) \ominus_g f(t-h), f^{\nabla_H}(t)(h-\eta(t))] \leq \epsilon^*|h-\eta(t)| \end{cases}$$

for all $t-h, t+h \in U$. Therefore, for all $t-h, t+h \in U \cap (t-\epsilon^*, t+\epsilon^*)$ we have

$$\begin{aligned}
& D[f(t+h), f(t)] \\
= \;& D[f(t+h) \ominus_g f^\rho(t), f(t) \ominus_g f^\rho(t)] \\
\leq \;& D[f(t+h) \ominus_g f^\rho(t), \tilde{0}] + D[f(t) \ominus_g f^\rho(t), \tilde{0}] \\
\leq \;& D[f(t+h) \ominus_g f^\rho(t), f^{\nabla_H}(t)(h+\eta(t))] + D[f^{\nabla_H}(t)(h+\eta(t)), \tilde{0}] \\
& + D[f(t) \ominus_g f^\rho(t), f^{\nabla_H}(t)\eta(t)] + D[f^{\nabla_H}(t)\eta(t), \tilde{0}] \\
\leq \;& D[f(t+h) \ominus_g f^\rho(t), f^{\nabla_H}(t)(h+\eta(t))] + D[f(t) \ominus_g f^\rho(t), f^{\nabla_H}(t)\eta(t)] \\
& + (h+2\eta(t))D[f^{\nabla_H}(t), \tilde{0}] \\
\leq \;& \epsilon^*|h+\eta(t)| + \epsilon^*|\eta(t)| + (h+2\eta(t))\|f^{\nabla_H}(t)\|_{\mathcal{F}} \\
\leq \;& 2h\epsilon^* + h\epsilon^* + 3h\|f^{\nabla_H}(t)\|_{\mathcal{F}} \leq \epsilon^*\big(3h + 3\|f^{\nabla_H}(t)\|_{\mathcal{F}}\big) = \epsilon.
\end{aligned}$$

Similarly, we can prove that $D[f(t), f(t-h)] < \epsilon$. Therefore, f is continuous at t.

(iii) Assume that f is continuous at t and t is left-scattered. By the continuity,

$$\lim_{h\to 0^+} \frac{f(t+h) \ominus_g f^\rho(t)}{h+\eta(t)} = \frac{f(t) \ominus_g f^\rho(t)}{\eta(t)} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f^\rho(t) \ominus_g f(t-h)}{h-\eta(t)} = \frac{f^\rho(t) \ominus_g f(t)}{-\eta(t)}.$$

Moreover, $D[\frac{f(t)\ominus_g f^\rho(t)}{\eta(t)}, \frac{f^\rho(t)\ominus_g f(t)}{-\eta(t)}] = 0$. Hence, given $\epsilon > 0$, there exists a neighborhood U of t such that

$$D\Big[\frac{f(t+h) \ominus_g f^\rho(t)}{h+\eta(t)}, \frac{f(t) \ominus_g f^\rho(t)}{\eta(t)}\Big] \leq \epsilon, \quad D\Big[\frac{f^\rho(t) \ominus_g f(t-h)}{h-\eta(t)}, \frac{f(t) \ominus_g f^\rho(t)}{\eta(t)}\Big] \leq \epsilon$$

for all $t-h, t+h \in U$. It follows that

$$\begin{cases} D[f(t+h) \ominus_g f^\rho(t), \frac{f(t)\ominus_g f^\rho(t)}{\eta(t)}(h+\eta(t))] \leq \epsilon|h+\eta(t)|, \\ D[f^\rho(t) \ominus_g f(t-h), \frac{f(t)\ominus_g f^\rho(t)}{\eta(t)}(h-\eta(t))] \leq \epsilon|h-\eta(t)| \end{cases}$$

for all $t-h, t+h \in U$. Hence we get the desired result

$$f^{\nabla_H}(t) = \frac{f(t) \ominus_g f^\rho(t)}{\eta(t)}.$$

(iv) Assume that f is ∇_H-differentiable at t and t is left-dense. Then for each $\epsilon > 0$, there exists a neighborhood U of t such that

$$\begin{cases} D[f(t+h) \ominus_g f^\rho(t), f^{\nabla_H}(t)(h+\eta(t))] \leq \epsilon|h+\eta(t)|, \\ D[f^\rho(t) \ominus_g f(t-h), f^{\nabla_H}(t)(h-\eta(t))] \leq \epsilon|h-\eta(t)| \end{cases}$$

for all $t-h, t+h \in U$. Since $\rho(t) = t, \eta(t) = 0$, we have that

$$D[f(t+h) \ominus_g f(t), hf^{\nabla_H}(t)] \leq \epsilon h \quad \text{and} \quad D[f(t) \ominus_g f(t-h), hf^{\nabla_H}(t)] \leq \epsilon h.$$

It follows that

$$D\Big[\frac{f(t+h) \ominus_g f(t)}{h}, f^{\nabla_H}(t)\Big] \leq \epsilon \quad \text{and} \quad D\Big[\frac{f(t) \ominus_g f(t-h)}{h}, f^{\nabla_H}(t)\Big] \leq \epsilon.$$

Hence we get the desired result

$$f^{\nabla_H}(t)=\lim_{h\to 0^+}\frac{f(t+h)\ominus_g f(t)}{h}=\lim_{h\to 0^+}\frac{f(t)\ominus_g f(t-h)}{h}.$$

On the other hand, if the limits

$$\lim_{h\to 0^+}\frac{f(t+h)\ominus_g f(t)}{h} \quad\text{and}\quad \lim_{h\to 0^+}\frac{f(t)\ominus_g f(t-h)}{h}$$

exist as a finite number and are equal to $u\in\mathbb{R}_{\mathcal{F}}$, then for each $\epsilon>0$, there exists a neighborhood U of t such that

$$D\Big[\frac{f(t+h)\ominus_g f(t)}{h},u\Big]\le\epsilon \quad\text{and}\quad D\Big[\frac{f(t)\ominus_g f(t-h)}{h},u\Big]\le\epsilon$$

for all $t-h,t+h\in U$.

Since t is left-dense, we have

$$D[f(t+h)\ominus_g f^{\rho}(t),u(h+\eta(t))]\le\epsilon|h+\eta(t)| \quad\text{and}\quad D[f^{\rho}(t)\ominus_g f(t-h),u(h-\eta(t))]\le\epsilon|h-\eta(t)|.$$

Hence f is ∇_H-differentiable at t and

$$f^{\nabla_H}(t)=\lim_{h\to 0^+}\frac{f(t+h)\ominus_g f(t)}{h}=\lim_{h\to 0^+}\frac{f(t)\ominus_g f(t-h)}{h}.$$

(v) If t is a left-dense point, then $\rho(t)=t,\eta(t)=0$ and we have

$$f(t)=f^{\rho}(t)\oplus f^{\nabla_H}(t)\eta(t).$$

If t is left-scattered, then by (iii), we have $f(t)=f^{\rho}(t)\oplus f^{\nabla_H}(t)\eta(t)$, and the proof of part (v) is completed.

Now, we present two examples to show that the Nabla-Hukuhara derivative is more general than the generalized derivative proposed in [7].

Example 3.3 We consider the two cases $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.

(i) If $\mathbb{T}=\mathbb{R}$, then $f:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ is ∇_H-differentiable at t if and only if the limits

$$\lim_{h\to 0^+}\frac{f(t+h)\ominus_g f(t)}{h} \quad\text{and}\quad \lim_{h\to 0^+}\frac{f(t)\ominus_g f(t-h)}{h}$$

exist as a finite number and are equal. In this case,

$$f^{\nabla_H}(t)=\lim_{h\to 0^+}\frac{f(t+h)\ominus_g f(t)}{h}=\lim_{h\to 0^+}\frac{f(t)\ominus_g f(t-h)}{h}.$$

If $\mathbb{T}=[a,b]$, the Nabla-Hukuhara derivative reduces to the generalized derivative proposed in [7].

(ii) If $\mathbb{T}=\mathbb{Z}$, then $f:\mathbb{Z}\to\mathbb{R}_{\mathcal{F}}$ is ∇_H-differentiable at t with

$$f^{\nabla_H}(t)=\frac{f(t)\ominus_g f^{\rho}(t)}{\eta(t)}=f(t)\ominus_g f(t-1).$$

Example 3.4 If $f:\mathbb{T}\to\mathbb{R}_{\mathcal{F}}$ is defined by $f(t)=[0,t^2]$ for all $t\in\mathbb{T}:=\{\frac{n}{2}:n\in\mathbb{N}_0\}$, then from Theorem 3.2 (ii), we have that f is ∇_H-differentiable at t with

$$f^{\nabla_H}(t)=\frac{f(t)\ominus_g f^{\rho}(t)}{\eta(t)}=\frac{[0,t^2]-[0,t^2-t+\frac{1}{4}]}{\frac{1}{2}}=\Big[0,2t-\frac{1}{2}\Big].$$

Theorem 3.5 Assume that $f,g:\mathbb{T}\to\mathbb{R}_{\mathcal{F}}$ are ∇_H-differentiable at $t\in\mathbb{T}_k$. Then

(i) for any constants λ_1,λ_2, the sum $(\lambda_1 f\oplus\lambda_2 g):\mathbb{T}\to\mathbb{R}_{\mathcal{F}}$ is ∇_H-differentiable at t with

$$(\lambda_1 f\oplus\lambda_2 g)^{\nabla_H}(t)=\lambda_1 f^{\nabla_H}(t)\oplus\lambda_2 g^{\nabla_H}(t);$$

(ii) if f and g are continuous, then the product $fg:\mathbb{T}\to\mathbb{R}_{\mathcal{F}}$ is ∇_H-differentiable at t with

$$(fg)^{\nabla_H}(t)=f^{\nabla_H}(t)g(t)\oplus f^{\rho}(t)g^{\nabla_H}(t)=f(t)g^{\nabla_H}(t)\oplus f^{\nabla_H}(t)g^{\rho}(t).$$

Proof (i) Since f and g are ∇_H-differentiable at $t\in\mathbb{T}_k$, for any $\epsilon>0$, there exist neighborhoods U_1 and U_2 of t with

$$\begin{cases} D[\lambda_1 f(t+h)\ominus_g\lambda_1 f^{\rho}(t),\lambda_1 f^{\nabla_H}(t)(h+\eta(t))]\le|\lambda_1|\epsilon|h+\eta(t)|,\\ D[\lambda_1 f^{\rho}(t)\ominus_g\lambda_1 f(t-h),\lambda_1 f^{\nabla_H}(t)(h-\eta(t))]\le|\lambda_1|\epsilon|h-\eta(t)|\end{cases}$$

for all $t-h,t+h\in U_1$ with $0\le h<\delta_1$, and

$$\begin{cases} D[\lambda_2 g(t+h)\ominus_g\lambda_2 g^{\rho}(t),\lambda_2 g^{\nabla_H}(t)(h+\eta(t))]\le|\lambda_2|\epsilon|h+\eta(t)|,\\ D[\lambda_2 g^{\rho}(t)\ominus_g\lambda_2 g(t-h),\lambda_2 g^{\nabla_H}(t)(h-\eta(t))]\le|\lambda_2|\epsilon|h-\eta(t)|\end{cases}$$

for all $t-h,t+h\in U_2$ with $0\le h<\delta_2$.

Let $U=U_1\cap U_2$, $\delta=\min\{\delta_1,\delta_2\}$, $\lambda=\max\{\lambda_1,\lambda_2\}$. Then we have, for all $t\in U$,

$$\begin{aligned} & D[(\lambda_1 f\oplus\lambda_2 g)(t+h)\ominus_g(\lambda_1 f\oplus\lambda_2 g)^{\rho}(t),(\lambda_1 f^{\nabla_H}(t)\oplus\lambda_2 g^{\nabla_H}(t))(h+\eta(t))]\\ \le\ & D[\lambda_1 f(t+h)\ominus_g\lambda_1 f^{\rho}(t),\lambda_1 f^{\nabla_H}(t)(h+\eta(t))]\\ & +D[\lambda_2 g(t+h)\ominus_g\lambda_2 g^{\rho}(t),\lambda_2 g^{\nabla_H}(t)(h+\eta(t))]\\ \le\ & |\lambda_1|\epsilon|h+\eta(t)|+|\lambda_2|\epsilon|h+\eta(t)|\le|\lambda|\epsilon|h+\eta(t)|,\end{aligned}$$

and

$$\begin{aligned} & D[(\lambda_1 f\oplus\lambda_2 g)^{\rho}(t)\ominus_g(\lambda_1 f\oplus\lambda_2 g)(t-h),(\lambda_1 f^{\nabla_H}(t)\oplus\lambda_2 g^{\nabla_H}(t))(h-\eta(t))]\\ \le\ & D[\lambda_1 f^{\rho}(t)\ominus_g\lambda_1 f(t-h),\lambda_1 f^{\nabla_H}(t)(h-\eta(t))]\\ & +D[\lambda_2 g^{\rho}(t)\ominus_g\lambda_2 g(t-h),\lambda_2 g^{\nabla_H}(t)(h-\eta(t))]\\ \le\ & |\lambda_1|\epsilon|h-\eta(t)|+|\lambda_2|\epsilon|h-\eta(t)|\le|\lambda|\epsilon|h-\eta(t)|.\end{aligned}$$

Therefore $(\lambda_1 f\oplus\lambda_2 g)$ is ∇_H-differentiable with

$$(\lambda_1 f\oplus\lambda_2 g)^{\nabla_H}(t)=\lambda_1 f^{\nabla_H}(t)\oplus\lambda_2 g^{\nabla_H}(t).$$

(ii) Let $0<\epsilon<1$. Denote

$$\epsilon^*=\frac{\epsilon}{1+\|g(t)\|_{\mathcal{F}}+\|f^{\rho}(t)\|_{\mathcal{F}}+\|f^{\nabla_H}(t)\|_{\mathcal{F}}},$$

then $0<\epsilon^*<1$. Since $f,g:\mathbb{T}\to\mathbb{R}$ are ∇_H-differentiable at t, there exist neighborhoods U_1 and U_2 of t with

$$\begin{cases} D[f(t+h)\ominus_g f^{\rho}(t), f^{\nabla_H}(t)(h+\eta(t))]\le \epsilon^*|h+\eta(t)|,\\ D[f^{\rho}(t)\ominus_g f(t-h), f^{\nabla_H}(t)(h-\eta(t))]\le \epsilon^*|h-\eta(t)|\end{cases}$$

for all $t-h,t+h\in U_1$ with $0\le h<\delta_1$, and

$$\begin{cases} D[g(t+h)\ominus_g g^{\rho}(t), g^{\nabla_H}(t)(h+\eta(t))]\le \epsilon^*|h+\eta(t)|,\\ D[g^{\rho}(t)\ominus_g g(t-h), g^{\nabla_H}(t)(h-\eta(t))]\le \epsilon^*|h-\eta(t)|\end{cases}$$

for all $t-h,t+h\in U_2$ with $0\le h<\delta_2$, and there exists neighborhoods U_3 of t such that

$$\begin{aligned}&D[g(t+h),g(t)]\le\epsilon^*,\quad D[g(t),g(t-h)]\le\epsilon^*,\\ &D[g(t+h),\widetilde{0}]\le D[g(t),\widetilde{0}]+1,\quad D[g(t-h),\widetilde{0}]\le D[g(t),\widetilde{0}]+1\end{aligned}$$

for all $t-h,t+h\in U_3$ with $0\le h<\delta_3$.

Let $U=U_1\cap U_2\cap U_3,\delta=\min\{\delta_1,\delta_2,\delta_3\}$. Then

$$\begin{aligned}
&D\Big[f(t+h)g(t+h)\ominus_g f^{\rho}(t)g^{\rho}(t),\Big(f^{\nabla_H}(t)g(t)\oplus f^{\rho}(t)g^{\nabla_H}(t)\Big)(h+\eta(t))\Big]\\
=\ &D\Big[f(t+h)g(t+h)\ominus_g f^{\rho}(t)g^{\rho}(t)\oplus g(t+h)f^{\rho}(t),\Big(f^{\nabla_H}(t)g(t)\oplus f^{\rho}(t)g^{\nabla_H}(t)\Big)\\
&(h+\eta(t))\oplus g(t+h)f^{\rho}(t)\Big]\\
\le\ &D\Big[\Big(f(t+h)\ominus_g f^{\rho}(t)\Big)g(t+h),g(t+h)f^{\nabla_H}(t)(h+\eta(t))\Big]+D\Big[\Big(g(t+h)\ominus_g g^{\rho}(t)\Big)f^{\rho}(t),\\
&f^{\rho}(t)g^{\nabla_H}(t)(h+\eta(t))\Big]+D\Big[\widetilde{0},g(t)f^{\nabla_H}(t)(h+\eta(t))\ominus_g g(t+h)f^{\nabla_H}(t)(h+\eta(t))\Big]\\
\le\ &D\Big[f(t+h)\ominus_g f^{\rho}(t),f^{\nabla_H}(t)(h+\eta(t))\Big]D[g(t+h),\widetilde{0}]+D\Big[g(t+h)\ominus_g g^{\rho}(t),\\
&g^{\nabla_H}(t)(h+\eta(t))\Big]D[f^{\rho}(t),\widetilde{0}]+D\Big[\widetilde{0},g(t)\ominus_g g(t+h)\Big]D[f^{\nabla_H}(t),\widetilde{0}]|h+\eta(t)|\\
\le\ &\epsilon^*|h+\eta(t)|\cdot\Big(D[g(t+h),\widetilde{0}]+D[f^{\rho}(t),\widetilde{0}]+D[f^{\nabla_H}(t),\widetilde{0}]\Big)\\
\le\ &\epsilon^*|h+\eta(t)|\cdot\big(1+\|g(t)\|_{\mathcal{F}}+\|f^{\rho}(t)\|_{\mathcal{F}}+\|f^{\nabla_H}(t)\|_{\mathcal{F}}\big)\\
\le\ &\epsilon|h+\eta(t)|
\end{aligned}$$

for all $t-h, t+h \in U$ with $0 \le h < \delta$. We also have

$$
\begin{aligned}
& D\Big[f^{\rho}(t)g^{\rho}(t) \ominus_g f(t-h)g(t-h), \Big(f^{\nabla_H}(t)g(t) \oplus f^{\rho}(t)g^{\nabla_H}(t)\Big)(h-\eta(t))\Big] \\
= \ & D\Big[f^{\rho}(t)g^{\rho}(t) \ominus_g f(t-h)g(t-h) \oplus g(t-h)f^{\rho}(t), \Big(f^{\nabla_H}(t)g(t) \oplus f^{\rho}(t)g^{\nabla_H}(t)\Big) \\
& (h-\eta(t)) \oplus g(t-h)f^{\rho}(t)\Big] \\
\le \ & D\Big[\Big(f^{\rho}(t) \ominus_g f(t-h)\Big)g(t-h), g(t-h)f^{\nabla_H}(t)(h-\eta(t))\Big] + D\Big[\Big(g^{\rho}(t) \ominus_g g(t-h)\Big)f^{\rho}(t), \\
& f^{\rho}(t)g^{\nabla_H}(t)(h-\eta(t))\Big] + D\Big[\widetilde{0}, g(t)f^{\nabla_H}(t)(h-\eta(t)) \ominus_g g(t-h)f^{\nabla_H}(t)(h-\eta(t))\Big] \\
\le \ & D\Big[f^{\rho}(t) \ominus_g f(t-h), f^{\nabla_H}(t)(h-\eta(t))\Big] D[g(t-h), \widetilde{0}] + D\Big[g^{\rho}(t) \ominus_g g(t-h), \\
& g^{\nabla_H}(t)(h-\eta(t))\Big] D[f^{\rho}(t), \widetilde{0}] + D\Big[\widetilde{0}, g(t) \ominus_g g(t-h)\Big] D[f^{\nabla_H}(t), \widetilde{0}]|h-\eta(t)| \\
\le \ & \epsilon^*|h-\eta(t)| \cdot \Big(D[g(t-h), \widetilde{0}] + D[f^{\rho}(t), \widetilde{0}] + D[f^{\nabla_H}(t), \widetilde{0}]\Big) \\
\le \ & \epsilon^*|h-\eta(t)| \cdot \big(1 + \|g(t)\|_{\mathcal{F}} + \|f^{\rho}(t)\|_{\mathcal{F}} + \|f^{\nabla_H}(t)\|_{\mathcal{F}}\big) \\
\le \ & \epsilon|h-\eta(t)|.
\end{aligned}
$$

Thus $(fg)^{\nabla_H}(t) = f^{\nabla_H}(t)g(t) \oplus f^{\rho}(t)g^{\nabla_H}(t)$.

The other product rule formula follows by interchanging the roles of functions f and g.

Conclusions This paper investigate the Nabla-Hukuhara derivative of fuzzy-valued functions on time scales, which extends and improve the related ones in [12]. Another research is to investigate the Nabla-Hukuhara derivative of fuzzy-valued functions on time scales in other different directions rather than the one considered here. For instance, instead of following the Nabla approach that we adopt, one can develop a diamond, or a symmetric Hukuhara derivative. These problems will be subject of future research.

References

[1] Zadeh L A. Fuzzy Sets [J]. Inform. Control, 1965, 8: 338–353.

[2] Hukuhara M. Intégration des applications mesurables dont la valeur est un compact convex [J]. Funkcial. Ekvac., 1967, 10: 205–229.

[3] Bede B, Gal S G. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations [J]. Fuzzy Sets Sys., 2005, 151: 581–599.

[4] Bede B, Rudas I J, Bencsik A. First order linear differential equations under generalized differentiability [J]. Inform. Sci., 2007, 177: 1648–1662.

[5] Bede B, Stefanini L. Generalized differentiability of fuzzy-valued functions [J]. Fuzzy Sets Sys., 2013, 230(1): 119–141.

[6] Stefanini L, Bede B. Generalized fuzzy differentiability with LU-parametric representation [J]. Fuzzy Sets Sys., 2014, 257(16): 184–203.

[7] Li Jinxian, Zhao Aimin, Yan Jurang. The Cauchy problem of fuzzy differential equations under generalized differentiability [J]. Fuzzy Sets Sys., 2012, 200(1): 1–24.

[8] Hong S. Differentiability of multivalued functions on time scales and applications to multivalued dynamic equations [J]. Nonl. Anal., 2009, 71: 3622–3637.

[9] Fard O S, Bidgoli T A. Calculus of fuzzy functions on time scales (I) [J]. Soft Comput., 2015, 19(2): 293–305.

[10] Hilger S. Ein Makettenkalkl mit Anwendung auf Zentrumsmannigfaltigkeiten [D]. Wurzburg: Universtat Wurzburg, 1988.

[11] Bohner M, Peterson A. Dynamic equations on time scales: an introduction with applications [M]. Boston: Birkhauser, 2001.

[12] Bohner M, Peterson A. Advances in dynamic equations on time scales [M]. Boston: Birkhauser, 2004.

[13] Zhao D F, Ye G J. C-integral and denjoy-C integral [J]. Comm. Korean. Math. Soc., 2007, 22(1): 27–39.

[14] Zhao D F, Ye G J. On strong C-integral of Banach-valued functions [J]. J. Chungcheong Math. Soc., 2007, 20(1): 1–10.

[15] Zhao D F. On the C_1-integral [J]. J. Math., 2011, 31(5): 823–828.

[16] Zhao D F, Li B W. A note on the C-integral [J]. J. Math., 2011, 31(4): 594–598.

时标上模糊值函数的Nabla-Hukuhara 导数

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摘要: 本文研究了时标上模糊值函数的Nabla-Hukuhara 导数的问题. 利用时标理论, 获得了关于模糊值函数的Nabla-Hukuhara 导数的若干重要性质. 这些结果推广并改进了文献[12] 中的有关结论.

关键词: Nabla-Hukuhara 导数; 模糊值函数; 时标

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