Vol. 38 (2018)
No. 3

数 学 杂 志
J. of Math. (PRC)

ANALYTIC APPROACH TO SPHERICALLY CONVEX SETS IN $\mathbb{S}^{n-1}$

SHAO Yu-cheng, GUO Qi

(*Department of Mathematics, Suzhou University of Science and Technology, Suzhou 215009, China*)

Abstract: This article concerns with the definition and elementary properties of spherically convex sets in the Euclidean unit sphere $\mathbb{S}^{n-1}$. First, we define the spherically convex combination of finitely many elements in the Euclidean spaces, in terms of which the spherically convex set and the spherically convex hull of sets in $\mathbb{S}^{n-1}$ are defined, then study the properties of the spherically convex sets and hulls. Finally, we prove that each closed spherically convex set can be expressed as the spherically convex hull of its extreme point set, the formulation and proof of which benifit from the analytic approach adopted in this paper.

Keywords: spherically convex set; spherically convex combination; radial function; convex hull; conical hull

2010 MR Subject Classification: 52A01; 52A30; 52A38

Document code: A **Article ID:** 0255-7797(2018)03-0473-08

1 Introduction

In light of the significance and the successful applications of the convexity of sets (in linear spaces) in many different mathematics branches, it is not unexpected for mathematicians to try to find its counterpart for sets in manifolds. Such an attempt started from the early 1940's with sets in $\mathbb{S}^{n-1}$, the unit sphere in the Euclidean space $\mathbb{R}^n$, since it is a typical manifold (see [2, 6, 9–11, 13] and the references therein).

Several different definitions of convex sets in $\mathbb{S}^{n-1}$ were proposed since 1940's. For instance, a set in $\mathbb{S}^{n-1}$ is called strongly convex if it contains no antipodal points ($x_1, x_2 \in \mathbb{S}^{n-1}$ are called (a pair of) antipodal points if $x_2 = -x_1$) and it contains, with each pair of its points, the shorter arc of the great circle determined by them (see [2]); weakly convex if it contains, with each pair of its points, the shorter arc or a semicircular arc of a great circle determined by them (see [10]); Robinson-convex if it contains, with each pair of its non-antipodal points, the shorter arc of the great circle determined by them (see [6]); Horn-convex if it contains, with each pair of its non-antipodal points, at least one of the great circle arcs determined by them (see [6]).

* **Received date:** 2017-02-19 **Accepted date:** 2017-03-08

Foundation item: Supported by the National Natural Science Foundation of China (11671293; 11271282).

Biography: Shao Yucheng (1985–), male, born at Nantong, Jiangsu, graduate, major in convex geometric analysis.

Corresponding author: Guo Qi.

Clearly, every strongly convex set is weakly convex and the weak convexity implies the Robinson-convexity, and the Robinson-convexity implies the Horn-convexity (see [2]). Also, simple examples show that all the inclusion relations here are proper.

Although various convexities for sets in $\mathbb{S}^{n-1}$ were proposed for different purposes and a great progress in the research was made in the early stage (see [2, 6, 9, 10, 13]) and recently this topic regains the attentions of mathematicians (see [1, 3, 4, 7, 8] and the references therein), the progress in this area (in particular in the aspect of geometric invariants) seems not as fast as expected. One of the main obstacles of progressing is that all these definitions of convexity for sets in $\mathbb{S}^{n-1}$ are in geometric forms, which made it complicated and even impossible to formulate or demonstrate some conclusions in the high dimension cases. This situation can be seen from several recent important work: Lassak in 2015 defined the width of a strongly (or spherically) convex body (see below for definition) and studied its elementary properties (see [7]), and further investigated the properties of (spherically) reduced bodies on spheres (see [8]), but the arguments he presented were somehow complicated due to his pure geometric methods. Similarly, Vigodsky's early work on demonstrating an analogue of Carathéodory's theorem for strongly convex sets in $\mathbb{S}^2$ (see [13]) was also done by a geometric approach and thus hard to extend to the high dimension cases. Also, some authors have to deal with sets in $\mathbb{S}^{n-1}$ in the frame of Euclidean spaces (see [14]).

In this paper, taking strong convexity as an example, we present a pure analytic approach to defining the convexity for sets in $\mathbb{S}^{n-1}$ and studying the basic properties of strongly convex sets in $\mathbb{S}^{n-1}$. As an illustration, at the end we prove a structure theorem of Minkowski type for strongly convex sets in $\mathbb{S}^{n-1}$.

2 Notation and Definitions

We work in the n-dimensional Euclidean space $\mathbb{R}^n$ with the classical inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. For any non-zero $u\in\mathbb{R}^n$, H_u denotes the hyperplane $\{x\in\mathbb{R}^n \mid \langle u,x\rangle=0\}$, H_u^- (or H^- simply) denotes the open half space $\{x\in\mathbb{R}^n \mid \langle u,x\rangle<0\}$, and $\bar{H}_u^-$ (or $\bar{H}^-$ simply) denotes the closed half space $\{x\in\mathbb{R}^n \mid \langle u,x\rangle\leqslant 0\}$.

For non-empty $C\subset\mathbb{R}^n$, $\operatorname{co}(C)$ and $\operatorname{cone}(C)$ denote the convex hull, the convex conical hull of C, respectively. For other notation and terms refer to [5].

A subset of the form $H^-\cap\mathbb{S}^{n-1}$ (resp. $\bar{H}^-\cap\mathbb{S}^{n-1}$) is call an open (resp. a closed) semi-sphere and a set of the form $H\cap\mathbb{S}^{n-1}$ is called a hypercircle. Generally, if $H_k\subset\mathbb{R}^n$ is a k-dimensional subspace ($1\leqslant k\leqslant n-1$), then $H_k\cap\mathbb{S}^{n-1}$ is called a k-circle. Thus a hypercircle is an $(n-1)$-circle and a 2-circle is a great circle named by other authors.

The main tool we used in our analytic approach is the radial function

$$\varphi:\mathbb{R}^n\longrightarrow\mathbb{S}^{n-1}\cup\{o\},$$

where o stands for the origin of $\mathbb{R}^n$, defined by

$$\varphi(x):=\begin{cases}\frac{x}{\|x\|}, & x\neq o,\\ o, & x=o.\end{cases}$$

Observe that the radial function has the following properties

(1) $\varphi \circ \varphi = \varphi$;

(2) $\varphi(\lambda x) = \varphi(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$;

(3) $\varphi(x) = x$ iff $x \in \mathbb{S}^{n-1}$ or $x = o$.

In terms of the radial function, we have the following definition.

Definition 2.1 For $x, y \in \mathbb{S}^{n-1}$ and $0 \leqslant \lambda \leqslant 1$, we define

$$\lambda x +_s (1-\lambda)y := \varphi(\lambda x + (1-\lambda)y)$$

called a spherically convex combination of x, y (s-convex combination for brevity).

Generally, given $x_1, x_2, \cdots, x_k \in \mathbb{S}^{n-1}$, $\lambda_1, \lambda_2, \cdots, \lambda_k \in [0,1]$ with $\sum\limits_{i=1}^{k} \lambda_i = 1$, we define their s-convex combination

$$(s)\sum_{i=1}^{k} \lambda_i x_i := \varphi(\sum_{i=1}^{k} \lambda_i x_i).$$

Natrually, when $k = 2$, we write $\lambda x +_s (1-\lambda)y$ instead of $(s)(\lambda x + (1-\lambda)y)$.

Remark 2.1 i) $x, y \in \mathbb{S}^{n-1}$ are non-antipodal points iff $\lambda x +_s (1-\lambda)y \neq o$ or equivalently $\lambda x + (1-\lambda)y \neq o$ for any $0 \leqslant \lambda \leqslant 1$.

ii) In terms of Definition 2.1, the short arc of the great circle determined by non-antipodal points $x, y \in \mathbb{S}^{n-1}$ is $\{\lambda x +_s (1-\lambda)y \mid 0 \leqslant \lambda \leqslant 1\}$.

Now, we give the definition of strongly convex sets in $\mathbb{S}^{n-1}$ in an analytic form, and we adopt "spherically convex" instead of "strongly convex".

Definition 2.2 A set $C \subset \mathbb{S}^{n-1}$ is called spherically convex (s-convex for brevity) if $\lambda x +_s (1-\lambda)y \in C$ for any $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$.

Remark 2.2 i) By Remark 2.1, an s-convex set contains no antipodal points for sure, and $C \subset \mathbb{S}^{n-1}$ is s-convex iff it contains, with each pair of its points, the short arc of the great circle determined by them.

ii) It is easy to check that an open semi-sphere is s-convex while a closed semi-sphere, a k-circle and $\mathbb{S}^{n-1}$ are not since they all contain antipodal points.

The following simple property of s-convex sets will be needed later.

Proposition 2.1 Let $C \subset \mathbb{S}^{n-1}$ be an s-convex set. Then we have

$$\operatorname{cone}(C) = \bigcup_{t \geqslant 0} tC = \{tx \mid x \in C, t \geqslant 0\}.$$

Moreover, $\operatorname{cone}(C)$ is closed iff C is closed (so compact).

Proof For the first conclusion, since $\bigcup\limits_{t \geqslant 0} tC$ is clearly a cone, we need only to show that it is convex. More precisely, we need only to show $x_1 + x_2 \in \bigcup\limits_{t \geqslant 0} tC$ whenever $x_1, x_2 \in \bigcup\limits_{t \geqslant 0} tC$. Without loss of generality, we may assume that both x_1 and x_2 are non-zero. Let $x_1 = t_1 x_1'$ and $x_2 = t_2 x_2'$ for some $x_1', x_2' \in C$ and $t_1, t_2 > 0$. Thus by the s-convexity of C,

$$x := \frac{x_1 + x_2}{\|x_1 + x_2\|} = \frac{t_1 x_1' + t_2 x_2'}{\| t_1 x_1' + t_2 x_2' \|} = \frac{\frac{t_1}{t_1+t_2} x_1' + \frac{t_2}{t_1+t_2} x_2'}{\| \frac{t_1}{t_1+t_2} x_1' + \frac{t_2}{t_1+t_2} x_2' \|} = \frac{t_1}{t_1+t_2} x_1' +_s \frac{t_2}{t_1+t_2} x_2' \in C,$$

where we used the fact that $\frac{t_1}{t_1+t_2}+\frac{t_2}{t_1+t_2}=1$, and so $x_1+x_2=\|x_1+x_2\|x\in\bigcup\limits_{t\geqslant 0}tC$.

The argument for the second conclusion is standard and straightforward.

3 Properties of s-Convex Sets and s-Convex Hull of Sets

In this section, we study basic properties of s-convex sets and in turn define the spherically convex hull (s-convex hull for brevity) of a set in $\mathbb{S}^{n-1}$. We start with the following simple property.

Proposition 3.1 i) If $\{C_i\}_{i\in\Lambda}$ is a family of s-convex sets in $\mathbb{S}^{n-1}$ with $C:=\bigcap\limits_{i\in\Lambda}C_i\neq\emptyset$, then C is s-convex.

ii) If C is s-convex and $\mathbb{S}_k:=H_k\cap\mathbb{S}^{n-1}$ is a k-circle ($2\leqslant k\leqslant n-1$), then $C\cap\mathbb{S}_k$ is s-convex provided that the intersection is not empty.

Proof i) For any $x,y\in C$, we have $x,y\in C_i$ for each $i\in\Lambda$. So $x+_s y\in C_i$ for each $i\in\Lambda$ by the s-convexity of C_i and in turn $x+_s y\in C$.

ii) For any $x,y\in C\cap\mathbb{S}_k$, we have $x+_s y\in C$ by the s-convexity of C, $x+_s y=\frac{x+y}{\|x+y\|}\in H_k$ and $x+_s y\in\mathbb{S}^{n-1}$ by the definition. So $x+_s y\in C\cap\mathbb{S}_k$.

Next is an analogue of the conclusion for convex sets in $\mathbb{R}^n$, which is not easy to formulate and prove in a geometric approach.

Theorem 3.1 If $C\subset\mathbb{S}^{n-1}$ is s-convex, then for any $x_1,x_2,\cdots,x_k\in C$ and $\lambda_1,\lambda_2,\cdots,\lambda_k\in[0,1]$ with $\sum\limits_{i=1}^{k}\lambda_i=1$, $k=1,2,\cdots$, we have $(s)\sum\limits_{i=1}^{k}\lambda_i x_i\in C$. In particular, $o\notin\mathrm{co}(C)$.

Proof It is trivial for $k=1,2$ by Definition 2.2. Suppose the conclusion holds for all $1\leqslant j\leqslant k-1$ (which implies, in particular, that $o\notin\mathrm{co}\{x_1,x_2,\cdots,x_j\}$ for any $x_1,x_2,\cdots,x_j\in C$ and $\mu_1,\mu_2,\cdots,\mu_j>0$ with $\sum\limits_{i=1}^{j}\mu_i=1$). Now for $x_1,x_2,\cdots,x_k\in C$ and $\lambda_i\geqslant 0$, $\sum\limits_{i=1}^{k}\lambda_i=1$ (we may clearly assume $\lambda_i>0$ for all i), we denote $z=\sum\limits_{i=1}^{k-1}\frac{\lambda_i}{\sum\limits_{j=1}^{k-1}\lambda_j}x_i$. Clearly, $z\neq o$ and $\varphi(z)=\frac{z}{\|z\|}\in C$ by the induction hypothesis.

Thus we have, by Definition 2.1 and the properties of φ,

$$\begin{aligned}
(s)\sum_{i=1}^{k}\lambda_i x_i &= \varphi(\sum_{i=1}^{k}\lambda_i x_i)=\varphi(\sum_{i=1}^{k-1}\frac{\lambda_i}{\sum\limits_{j=1}^{k-1}\lambda_j}x_i+\frac{\lambda_k}{\sum\limits_{j=1}^{k-1}\lambda_j}x_k)\\
&= \varphi(z+\frac{\lambda_k}{\sum\limits_{j=1}^{k-1}\lambda_j}x_k)=\varphi(\frac{z}{\|z\|}+\frac{\lambda_k}{\|z\|\sum\limits_{j=1}^{k-1}\lambda_j}x_k)\\
&= \varphi((\sum_{j=1}^{k-1}\lambda_j)\frac{z}{\|z\|}+\frac{\lambda_k}{\|z\|}x_k)=\varphi(\|z\|(\sum_{j=1}^{k-1}\lambda_j)\frac{z}{\|z\|}+\lambda_k x_k)\\
&= \varphi(\frac{\|z\|\sum\limits_{j=1}^{k-1}\lambda_j}{\|z\|\sum\limits_{j=1}^{k-1}\lambda_j+\lambda_k}\frac{z}{\|z\|}+\frac{\lambda_k}{\|z\|\sum\limits_{j=1}^{k-1}\lambda_j+\lambda_k}x_k)\\
&= \frac{\|z\|\sum\limits_{j=1}^{k-1}\lambda_j}{\|z\|\sum\limits_{j=1}^{k-1}\lambda_j+\lambda_k}\frac{z}{\|z\|}+_s\frac{\lambda_k}{\|z\|\sum\limits_{j=1}^{k-1}\lambda_j+\lambda_k}x_k\in C,
\end{aligned}$$

where we used the fact that $\frac{z}{\|z\|}$, $x_k \in C$ and

$$\frac{\|z\| \sum\limits_{j=1}^{k-1} \lambda_j}{\|z\| \sum\limits_{j=1}^{k-1} \lambda_j + \lambda_k} + \frac{\lambda_k}{\|z\| \sum\limits_{j=1}^{k-1} \lambda_j + \lambda_k} = 1.$$

Now, $o \notin \mathrm{co}(C)$ is trivial.

Next, we consider the spherically convex hull of a set in $\mathbb{S}^{n-1}$, i.e., the smallest s-convex set containing a set. Since if $C \subset \mathbb{S}^{n-1}$ is s-convex, then $o \notin \mathrm{co}(C)$ by the definition and in turn $o \notin \mathrm{co}(S)$ if $S \subset C$, so we see that a set $S \subset \mathbb{S}^{n-1}$ has a spherically convex hull iff $o \notin \mathrm{co}(S)$.

Definition 3.1 Given $S \subset \mathbb{S}^{n-1}$ with $o \notin \mathrm{co}(S)$, we define its spherically convex hull $\mathrm{Sco}(S)$ (s-convex hull for brevity) by

$$\mathrm{Sco}(S) := \varphi(\mathrm{co}(S)) = \{(s)\sum_{i=1}^{k}\lambda_i x_i \mid x_i \in S, \lambda_i \geqslant 0, \sum_{i=1}^{k}\lambda_i = 1, k = 1, 2, \cdots\}.$$

Theorem3.2 Let $S \subset \mathbb{S}^{n-1}$ with $o \notin \mathrm{co}(S)$. Then

i) $\mathrm{Sco}(S)$ is s-convex.

ii) $\mathrm{Sco}(S) = \cap\{C \subset \mathbb{S}^{n-1} \mid C \supset S$ and C is s-convex$\}$, i.e., $\mathrm{Sco}(S)$ is the smallest s-convex set containing S.

iii) $\mathrm{Sco}(S) = \mathrm{cone}(S) \cap \mathbb{S}^{n-1}$.

Proof i) Let $x := (s)\sum\limits_{i=1}^{k}\lambda_i x_i, y := (s)\sum\limits_{j=1}^{l}\mu_i y_j \in \mathrm{Sco}(S)$ and $0 < \lambda < 1$. Then by the properties of φ (write $a :=\| \sum\limits_{i=1}^{k} \lambda_i x_i \|,\ b :=\| \sum\limits_{j=1}^{l} \mu_j y_j \|$),

$$\begin{aligned} \lambda x +_s (1-\lambda) y &= \varphi\left(\lambda x + (1-\lambda) y\right) = \varphi\left(\tfrac{\lambda}{a}\sum_{i=1}^{k}\lambda_i x_i + \tfrac{(1-\lambda)}{b}\sum_{j=1}^{l}\mu_i y_j\right) \\ &= \varphi\left(\lambda b \sum_{i=1}^{k}\lambda_i x_i + (1-\lambda) a \sum_{j=1}^{l}\mu_i y_j\right) = \varphi\left(\sum_{i=1}^{k}\tfrac{\lambda b \lambda_i}{\lambda b + (1-\lambda)a} x_i + \sum_{j=1}^{l}\tfrac{(1-\lambda)a\mu_i}{\lambda b + (1-\lambda)a} y_j\right) \\ &= (s)\left(\sum_{i=1}^{k}\tfrac{\lambda b \lambda_i}{\lambda b + (1-\lambda)a} x_i + \sum_{j=1}^{l}\tfrac{(1-\lambda)a\mu_i}{\lambda b + (1-\lambda)a} y_j\right) \in \mathrm{Sco}(S), \end{aligned}$$

where we used the definition of $\mathrm{Sco}(S)$ and the fact that

$$\frac{\lambda b \lambda_i}{\lambda b + (1-\lambda)a} \geqslant 0,\ \frac{(1-\lambda)a\mu_i}{\lambda b + (1-\lambda)a} \geqslant 0$$

and

$$\sum_{i=1}^{k}\frac{\lambda b \lambda_i}{\lambda b + (1-\lambda)a} + \sum_{j=1}^{l}\frac{(1-\lambda)a\mu_i}{\lambda b + (1-\lambda)a} = 1.$$

ii) Denote $D := \cap\{C \subset \mathbb{S}^{n-1} \mid C \supset S$ and C is s-convex$\}$ (D is well-defined since $\mathrm{Sco}(S) \supset S$ and $\mathrm{Sco}(S)$ is s-convex by i)). Then $D \subset \mathrm{Sco}(S)$ clearly since $\mathrm{Sco}(S) \supset S$ and it is s-convex by i) above. Conversely, since for each s-convex $C \supset S$, $\mathrm{Sco}(S) \subset C$ by Theorem 3.1, we have $\mathrm{Sco}(S) \subset D$. Therefore $\mathrm{Sco}(S) = D$.

iii) Since $\mathrm{cone}(S) = \bigcup\limits_{t\geqslant 0} t\mathrm{co}(S)$, $\mathrm{Sco}(S) \subset \mathrm{cone}(S) \cap \mathbb{S}^{n-1}$ clearly by the definition of $\mathrm{Sco}(S)$. Conversely, if $x \in \mathrm{cone}(S)\cap\mathbb{S}^{n-1}$, then $x = \sum\limits_{i=1}^{k} \lambda_i x_i$ with $x_i \in S$, $\lambda_i \geqslant 0$ and $\|x\| = 1$ (so $\sum\limits_{i=1}^{k} \lambda_i \neq 0$). Thus

$$x = \varphi(x) = \varphi\Big(\frac{x}{\sum\limits_{j=1}^{k}\lambda_j}\Big) = (s)\sum_{i=1}^{k} \frac{\lambda_i}{\sum\limits_{j=1}^{k}\lambda_j} x_i \in \mathrm{Sco}(S)$$

by the definition of $\mathrm{Sco}(S)$ again. Hence $\mathrm{cone}(S)\cap\mathbb{S}^{n-1} \subset \mathrm{Sco}(S)$ and so $\mathrm{Sco}(S) = \mathrm{cone}(S)\cap \mathbb{S}^{n-1}$.

4 A Structure Theorem of Minkowski's Type for Closed s-Convex Sets

In this section, we establish a theorem of Minkowski type for s-convex sets, which states that a closed s-convex set can be expressed as the s-convex hull of its s-extreme points (see below for definition). We start with some necessary notation and terms.

For an s-convex $C \subset \mathbb{S}^{n-1}$, we denote $\dim C = \dim(\mathrm{cone}(C)) - 1$ (notice that $\mathrm{cone}(C)$ is convex), called the dimension of C, $\mathrm{rbd}(C) := \mathrm{rbd}(\mathrm{cone}(C)) \cap \mathrm{cl}C$, where "rbd" in the right-handed side denotes the relative boundary, called the relative boundary of C and $\mathrm{ri}(C) := \mathrm{ri}(\mathrm{cone}(C))\cap C = \mathrm{ri}(\mathrm{cone}C)\cap\mathbb{S}^{n-1}$ (the latter equality can be checked by Proposition 2.1), where "ri" in the right-handed side denotes the relative interior, called the relative interior of C. It is easy to show that if $C \subset \mathbb{S}^{n-1}$ is s-convex, then so is $\mathrm{ri}(C)$ and that if C is closed and s-convex, then $C = \mathrm{rbd}(C) \cup \mathrm{ri}(C)$.

Definition 4.1 Let $C \subset \mathbb{S}^{n-1}$ be a closed s-convex set. A point $x \in C$ is called an s-extreme point of C if $x = \frac{1}{2}x_1 +_s \frac{1}{2}x_2$ for some $x_1, x_2 \in C$, then $x_1 = x_2$. The set of s-extreme points of C is denoted by $\mathrm{Sext}(C)$.

Remark 4.1 It is easy to check that if x is an s-extreme point of C, then $x \in \mathrm{rbd}(C)$.

Proposition 4.1 If $C \subset \mathbb{S}^{n-1}$ is a closed and s-convex set, then $\mathrm{Sext}(C) \neq \emptyset$.

Proof Observe first that $\mathrm{cone}(C)$ is closed by Proposition 2.1. Now we show that $\mathrm{cone}(C)$ is also line-free: Suppose there is a line $l := \{x_0 + \mu u \mid \mu \in \mathbb{R}\} \subset \mathrm{cone}(C)$ for some $x_0 \in \mathbb{R}^n$ and $u \in \mathbb{S}^{n-1}$. Then $x_0 + \frac{1}{t}u, x_0 - \frac{1}{t}u \in \mathrm{cone}(C)$ for all $t > 0$ and in turn $tx_0 + u = t(x_0 + \frac{1}{t}u), tx_0 - u = t(x_0 - \frac{1}{t}u) \in \mathrm{cone}(C)$ for all $t > 0$ since $\mathrm{cone}(C)$ is a cone. Thus, by the closedness of $\mathrm{cone}(C)$, we have

$$u = \lim_{t\to 0^+}(tx_0 + u) \in \mathrm{cone}(C) \text{ and } -u = \lim_{t\to 0^+}(tx_0 - u) \in \mathrm{cone}(C),$$

which together with $\|u\| = \|-u\| = 1$ leads to (the antipodal points) $u, -u \in C$ by Proposition 2.1, a contradiction.

Since cone(C) is closed and line-free, it has at least one extreme ray by Theorem 1.4.3 in [12]. Let L be an extreme ray of cone(C) and let $x = L \cap C$. Then $L = \{tx \mid t \geqslant 0\}$. Now, if $x = \frac{1}{2}x_1 +_s \frac{1}{2}x_2 = (\frac{1}{2}x_1 + \frac{1}{2}x_2)/\|\frac{1}{2}x_1 + \frac{1}{2}x_2\|$ for some $x_1, x_2 \in C$, then $\frac{1}{2}x_1 + \frac{1}{2}x_2 = \|\frac{1}{2}x_1 + \frac{1}{2}x_2\|x \in L$. So $x_1, x_2 \in L$ since L is an extreme ray and in turn $x_1, x_2 \in L \cap C$ which leads to $x_1 = x_2 = x$ clearly. Thus, x is an s-extreme point.

The following is a theorem of Minkowski type for closed s-convex sets in $\mathbb{S}^{n-1}$, which, again, it is hard to prove in geometric methods.

Theorem 4.1 Let $C \subset \mathbb{S}^{n-1}$ be a closed and s-convex set. Then $C = \mathrm{Sco}\,(\mathrm{Sext}(C))$.

Proof Since cone(C) is closed by Proposition 2.1 and line-free (see the argument for Proposition 4.1), we have cone(C) = co(extr(cone(C))) by Theorem 1.4.3 in [12] again, where extr(cone(C)) denotes the union of extreme rays of cone(C). Thus, if $x \in C \subset$ cone(C), then $x = \sum\limits_{i=1}^{m} \lambda_i x_i$ with non-zero $x_i \in L_i$ for some extreme ray L_i and $\lambda_i > 0$, $\sum\limits_{i=1}^{m} \lambda_i = 1$. Hence, by the properties of φ,

$$
\begin{aligned}
x &= \varphi(x) = \varphi(\sum_{i=1}^{m} \lambda_i x_i) = \varphi(\sum_{i=1}^{m} \lambda_i \|x_i\| \tfrac{x_i}{\|x_i\|}) \\
&= \varphi(\sum_{i=1}^{m} \tfrac{\lambda_i \|x_i\|}{\sum_{j=1}^{m} \lambda_j \|x_j\|} \tfrac{x_i}{\|x_i\|}) = (s)(\sum_{i=1}^{m} \tfrac{\lambda_i \|x_i\|}{\sum_{j=1}^{m} \lambda_j \|x_j\|} \tfrac{x_i}{\|x_i\|}) \in \mathrm{Sco}\,(\mathrm{Sext}(C))\,,
\end{aligned}
$$

where we used the fact that all $\frac{x_i}{\|x_i\|} = L_i \cap C$ are in Sext(C) (see the argument for Proposition 4.1), and $\sum\limits_{i=1}^{m} \frac{\lambda_i \|x_i\|}{\sum_{j=1}^{m} \lambda_j \|x_j\|} = 1$.

Final Remark From Theorem 3.1 and Theorem 4.1, one can see that our analytic approach indeed makes it possible to formulate and demonstrate some conclusions which can hardly be formulated and proved by pure geometric methods. It is expectable that, with this analytic approach, more conclusions will be established and more concepts, such as the meaningful geometric invariants for s-convex sets etc., will be proposed later. We leave these topics to other papers.

References

[1] Brummelen G V. Heavenly mathematics, the forgotten art of spherical trigonometry[M]. Priceton: Priceton Univ. Press, 2013.

[2] Danzer L, Grünbaum B, Klee V. Helly's theorem and its relatives[J]. Conv. Proc. Symp. Pure Math., 1963, 7: 99–180.

[3] Ferreira O P, Iusem A N, Németh S Z. Projections on to convex sets on the sphere[J]. Glob. Optim., 2013, 57: 663–676.

[4] Gao F, Hug D, Schneider R. Intrisic volumes and polar sets in spherical spaces[J]. Homage Luis Santaló Vol. I, Math. Not., 2001, 41(2): 159–176.

[5] Hiriart-Urruty J-B, Lemaréchal C. Fundamentals of convex analysis[M]. Berlin, Heidelberg, New York: Springer-Verlag, 2001.

[6] Horn A. Some generalizations of Helly's theorem on convex sets[J]. Bull. Amer. Math. Soc., 1949, 55: 923–929.

[7] Lassak M. Width of spherical convex bodies[J]. Aequ. Math., 2015, 89: 555–567.

[8] Lassak M. Reduced spherical polygons[J]. Coll. Math., 2015, 138(2): 205–216.

[9] Robinson C V. Spherical theorems of Helly type and congruence indices of spherical caps[J]. Amer. J. Math., 1942, 64(1): 260–272.

[10] Santal L Ó. Propiedades de las figuras convexas sóbre la esfera[J]. Math. Not., 1944, 4: 11–40.

[11] Santal L Ó. Convex regions on the n-dimensional spherical surface[J]. Ann. Math, 1946, 47: 448–459.

[12] Schneider R. Convex Bodies: the Brunn-Minkowski theory (2nd ed.)[M]. Cambridge: Cambridge University Press, 2014.

[13] Vigodsky M. Sur les courbes fermées à indicatrice des tangentes donnée[J]. Rec. Math. [Mat. Sbornik] N.S., 1945: 73–80.

[14] Yang S G. Two classes of geometric inequalities of finite point set in space with constant curvature[J]. J. Math., 2006, 26(6): 665–668.

单位球面$\mathbb{S}^{n-1}$上球凸集的分析研究方法

邵煜骋, 国 起

(苏州科技大学数学系, 江苏 苏州 215009)

摘要: 本文研究了欧式空间单位球面$\mathbb{S}^{n-1}$上秋凸集的定义与基本性质. 利用径向函数, 定义了空间中有限个点的凸组合运算, 并由此给出了$\mathbb{S}^{n-1}$上球凸集的分析定义和集合球凸包的定义. 讨论了球凸集和球凸包的基础性质. 最后证明了任一闭球凸集都可以表示为其端点集的球凸包. 这个结论的形成与获证完全得益于本文采用的分析方法.

关键词: 球面凸集; 球面凸组合; 径向函数; 凸包; 锥包

MR(2010)**主题分类号**: 52A01; 52A30; 52A38 **中图分类号**: O184; O177.99