Vol. 38 (2018)
No. 3

数 学 杂 志
J. of Math. (PRC)

ON THE π-REGULAR CLASS FUNCTIONS OF A FINITE π-SEPARABLE GROUP

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Abstract: In this paper, we study the principal indecomposable $B_{\pi'}$-characters of a finite π-separable group. By using the decomposition matrices in the π-theory of characters, we obtain some important results on π-regular class functions of a finite π-separable group, which generalizes some well-known theorems.

Keywords: π-separable group; π-regular class functions; B_π-character

2010 MR Subject Classification: 20C15; 20C20

Document code: A **Article ID:** 0255-7797(2018)03-0453-06

1 Introduction

In literature [1], Isaacs introduced the concept of B_π-characters of a π-separable group G, where π denotes a set of primes. B_π-characters of G are also called π-Brauer characters. Denote by $B_\pi(G)$ the set of all π-Brauer characters of G. Let χ be a character of G and denote by χ^* the restriction of χ to the set of all π-elements of G. In [1], Isaacs proved that the set of χ^* forms a basis of the space of π-class functions of G. In addition, he also proved that the number of π-characters of G equals the number of conjugacy classes of π-elements of G.

Motivated by these results due to Isaacs, we want to know if other properties of Brauer characters can be generalized. In this note, by using the decomposition matrices in the π-theory of characters, the concept of principal indecomposable $B_{\pi'}$-characters of G is introduced, where π' denotes the complement set of π. Denote by $CF(G_{\pi'})$ the space of π-regular class functions of G. One of our main results is as follows.

Theorem A Let G be a π-separable group and let η_i $(1 \le i \le l)$ be the all the principal indecomposable $B_{\pi'}$-characters of G. Then the set $\{\eta_i | 1 \le i \le l\}$ forms a basis of $CF(G_{\pi'})$.

In the theory of Brauer character, generalized Brauer characters is introduced. It is known that the set of principal indecomposable characters forms a $\mathbb{Z}$-basis of $X(G|G_{p'})$, where $X(G)$ is the ring of all generalized characters of G. Let $\widetilde{\varphi} \in X(G|G_{p'})$, then $\widetilde{\varphi}(x) = 0$

* **Received date:** 2016-05-13 **Accepted date:** 2017-03-14

Foundation item: Supported by the Natural Science Foundation of Xinjiang Uygurzizhiqu (2016D01C387).

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if $x \in G - G_{p'}$. $X(G|G_{p'})$ is a $\mathbb{Z}$-module. Similarly, we can introduce the notation $X(G|G_{\pi'})$ and the concept of generalized $B_{\pi'}$-characters of G. The another main result is as follows.

Theorem B Let G be a π-separable group and let η_i $(1 \le i \le l)$ be the all the principal indecomposable $B_{\pi'}$-characters of G. Then the set $\{\eta_i | 1 \le i \le l\}$ forms a $\mathbb{Z}$-basis of $X(G|G_{\pi'})$.

Throughout this paper, groups considered are finite π-separable. $Irr(G)$ denotes the set of all irreducible ordinary characters of G; $G_{\pi'}$ denotes the set of all π-regular elements of G; χ^* denotes the restriction of χ to $G_{\pi'}$. Let H be a subgroup of G. Let χ and φ be characters of G and H respectively. Then we write χ_H the restriction of χ to H and φ^G the lift of φ to G. Unless stated otherwise, other notation and terminologies used are standard, refer to the literature [1, 4] or [9].

2 Preliminaries

First we recall some basis concepts and present some lemmas which will be used in the sequel.

Definition 2.1 Let $\chi \in Irr(G)$. Then χ^* is said to be irreducible if χ^* can not be expressed as $\chi^* = \mu^* + \nu^*$, where μ, ν are two class functions of G. $I^\pi(G)$ denotes the set of all irreducible χ^*.

Lemma 2.2 [1, Corollary 9.1] Let G be a π-separable group. Then $\{\chi^* | \chi \in B_{\pi'}(G)\}$ are linearly independent.

Lemma 2.3 [1, Corollary 9.2] Let G be a finite π-separable group. Then $|B_{\pi'}(G)|$ equals the number of π-regular classes of G.

Lemma 2.4 [1, Corollary 10.2] Let G be a finite π-separable group. Then the map $\phi : B_{\pi'}(G) \to I^\pi(G)$ defined by $\Psi \mapsto \Psi^*$ is a bijection. In particular, $I^\pi(G)$ is a basis of $CF(G_{\pi'})$.

Lemma 2.5 [1, Corollary 10.3] Let G be a p-solvable. Then $\phi : B_{p'}(G) \to IBr(G)$ defined by $\Psi \mapsto \Psi^*$ is a bijection, where $IBr(G)$ denotes the set of all irreducible Brauer characters of G.

Remark 2.6 For simplicity of notation, in view of Lemmas 2.4 and 2.5, we may write $B_{\pi'}(G) = I^\pi(G)$. In particular, $B_{p'}(G) = IBr(G)$ when $\pi = \{p\}$. Thus the space of all π-regular class functions are a generalization of the space of Brauer p-regular class functions of G.

Lemma 2.7 [1, Corollary 10.1] Let G be a finite π-separable group, $\xi \in Irr(G)$ and $\eta \in B_{\pi'}(G)$. Then there exists a nonnegative integral "decomposition number" $d_{\xi\eta}$ such that $\xi^* = \sum\limits_{\eta \in B_{\pi'}(G)} d_{\xi\eta}\eta$ for any $\xi \in Irr(G)$.

Set $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$, $B_{\pi'}(G) = \{\varphi_1, \varphi_2, \cdots, \varphi_l\}$. By Lemma 2.7,

$$(\chi_1^*, \chi_2^*, \cdots, \chi_k^*)^t = (d_{ij})_{k\times l}(\varphi_1, \varphi_2, \cdots, \varphi_l)^t.$$

Write $D = (d_{ij})_{k\times l}$, $C = D^t D$. Then D and C are said to be the decomposition matrix and the Cartan matrix of G, respectively.

Let $(\eta_1, \eta_2, \cdots, \eta_l)^t = ^t D(\chi_1, \chi_2, \cdots, \chi_k)^t$, then η_j $(1 \leq j \leq l)$ are said to be the principal indecomposable $B_{\pi'}$-characters of G. It is easy to verify that the relation of the principal indecomposable $B_{\pi'}$-characters and the $B_{\pi'}$-characters of G is as follows

$$(\eta_1^*, \eta_2^*, \cdots, \eta_l^*)^t = C(\varphi_1, \varphi_2, \cdots, \varphi_l)^t.$$

In particular, if G is a π'-group, then $\chi_i = \eta_i = \varphi_i$, where $1 \leq i \leq k$.

Lemma 2.8 Let G be a finite π-separable group and notation be as above, then the following statements hold

(1) $\sum\limits_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y) = \delta_{x^G y^G}|C_G(x)|$ for any $y \in G_{\pi'}$;

(2) $\eta_i(x) = 0$ for any $x \in G - G_{\pi'}$, where $1 \leq i \leq l$.

Proof (1) For any $y \in G_{\pi'}$, we have $\chi_i(y) = \sum\limits_{j=1}^{l} d_{ij}\varphi_j(y)$. By the second orthogonal relation of characters, one gets that

$$\begin{aligned}\delta_{x^G y^G}|C_G(x)| &= \sum_{i=1}^{k} \overline{\chi_i(x)}\chi_i(y) = \sum_{i=1}^{k} \overline{\chi_i(x)} \sum_{j=1}^{l} d_{ij}\varphi_j(y) = \sum_{i=1}^{k}\sum_{j=1}^{l} d_{ij}\overline{\chi_i(x)}\varphi_j(y) \\ &= \sum_{j=1}^{k} d_{ji} \sum_{i=1}^{l} \overline{\chi_j(x)}\varphi_i(y) = \sum_{i=1}^{l} d_{ji}(\sum_{j=1}^{k} \overline{\chi_j(x)})\varphi_i(y) = \sum_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y).\end{aligned}$$

Consequently, we have $\sum\limits_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y) = \delta_{x^G y^G}|C_G(x)|$.

(2) For any $x \in G - G_{\pi'}$ and $y \in G_{\pi'}$, by (1), we have $\sum\limits_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y) = 0$. As y is arbitrary, we obtain that $\sum\limits_{i=1}^{l} \overline{\eta_i(x)}\varphi_i = 0$. Note that $\{\varphi_i | 1 \leq i \leq l\}$ is linearly independent, so the previous equality yields that $\eta_i(x) = 0$ for each $1 \leq i \leq l$. This completes the proof of Lemma 2.8.

Let G be a π-separable group and let φ and η be class functions defined on $G_{\pi'}$. Write $(\varphi, \eta)' = \frac{1}{|G|} \sum\limits_{x \in G_{\pi'}} \varphi(x)\overline{\eta(x)}$.

Lemma 2.9 Let G be a π-separable group and notation be as above, then $(\eta_i, \varphi_j)' = \delta_{ij}$.

Proof By Lemma 2.3 we may assume that $Cl(G_{\pi'}) = \{C_1, C_2, \cdots, C_l\}$, where $x_i \in C_i$ $(1 \leq i \leq l)$.

Write $\Phi = (\varphi_i(x_j))_{l\times l}$, $Y = (\eta_i(x_j))_{l\times l}$ and $S = \begin{pmatrix} |C_G(x_1)| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & |C_G(x_l)| \end{pmatrix}$.

By Lemma 2.8(1), we have $\overline{Y}^t\Phi = S$ and thus $\overline{Y}^t(\Phi S^{-1}) = I$, i.e., $(\Phi S^{-1})^t\overline{Y} = I$. Therefore $\sum\limits_{\nu=1}^{l} \varphi_i(x_\nu)\frac{1}{|C_G(x_\nu)|}\overline{\eta_j(x_\nu)} = \delta_{ij}$, i.e.,

$$\frac{1}{|G|}\sum_{\nu=1}^{l} \varphi_i(x_\nu)\frac{|G|}{|C_G(x_\nu)|}\overline{\eta_j(x_\nu)} = \delta_{ij}.$$

On the other hand,

$$\begin{aligned}\frac{1}{|G|}\sum_{\nu=1}^{l}\varphi_i(x_\nu)\frac{|G|}{|C_G(x_\nu)|}\overline{\eta_j(x_\nu)} &= \frac{1}{|G|}\sum_{\nu=1}^{l}|Cl(x_\nu)|\varphi_i(x_\nu)\overline{\eta_j(x_\nu)}\\ &= \frac{1}{|G|}\sum_{x\in G_{\pi'}}\varphi_i(x)\overline{\eta_j(x)}=(\varphi_i,\eta_j)'.\end{aligned}$$

Consequently, $(\varphi_i,\eta_j)'=\delta_{ij}$.

Lemma 2.10 Let G be a π-separable group and notation be as above, then the following statements hold

(1) $C_{ij}=(\eta_i,\eta_j)_G$;

(2) Let $Z=((\varphi_i,\varphi_j)')_{l\times l}$, then $CZ=I$.

Proof (1) Since $C={}^tDD$, it follows that $C_{ij}=\sum\limits_{v=1}^{k}d_{vi}d_{vj}$. But note that $\eta_i=\sum\limits_{j=1}^{k}d_{ji}\chi_j$ $(1\le i\le l)$, so

$$\begin{aligned}(\eta_i,\eta_j)_G &= \frac{1}{|G|}\sum_{x\in G}\eta_i(x)\overline{\eta_j(x)}=\frac{1}{|G|}\sum_{x\in G}(\sum_{u=1}^{k}d_{ui}\chi_u(x)\sum_{v=1}^{k}d_{vj}\overline{\chi_v(x)})\\ &= \sum_{u=1}^{k}\sum_{v=1}^{k}d_{ui}d_{vj}(\frac{1}{|G|}\sum_{x\in G}\chi_u(x)\overline{\chi_v(x)})=\sum_{u=1}^{k}\sum_{v=1}^{k}d_{ui}d_{vj}(\chi_u,\chi_v)_G\\ &= \sum_{u=1}^{k}\sum_{v=1}^{k}d_{ui}d_{vj}\delta_{uv}=\sum_{u=1}^{k}d_{ui}d_{vj}=C_{ij}.\end{aligned}$$

(2) Since $CZ=\begin{pmatrix}C_{11}&\cdots&C_{1l}\\ \vdots&\ddots&\vdots\\ C_{l1}&\cdots&C_{ll}\end{pmatrix}\begin{pmatrix}(\varphi_1,\varphi_1)'&\cdots&(\varphi_1,\varphi_1)'\\ \vdots&\ddots&\vdots\\ (\varphi_l,\varphi_1)'&\cdots&(\varphi_l,\varphi_l)'\end{pmatrix}$, it follows that the (i,j)-element of CZ is $\sum\limits_{\nu=1}^{l}C_{i\nu}(\varphi_v,\varphi_j)'$. It suffices to show that $\sum\limits_{\nu=1}^{l}C_{i\nu}(\varphi_v,\varphi_j)'=\delta_{ij}$. In fact, we have

$$\begin{aligned}&\sum_{\nu=1}^{l}C_{i\nu}(\varphi_v,\varphi_j)'=\sum_{\nu=1}^{l}(\eta_i,\eta_\nu)_G(\varphi_v,\varphi_j)'\\ =&\sum_{\nu=1}^{l}\frac{1}{|G|}\sum_{x\in G}\eta_i(x)\overline{\eta_\nu(x)}\frac{1}{|G|}\sum_{y\in G_{\pi'}}\varphi_\nu(y)\overline{\varphi_j(y)}=\frac{1}{|G|^2}\sum_{\nu=1}^{l}\sum_{x\in G_{\pi'}}\sum_{y\in G_{\pi'}}\eta_i(x)\overline{\eta_\nu(x)}\varphi_\nu(y)\overline{\varphi_j(y)}\\ =&\frac{1}{|G|^2}\sum_{x\in G_{\pi'}}\sum_{y\in G_{\pi'}}(\sum_{\nu=1}^{l}\overline{\eta_\nu(x)}\varphi_\nu(y))\eta_i(x)\overline{\varphi_j(y)}=\frac{1}{|G|^2}\sum_{x\in G_{\pi'}}\sum_{y\in G_{\pi'}}(\delta_{x^G,y^G}|C_G(x)|)\eta_i(x)\overline{\varphi_j(y)}\\ =&\frac{1}{|G|^2}\sum_{x\in G_{\pi'}}\frac{|G|}{|C_G(x)|}|C_G(x)|\eta_i(x)\overline{\varphi_j(x)}=\frac{1}{|G|}\sum_{x\in G_{\pi'}}\eta_i(x)\overline{\varphi_j(x)}=(\eta_i,\varphi_j)'=\delta_{ij}.\end{aligned}$$

Lemma 2.11 [9, Theorem 4.2] Let $\varphi \in CF(G)$. Then $\varphi \in X(G)$ if and only if $\varphi_E \in X(E)$ for any $E \in \varepsilon(G)$, where $\varepsilon(G)$ denotes the set of elementary subgroups of G.

3 Proof of Main Theorem

Theorem 3.1 Let G be a π-separable group and $\{\eta_i | 1 \le i \le l\}$ be principal indecomposable $B_{\pi'}$-characters. Then $\{\eta_i | 1 \le i \le l\}$ is a basis of the space $CF(G_{\pi'})$ of all π-regular functions.

Proof We first show that $\{\eta_i | 1 \le i \le l\}$ is linearly independent. Let $\sum\limits_{i=1}^{l} k_i \eta_i = 0$, where $k_i \in \mathbb{C}$. We shall show that $k_i = 0$.

By Lemma 2.9, $(\varphi, \sum\limits_{j=1}^{l} k_j \eta_j)' = \sum\limits_{j=1}^{l} k_j (\varphi_i, \eta_j)' = \sum\limits_{j=1}^{l} k_j \delta_{ij} = k_i$, it follows that $k_i = 0$. Therefore $\{\eta_i | 1 \le i \le l\}$ is linearly independent. Note that $B_{\pi'}(G)$ is a basis of $CF(G_{\pi'})$, so $|B_{\pi'}(G)| = l$ and hence $\{\eta_i | 1 \le i \le l\}$ is also a basis of the space $CF(G_{\pi'})$ of all π-regular functions.

Theorem 3.2 Let G be a π-separable group, $\varphi \in B_{\pi'}(G)$ and $x \in G$. Define $\widetilde{\varphi}$ as follows: $\widetilde{\varphi}(x) = \varphi(x_{\pi'})$. Then $\widetilde{\varphi}$ is a generalized character of G. It follows that there exists $m_i \in \mathbb{Z}$ such that $\widetilde{\varphi} = \sum\limits_{i=1}^{k} m_i \chi_i$. We call $\widetilde{\varphi}$ a generalized $B_{\pi'}$-character of G.

Proof Let E be a elementary group of G. Then E is nilpotent and hence $E = P \times Q$, where P is a π-group and Q is a π'-group. By the definition of $\widetilde{\varphi}$, $\widetilde{\varphi}_E = 1_P \times \varphi_Q$. In fact, for any $x \in E$, we have $x = pq$ with $p \in P$ and $q \in Q$. Note that $\widetilde{\varphi}_E(x) = \widetilde{\varphi}_E(pq) = \varphi(q)$ and $(1_p \times \varphi_Q)(x) = (1_p \times \varphi_Q)(pq) = 1_p(p) \times \varphi_Q(q) = \varphi(q)$, so $\widetilde{\varphi}_E = 1_p \times \varphi_Q$. Hence $\widetilde{\varphi}_E$ is a character on E, i.e., $\widetilde{\varphi}_E \in X(E)$. By Lemma 2.11, $\widetilde{\varphi} \in X(G)$. It follows that there exists $m_i \in \mathbb{Z}$ such that $\widetilde{\varphi} = \sum\limits_{i=1}^{l} m_i \chi_i$.

Theorem 3.3 Let G be a π-separable group and notation as above. Then the rank of the decomposition matrix D equals l.

Proof Write $\widetilde{\varphi_i} = \sum\limits_{j} m_{ij} \chi_j$ with $m_{ij} \in \mathbb{Z}$ and let $M = (m_{ij})_{l \times k}$. Since

$$\varphi_i = \widetilde{\varphi_i}|_{G_{\pi'}} = \sum_{j=1}^{k} m_{ij} (\chi_j|_{G_{\pi'}}) = \sum_{j=1}^{k} m_{ij} (\sum_{r=1}^{l} d_{jr} \varphi_r) = \sum_{r=1}^{l} (\sum_{j=1}^{k} m_{ij} d_{jr}) \varphi_r.$$

Since $\{\varphi_i\}$ is linearly independent, it follows that $\sum\limits_{j=1}^{k} m_{ij} d_{jr} = 1$, i.e., $MD = I$. Therefore the rank of D is l.

Theorem 3.4 Let G be a π-separable group and $\{\eta_i | 1 \le i \le l\}$ be principal indecomposable $B_{\pi'}$-characters. Then $\{\eta_i | 1 \le i \le l\}$ is a $\mathbb{Z}$-basis of $X(G|G_{\pi'})$.

Proof Note that by Lemma 2.8(2), $\eta_i(x) = 0$ for any $x \in G - G_{\pi'}$, so $\eta_i \in X(G|G_{\pi'}) \ne \emptyset$. Since $\eta_i = \sum\limits_{j=1}^{k} d_{ji} \chi_j$ $(1 \le i \le l)$, $\text{rank}(D) = l$ and $\{\chi_i | 1 \le i \le k\}$ is linearly independent, it follows that $\{\eta_i | 1 \le i \le l\}$ is also linearly independent. Since $C \otimes_{\mathbb{Z}} X(G|G_{\pi'})$ is a subspace of $CF((G_{\pi'}))$, so $\dim_C(C \otimes_{\mathbb{Z}} X(G|G_{\pi'})) \le l$ and hence $\{\eta_i|1 \le i \le l\}$ is a C-basis of $C \otimes_{\mathbb{Z}} X(G|G_{\pi'})$. For any $\theta \in X(G|G_{\pi'})$, we have $\theta = \sum\limits_{i=1}^{l} a_i\eta_i$ $(a_i \in C)$ and hence

$$\begin{aligned} a_i &= (\theta, \varphi_i)' = \frac{1}{|G|}\sum_{x\in G_{\pi'}} \theta(x)\overline{\varphi_i(x)} = \frac{1}{|G|}\sum_{x\in G}\theta(x)\overline{\widetilde{\varphi}_i(x)} \\ &= (\theta, \widetilde{\varphi}_i)_G = (\theta, \sum_j m_{ij}\chi_j)_G = \sum_j m_{ij}(\theta, \chi_j)_G. \end{aligned}$$

Note that $m_{ij} \in \mathbb{Z}$, $\theta \in X(G|G_{\pi'})$ and $(\theta, \chi_j)_G \in \mathbb{Z}$, so $a_i \in \mathbb{Z}$. Therefore $\{\eta_i|1 \le i \le l\}$ is a $\mathbb{Z}$-basis of $X(G|G_{\pi'})$.

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关于有限π-可分群的π-正则类函数

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摘 要: 本文研究了有限π-可分群的主不可分解$B_{\pi'}$-特征标. 利用特征标π-理论中的分解矩阵, 得到了关于有限π-可分群的π-正则类函数的一些重要结果, 推广了一些著名定理.

关键词: π-可分群; π-正则类函数; B_π-特征标

MR(2010)**主题分类号**: 20C15; 20C20 **中图分类号**: O152.6