Vol. 38 (2018)
No. 3

数 学 杂 志
J. of Math. (PRC)

GLOBAL HYPOELLIPTIC ESTIMATE FOR LANDAU OPERATOR WITH EXTERNAL POTENTIAL

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Abstract: In this paper, we study the hypoellipticity of Landau-type operator with external force. By using Fourier transform and symbolic class calculation, we get the global hypoelliptic estimate under suitable assumptions on the external potential.

Keywords: global hypoellipticity; Landau equation; external potential; Wick quantization

2010 MR Subject Classification: 35A25; 47J05

Document code: A **Article ID:** 0255-7797(2018)03-0427-26

1 Introduction and Main Results

We may recall here that the Landau equation reads as the evolution equation of the density of particles

$$\begin{cases}\partial_t f + y\cdot\nabla_x = Q_L(f,f),\\ f|_{t_0} = f_0,\end{cases}$$

where Q_L is the so-called Landau collision operator

$$Q_L(f,f) = \nabla_y\cdot\int_{R^3} a(y-y_*)\big((f(y_*)\nabla_y f)(y) - f(y)(\nabla_y f)(y_*)\big)dy_*,$$

here, $a(y)$ is a symmetric nonnegative matrix depending on a parameter $y\in R^3$,

$$a(y) = \big(\delta_{ij} - \frac{y_i y_j}{|y|^2}\big)\,|y|^2,\ \ 1\le i,j\le 3$$

and

$$\delta_{ij} = \begin{cases}1 & i=j,\\ 0 & i\ne j.\end{cases}$$

The original Landau collision operator describing collisions among charged particles interacting with Coulombic force. As in the Boltzmann equation, it is well-known that Maxwellians are steady states to the Landau equation $\mu(y) = e^{-\frac{|y|^2}{2}}$ and we linearize the Landau equation around μ by posing $f = \mu + \mu^{\frac12}u$ the perturbation u satisfies the equation

$$\partial_t u + y\partial_x u + \mu^{-\frac12}Q_L(\mu^{\frac12}u,\mu) + \mu^{-\frac12}Q_L(\mu,\mu^{\frac12}u) = -\mu^{-\frac12}Q_L(\mu^{\frac12},\mu^{\frac12}u).$$

* **Received date:** 2016-11-16 **Accepted date:** 2017-02-27

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We consider the Landau-type operator with external potential

$$\mathcal{L} = y \cdot \partial_x - \partial_x V(x) \cdot \partial_y + \mu^{-\frac{1}{2}} Q_L(\mu^{\frac{1}{2}} u, \mu) + \mu^{-\frac{1}{2}} Q_L(\mu, \mu^{\frac{1}{2}} u). \tag{1.1}$$

In the linear homogeneous case, Fokker-Planck equations, Landau equations and Boltzmann equations equations have then a parabolic behavior, and the study of the local smoothing properties in the velocity variable is rather direct. In the non-homogeneous case, the regularization in space variable is not so easy, but occurs anyway thanks to the so-called hypoelliptic structure of the equation. In this article, we are interested in global estimates of the following Landau operator

$$\begin{aligned}\mathcal{L} =& i(y \cdot D_x - \partial_x V(x) \cdot D_y) + p(y) + (y \wedge D_y) \cdot a(y)(y \wedge D_y) + D_y \cdot M(y) D_y \\ &+ (D_y \wedge b) \cdot (y \wedge D_y) + q(y) + \mu^{\frac{1}{2}} m(y),\end{aligned}$$

where $D_x = -i\partial_x$, $D_y = -i\partial_y$, and $x \in \mathbb{R}^3$ is the space variable and $y \in \mathbb{R}^3$ is the velocity variable, and $X \cdot Y$ stands for the standard dot-product on $\mathbb{R}^3$. The real-valued function $V(x)$ of space variable x stands for the macroscopic force, $M(y)$ is a metric and the functions $a(y)$, $b(y)$ and $p(y)$ of the variable y in the diffusion are smooth and real-valued with the properties subsequently listed below.

(1) There exists a constant $c > 0$ such that

$$\begin{aligned} &\forall\, y \in \mathbb{R}^3, \quad a(y) \geq c \langle y \rangle^{\gamma}, \ b(y) \geq c \langle y \rangle^{\gamma}, \ c \langle y \rangle^{\gamma} \leq |q(y)| \leq c \langle y \rangle^{\gamma+1}, \\ \text{and} \quad & p(y) \geq c \langle y \rangle^{2+\gamma}, c \langle y \rangle^{\gamma-1} \leq |m(y)| \leq c \langle y \rangle^{\gamma+2} \end{aligned} \tag{1.2}$$

with $\gamma \in [0,1]$ and $\langle y \rangle = \left(1 + |y|^2\right)^{1/2}$.

(2) For any $\alpha \in \mathbb{Z}_+^3$, there exists a constant C_α such that

$$\forall\, y \in \mathbb{R}^3, \quad |\partial^\alpha a(y)| + |\partial^\alpha b(y)| \leq C_\alpha \langle y \rangle^{\gamma - |\alpha|}, \quad \text{and} \quad |\partial^\alpha p(y)| \leq C_\alpha \langle y \rangle^{2+\gamma-|\alpha|}. \tag{1.3}$$

(3) $M(y)$ is a positive definite matrix with

$$M_{ij}(y) = \int_{R^3} |y - y_*|^{\gamma} (\delta_{ij} |y_*|^2 - y_{i_*} y_{j_*}) dy_*, \tag{1.4}$$

here we can substitute $D_y \cdot F(y) D_y$ for $D_y \cdot M(y) D_y$ with $F(y) \gtrsim \langle y \rangle^{\gamma}$. It is sometimes convenient to rewrite the operator as the form

$$\mathcal{L} = i(y \cdot D_x - \partial_x V(x) \cdot D_y) + \left(B(y) D_y\right)^* \cdot B(y) D_y + p(y) + q(y) + (D_y \wedge b) \cdot (y \wedge D_y) + \mu^{\frac{1}{2}} m(y), \tag{1.5}$$

where the matrix $B(y)$ is given by

$$B(y) = \left(B_{jk}(y)\right)_{1 \leq j,k \leq 3} = \begin{pmatrix} \sqrt{F(y)} & -y_3 \sqrt{b(y)} & y_2 \sqrt{b(y)} \\ -y_3 \sqrt{b(y)} & \sqrt{F(y)} & -y_1 \sqrt{b(y)} \\ -y_2 \sqrt{b(y)} & y_1 \sqrt{b(y)} & \sqrt{F(y)} \end{pmatrix}, \tag{1.6}$$

and $\big(B(y)D_y\big)^* = D_y B(y)^T$ with B^T the transpose of B, is the formal adjoint of $B(y)D_y$. By (1.2) and (1.3), one has, for any $y, \eta \in \mathbb{R}^3$ and any $\alpha \in \mathbb{Z}^3_+$,

$$\begin{aligned} &|\partial^\alpha B_{jk}(y)| \le C_\alpha \langle y\rangle^{1-|\alpha|+\gamma/2}, \\ &|B(y)\eta|^2 = a(y)\,|\eta|^2 + F(y)\,|y\wedge\eta|^2 \ge c\,|y|^\gamma \left(|\eta|^2 + |y\wedge\eta|^2\right). \end{aligned} \tag{1.7}$$

Denoting by (ξ, η) the dual variables of (x, y), we notice that the diffusion only occurs in the variables (y, η), but not in the other directions; and that the cross product term $y \wedge D_y$ improves this diffusion in specific directions of the phase space. In [3], the authors gave a estimate of the main term to the operator $\mathcal{L}$. In this work, we aim at dealing with the low order terms to linear Landau-type operators and giving a similar results. Our main results can be stated as follows.

Theorem 1.1 Let $V \in C^2(\mathbb{R}^3;\, \mathbb{R})$ satisfy that

$$\forall\ |\alpha| = 2,\ \exists\ C_\alpha > 0 \text{ such that } \forall x \in \mathbb{R}^3, \quad |\partial_x^\alpha V(x)| \le C_\alpha \langle \partial_x V(x)\rangle^{2/3}, \tag{1.8}$$

then there exists a constant C such that for any $u \in C_0^\infty(\mathbb{R}^6)$, one has

$$\begin{aligned} &\big\| \langle y\rangle^{\frac{\gamma}{6}} \langle \partial_x V(x)\rangle^{\frac{2}{3}} u\big\|^2_{L^2} + \big\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\big\|^2_{L^2} + \big\| |D_x|^{2/3} u\big\|^2_{L^2} + \big\| \langle y\rangle^{\gamma/2} |D_y|^2 u\big\|^2_{L^2} \\ &+ \big\| \langle y\rangle^{\gamma/2} |y\wedge D_y|^2 u\big\|^2_{L^2} \le\ C\Big\{ \big\|\mathcal{L}u\big\|^2_{L^2} + \big\|u\big\|^2_{L^2} \Big\}. \end{aligned} \tag{1.9}$$

Estimates of the type given in Theorem 1.1 can be analyzed through different point of views. At first they give at least local regularity estimates in the velocity direction, according to the term $|D_y|^2$ appearing in (1.9). Now one of the goal of this article was to give global estimates in order to identify the good functional spaces associated to the problems.

The second main feature of this result is to reflect the regularizing effect in space variable x, thanks to the hypoelliptic structure, which leads to terms involving e.g. $|D_x|^{2/3}$. Recall that the exponent $2/3$ here is optimal .

Now similarly to the case of elliptic directions, it may be interesting to get global weighted estimates in space direction. In [7, 9], the authors studied the Fokker-Planck case, in particular with a potential. In this direction, the work [4] also gave a first subelliptic global (optimal) estimate, concerning the Landau operator in the case when there is no potential; the main feature of that work was to show that subellipticity in space direction occured with anisotropic weights of type $\langle y\rangle^\gamma\, y\wedge D_x$. In the present article, we first give complete form of operator and recover the same type of behavior, with additional terms also involving wedges linked with the potential V.

The present work is a natural continuation of [1, 3, 4], and as there we will make a strong use of pseudodifferential calculus.

2 Notations and Some Basic Facts

We first list some notations used throughout the paper. Denote respectively by $(\cdot,\ \cdot)_{L^2}$ and $\|\cdot\|$ the inner product and the norm in $L^2(\mathbb{R}^n)$. For a vector-valued functions $U = \big(u_1, \cdots, u_n\big)$ the norm $\|U\|_{L^2}$ stands for $\big(\sum\limits_j \|u_j\|^2_{L^2}\big)^{1/2}$.

To simplify the notation, by $A \lesssim B$ we mean there exists a positive constant C, such that $A \leq CB$, and similarly for $A \gtrsim B$. While the notation $A \approx B$ means both $A \lesssim B$ and $B \lesssim A$ hold.

Now, we introduce some notations of phase space analysis and recall some basic properties of symbolic calculus, and refer to [8] and [11] for detailed discussions. Throughout the paper let g be the admissible metric $|dz|^2 + |d\zeta|^2$ and m be an admissible weight for g (see [8] and [11] for instance the definitions of admissible metric and weight). Given a symbol $p(z,\zeta)$, we say $p \in S\big(m,\ g\big)$ if

$$\forall\, \alpha,\ \beta \in \mathbb{Z}_+^n,\ \ \forall\,(z,\zeta) \in \mathbb{R}^{2n}, \quad \left|\partial_z^\alpha \partial_\zeta^\beta p(z,\zeta)\right| \leq C_{\alpha,\beta}\, m(z,\zeta)$$

with $C_{\alpha,\beta}$ a constant depending only on α,β. For such a symbol p we may define its Weyl quantization p^w by

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad p^w u(z) = \int e^{2i\pi(z-v)\cdot\zeta} p\left(\frac{z+v}{2}, \zeta\right) u(v) dv d\zeta.$$

The L^2 continuity theorem in the class $S\big(1,\ g\big)$, which will be used frequently, says that if $p \in S\big(1,\ g\big)$ then

$$\forall\, u \in L^2, \quad \left\|p^w u\right\|_{L^2} \lesssim \left\|u\right\|_{L^2}.$$

We shall denote by $Op(S(m,g))$ the set of operators whose symbols are in the class $S(m,g)$. Finally, let's recall some basic properties of the Wick quantization, and refer the reader to the works of Lerner [11] for thorough and extensive presentations of this quantization and some of its applications. Using the notation $Z = (z,\zeta) \in \mathbb{R}^{2n}$, the wave-packets transform of a function $u \in \mathcal{S}\big(\mathbb{R}^n\big)$ is defined by

$$Wu(Z) = \big(u, \varphi_Z\big)_{L^2(\mathbb{R}^n)} = 2^{n/4} \int_{\mathbb{R}^n} u(v) e^{-\pi|v-z|^2} e^{2i\pi(v-z/2)\cdot\zeta} dv$$

with $\varphi_Z(v) = 2^{n/4} e^{-\pi|v-z|^2} e^{2i\pi(v-z/2)\cdot\eta},\ \ v \in \mathbb{R}^n$, then one can verify that W is an isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$,

$$\left\|Wu\right\|_{L^{2n}} = \left\|u\right\|_{L^2}. \tag{2.1}$$

Moreover the operator $\pi_{\mathcal{H}} = WW^*$, with W^* the adjoint of W, is an orthogonal projection on a closed space in L^2 whose kernel is given by

$$K\big(Z, \tilde{Z}\big) = e^{-\frac{\pi}{2}\left(|z-\tilde{z}|^2 + |\zeta-\tilde{\zeta}|^2\right)} e^{i\pi\left(z-\tilde{z}\right)\cdot\left(\zeta+\tilde{\zeta}\right)}, \quad Z = (z,\zeta),\ \tilde{Z} = (\tilde{z},\tilde{\zeta}). \tag{2.2}$$

We define the Wick quantization of any L^∞ symbol p as $p^{\mathrm{Wick}} = W^* p W$. The main property of the Wick quantization is its positivity, i.e.,

$$p(Z) \geq 0 \ \text{ for all } Z \in \mathbb{R}^{2n} \text{ implies } \ p^{\mathrm{Wick}} \geq 0. \tag{2.3}$$

According to Proposition 2.4.3 in [11], the Wick and Weyl quantizations of a symbol p are linked by the following identities

$$p^{\mathrm{Wick}} = p^w + r^w \tag{2.4}$$

with

$$r(Z) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta)p''(Z+\theta Y)Y^2 e^{-2\pi|Y|^2} 2^n \, dY d\theta.$$

We also recall the following composition formula obtained in the proof of Proposition 3.4 in [10]

$$p^{\mathrm{Wick}} q^{\mathrm{Wick}} = \big[pq - \frac{1}{4\pi} p' \cdot q' + \frac{1}{4i\pi}\{p, q\}\big]^{\mathrm{Wick}} + T$$

with T a bounded operator in $L^2(\mathbb{R}^{2n})$, when $p \in L^\infty(\mathbb{R}^{2n})$ and q is a smooth symbol whose derivatives of order ≥ 2 are bounded on $\mathbb{R}^{2n}$. The notation $\{p, q\}$ denotes the Poisson bracket defined by

$$\big\{p,\ q\big\} = \frac{\partial p}{\partial \zeta} \cdot \frac{\partial q}{\partial z} - \frac{\partial p}{\partial z} \cdot \frac{\partial q}{\partial \zeta}. \tag{2.5}$$

3 The Proof of Theorem 1.1: Weighted Estimates

In this section, we are mainly concerned with the estimate in weighted L^2 norms, that is

Proposition 3.1 Let $V(x) \in C^2(\mathbb{R}^3;\ \mathbb{R})$ satisfy condition (1.8), then

$$\forall\, u \in C_0^\infty(\mathbb{R}^6), \quad \big\| \langle y \rangle^{\frac{\gamma}{6}} \left|\partial_x V(x)\right|^{\frac{2}{3}} u \big\|_{L^2}^2 + \big\| \langle y \rangle^{2+\frac{5\gamma}{6}} u \big\|_{L^2}^2 \lesssim \big\|\mathcal{L}u\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2. \tag{3.1}$$

In order to prove the proposition, we begin with

Lemma 3.2 Considerate the operator $\mathcal{L}$ in (1.5), in the elliptic direction we have an estimate. For all $u \in C_0^\infty(R^3)$,

$$\big\| \langle y \rangle^{\gamma/2} D_y u \big\|_{L^2}^2 + \big\| \langle y \rangle^{\gamma/2} \big(y \wedge D_y\big) u \big\|_{L^2}^2 \lesssim \big\| B(y) D_y u \big\|_{L^2}^2 \lesssim \mathrm{Re}\,(\mathcal{L}u, u)_{L^2} \tag{3.2}$$

and

$$\big\| \langle y \rangle^{1+\gamma/2} u \big\|_{L^2}^2 + \big\| \langle y \rangle^{\gamma/2} D_y u \big\|_{L^2}^2 + \big\| \langle y \rangle^{\gamma/2} \big(y \wedge D_y\big) u \big\|_{L^2}^2 \lesssim \mathrm{Re}\,(\mathcal{L}u, u)_{L^2}\,,$$

where $(\cdot,\ \cdot)_{L^2}$ and $\big\|\cdot\big\|_{L^2}$ standing for the inner product and norm in $L^2(\mathbb{R}^6_{x,y})$.

Proof Observing $i(y \cdot D_x - \partial_x V(x) \cdot D_y)$ is skew-adjoint, then

$$\begin{aligned}
\mathrm{Re}\,(\mathcal{L}u, u)_{L^2} = {} & \Big(\big(B(y)D_y\big)^* \cdot B(y) D_y u, u\Big)_{L^2} + (p(y)u, u)_{L^2} \\
& + (q(y)u, u)_{L^2} + \Big(\mu^{\frac{1}{2}} m(y) u, u\Big)_{L^2} + \left((D_y \wedge b) \cdot (v \wedge D_y)\, u, u\right)_{L^2} \\
\gtrsim {} & (B(y)D_y u, B(y) D_y u)_{L^2} \\
& (p(y), u) + ((y \wedge D_y) \cdot a(y)(y \wedge D_y) u, u) + (D_y \cdot M(y) D_y u, u) \\
& + ((D_y \wedge b) \cdot (y \wedge D_y) u, u) + (q(y)u, u) + \Big(\mu^{\frac{1}{2}} m(y) u, u\Big).
\end{aligned}$$

The inequality hold due to

$$(p(y)u,u)_{L^2}+(q(y)u,u)_{L^2}+\Big(\mu^{\frac{1}{2}}m(y)u,u\Big)_{L^2}\gtrsim\big\|\langle y\rangle^{1+\gamma/2}u\big\|_{L^2}^2$$

and

$$\Big(\big(B(y)D_y\big)^*\cdot B(y)D_yu,u\Big)_{L^2}\geq((D_y\wedge b)\cdot(v\wedge D_y)u,u)_{L^2}.$$

By (1.2) and (1.3) one has, for any $y,\eta\in\mathbb{R}^3$ and any $\alpha\in\mathbb{Z}_+^3$,

$$|B(y)\eta|^2=a(y)|\eta|^2+b(y)|y\wedge\eta|^2\geq c|y|^\gamma\left(|\eta|^2+|y\wedge\eta|^2\right),$$

then

$$\big\|\langle y\rangle^{\gamma/2}D_yu\big\|_{L^2}^2+\big\|\langle y\rangle^{\gamma/2}\big(y\wedge D_y\big)u\big\|_{L^2}^2\lesssim\big\|B(y)D_yu\big\|_{L^2}^2. \tag{3.3}$$

So we complete the lemma.

Lemma 3.3 Let $G\in S(1,|dy|^2+|d\eta|^2)$ and $B(y)$ be the matrix given in (1.6), we have $\forall\, u\in C_0^\infty(\mathbb{R}^6)$,

$$|(p(y)u,Gu)|+\Big|\big((B(y)D_y)^*\cdot B(y)D_yu,Gu\big)\Big|+|(\partial_y\wedge b)\cdot(y\wedge\partial_y)u,Gu)|\lesssim\big\|\mathcal{L}u\big\|_{L^2}^2+\big\|u\big\|_{L^2}^2. \tag{3.4}$$

Proof We notice that $p(y)^\alpha\in S(p(y)^\alpha,|dy|^2+|d\eta|^2)$, then

$$[G,p(y)^{\frac{1}{2}}]\cdot p(y)^{-\frac{1}{2}}\in S(1,|dy|^2+|d\eta|^2).$$

For the first term to (3.4),

$$\begin{aligned}|(p(y)u,Gu)|&=\Big|\Big(G\big(p(y)\big)^{\frac{1}{2}}u,\big(p(y)\big)^{\frac{1}{2}}u\Big)\Big|+\Big|\Big([G,\big(p(y)\big)^{\frac{1}{2}}]\big(p(y)\big)^{\frac{1}{2}}u,u\Big)\Big|\\&=\Big|\Big(G\big(p(y)\big)^{\frac{1}{2}}u,\big(p(y)\big)^{\frac{1}{2}}u)\Big)\Big|+\Big|\Big(\big(p(y)\big)^{\frac{1}{2}}u,[G,\big(p(y)\big)^{\frac{1}{2}}]u\Big)\Big|\\&\lesssim\big\|\big(p(y)\big)^{\frac{1}{2}}u\big\|_{L^2}^2+\big\|[G,\big(p(y)\big)^{\frac{1}{2}}]u\big\|_{L^2}^2\lesssim\big\|\big(p(y)\big)^{\frac{1}{2}}u\big\|_{L^2}^2,\end{aligned}$$

where the third holds,since $G\in S(1,|dy|^2+|d\eta|^2)$. As to the second term,

$$\begin{aligned}&\Big|\big((B(y)D_y)^*\cdot B(y)D_yu,Gu\big)_{L^2}\Big|=\big|(B(y)D_yu,B(y)D_y(Gu))_{L^2}\big|\\&\leq\big\|B(y)D_yu\big\|_{L^2}\big\|B(y)D_y(Gu)\big\|_{L^2}\lesssim\big\|B(y)D_yu\big\|_{L^2}^2+\big\|\big(p(y)\big)^{\frac{1}{2}}u\big\|_{L^2}^2,\end{aligned}$$

we get the third inequality from

$$\begin{aligned}&\big\|B(y)D_y(Gu)\big\|_{L^2}=\big\|B(y)\big[D_y,G\big]u+B(y)GD_yu\big\|_{L^2}\\&\lesssim\big\|B(y)\big[D_y,G\big]u+[B(y),G]D_yu+GB(y)D_yu\big\|_{L^2}\lesssim\big\|B(y)u\big\|_{L^2}+\big\|B(y)D_yu\big\|_{L^2}.\end{aligned}$$

Now, we will estimate the last term, similar to the above inequality, we get

$$\begin{aligned}&\big|(\partial_y\wedge b)\cdot(y\wedge D_y)u,Gu)_{L^2}\big|=\big|((y\wedge D_y)u,(D_y\wedge b)(Gu))_{L^2}\big|\\&\leq\big\|(y\wedge D_y)u\big\|_{L^2}\big\|(D_y\wedge b)(Gu)\big\|_{L^2}\lesssim\big\|B(y)D_yu\big\|_{L^2}^2+\big\|\big(p(y)\big)^{\frac{1}{2}}u\big\|_{L^2}^2.\end{aligned}$$

Together the above estimate and Lemma 3.2 give Lemma 3.3.

Lemma 3.4 For all $u \in C_0^\infty(R^3)$, we have

$$\begin{aligned}&\left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + \left\| \langle y\rangle^{1+\frac{5}{6}\gamma} D_y u\right\|_{L^2}^2 + \left\| \langle y\rangle^{1+\frac{5}{6}\gamma} (y\wedge D_y)u\right\|_{L^2}^2 \\ \lesssim & \left\| \langle y\rangle^{\frac{1}{6}\gamma} \langle \partial_x V\rangle^{\frac{2}{3}} u\right\|_{L^2}^2 + \left\|\mathcal{L}u\right\|_{L^2}^2.\end{aligned} \tag{3.5}$$

Proof In this proof, we let $u \in C_0^\infty(R^{2n})$, the conclusion will follow if one could prove

$$\left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + \left\| \langle y\rangle^{1+\frac{1}{3}\gamma} B(y)D_y u\right\|_{L^2}^2 \lesssim \left\| \langle y\rangle^{\frac{1}{6}\gamma} \langle \partial_x V\rangle^{\frac{2}{3}} u\right\|_{L^2}^2 + \left\|\mathcal{L}u\right\|_{L^2}^2 + \left\|u\right\|_{L^2}^2. \tag{3.6}$$

From (1.7), we have $\langle y\rangle^{1+\frac{1}{3}\gamma} |B(y)D_y u| \geq \langle y\rangle^{1+\frac{5}{6}\gamma} |D_y u| + \langle y\rangle^{1+\frac{5}{6}\gamma} |(y\wedge D_y)u|$. As a preliminary step, let's first show that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned}\left|\left(\mathcal{L}\langle y\rangle^{1+\frac{1}{3}\gamma} u, \langle y\rangle^{1+\frac{1}{3}\gamma} u\right)_{L^2}\right| \lesssim & \varepsilon\left(\left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + \left\| \langle y\rangle^{1+\frac{1}{3}\gamma} B(y)D_y u\right\|_{L^2}^2\right) \\ & + C_\varepsilon\left(\left\| \langle y\rangle^{\frac{1}{6}\gamma} \langle \partial_x V\rangle^{\frac{2}{3}} u\right\|_{L^2}^2 + \left\|\mathcal{L}u\right\|_{L^2}^2 + \left\|u\right\|_{L^2}^2\right).\end{aligned} \tag{3.7}$$

In fact, the estimate

$$\langle \partial_x V(x)\rangle \langle y\rangle^{1+\frac{2}{3}\gamma} \leq \varepsilon \langle y\rangle^{4+\frac{5}{3}\gamma} + C_\varepsilon \langle y\rangle^{\frac{1}{3}\gamma} \langle \partial_x V(x)\rangle^{\frac{4}{3}}$$

yields

$$\left(\langle y\rangle^{1+\frac{2}{3}\gamma} \langle \partial_x V(x)\rangle u, u\right)_{L^2} \leq \varepsilon \left\| (y)^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + C_\varepsilon \left\| \langle y\rangle^{\frac{1}{6}\gamma} \langle \partial_x V(x)\rangle^{\frac{2}{3}} u\right\|_{L^2}^2.$$

Consequently, using (1.5), we compute

$$\left|[\mathcal{L}, \langle y\rangle^{1+\frac{1}{3}\gamma}]u\right| \lesssim |\partial_x V(x)| \langle y\rangle^{\frac{1}{3}\gamma} |u| + \langle y\rangle^{1+\frac{5}{6}\gamma} |B(y)D_y u| + \langle y\rangle^{1+\frac{4}{3}\gamma} |(y\wedge \partial_y)u|.$$

And thus

$$\begin{aligned}&\left|\left([\mathcal{L}, \langle y\rangle^{1+\frac{1}{3}\gamma}]u, \langle y\rangle^{1+\frac{1}{3}\gamma} u\right)\right| \\ \lesssim & \left(|\partial_x V(x)| \langle y\rangle^{1+\frac{2}{3}\gamma} u, u\right)_{L_2} + \left(\langle y\rangle^{\frac{1}{2}\gamma} |B(y)D_y u|, \langle y\rangle^{2+\frac{5}{6}\gamma} |u|\right)_{L^2} + \left(\langle y\rangle^{1+\frac{5}{3}\gamma} |(y\wedge \partial_y)u|, |u|\right)_{L^2} \\ \lesssim & \left(|\partial_x V(x)| \langle y\rangle^{1+\frac{2}{3}\gamma} u, u\right)_{L^2} + \left(\langle y\rangle^{\frac{1}{3}\gamma} |B(y)D_y u|, \langle y\rangle^{2+\frac{5}{6}\gamma} |u|\right)_{L^2} \\ \lesssim & \varepsilon\left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + C_\varepsilon \left\| (y)^{\frac{1}{6}\gamma} \langle \partial_x V\rangle^{\frac{2}{3}} u\right\|_{L^2}^2 + C_\varepsilon \left\| \langle y\rangle^{\frac{1}{3}\gamma} B(y)D_y u\right\|_{L^2}^2 \\ \lesssim & \varepsilon\left(\left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + \left\| \langle y\rangle^{1+\frac{1}{3}\gamma} B(y)D_y u\right\|_{L^2}^2\right) + C_{\varepsilon,\gamma} \left\| \langle y\rangle^{-1} B(y)D_y u\right\|_{L^2}^2 \\ \lesssim & \varepsilon\left(\left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + \left\| \langle y\rangle^{1+\frac{1}{3}\gamma} B(y)D_y u\right\|_{L^2}^2\right) + C_\varepsilon\left(\left\| \langle y\rangle^{\frac{1}{6}\gamma} \langle \partial_x V\rangle^{\frac{2}{3}} u\right\|_{L^2}^2 + \left\|\mathcal{L}u\right\|_{L^2}^2 + \left\|u\right\|_{L^2}^2\right),\end{aligned}$$

where the second inquality follows from

$$\left(\langle y\rangle^{1+\frac{5}{3}\gamma} |(y\wedge \partial_y)u|, |u|\right)_{L_2} \lesssim \left(\langle y\rangle^{\frac{1}{3}\gamma} |B(y)D_y u|, \langle y\rangle^{2+\frac{5}{6}\gamma} |u|\right)_{L_2},$$

the third inquality holds due to interpolation inequality. The forth inquality holds because

$$\forall \delta > 0, \left\| \langle y\rangle^{\frac{1}{3}\gamma} B(y)D_y u\right\|_{L_2}^2 \lesssim \delta \left\| \langle y\rangle^{1+\frac{1}{3}\gamma} B(y)D_y u\right\|_{L_2}^2 + C_\delta \left\| \langle y\rangle^{-1} B(y)D_y u\right\|_{L_2}^2,$$

and the last inequality follows from

$$\left\| \langle y\rangle^{-1} B(y)D_y u\right\|_{L_2} \lesssim \left\| \langle y\rangle^{\frac{1}{2}\gamma} D_y u\right\|_{L_2}.$$

As a result

$$\left|\left(\mathcal{L}\langle y\rangle^{1+\frac{1}{3}\gamma} u, \langle y\rangle^{1+\frac{1}{3}\gamma} u\right)_{L_2}\right| \le \left|\left(\mathcal{L}u, \langle y\rangle^{2+\frac{2}{3}\gamma} u\right)_{L_2}\right| + \left|\left([\mathcal{L}, \langle y\rangle^{1+\frac{1}{3}\gamma}]u, \langle y\rangle^{1+\frac{1}{3}\gamma} u\right)_{L_2}\right|$$

and notice that $\gamma \ge 0$,

$$\left|\left(\mathcal{L}u, \langle y\rangle^{2+\frac{2}{3}\gamma} u\right)_{L_2}\right| \lesssim \varepsilon \left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + C_\varepsilon \left\|\mathcal{L}u\right\|_{L_2}^2.$$

Then we gain inequality (3.7).

Now we prove (3.6). Let's first write

$$\begin{aligned}
&\left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + \left\| \langle y\rangle^{1+\frac{\gamma}{3}} B(y)D_y u\right\|_{L^2}^2 \\
\lesssim\ & \left\| \langle y\rangle^{1+\frac{\gamma}{2}} \langle y\rangle^{1+\frac{\gamma}{3}} u\right\|_{L^2}^2 + \left\|B(y)D_y \langle y\rangle^{1+\frac{\gamma}{3}} u\right\|_{L^2}^2 + \left\|B(y)\left[D_y,\ \langle y\rangle^{1+\frac{\gamma}{3}}\right]u\right\|_{L^2}^2 \\
\lesssim\ & \left|\left(\mathcal{L}\langle y\rangle^{1+\frac{\gamma}{3}} u,\ \langle y\rangle^{1+\frac{\gamma}{3}} u\right)_{L^2}\right| + \left\|B(y)\left[D_y,\ \langle y\rangle^{1+\frac{\gamma}{3}}\right]u\right\|_{L^2}^2,
\end{aligned}$$

the second inequality using (3.2). For the last term, we have

$$\left\|B(y)\left[D_y,\ \langle y\rangle^{1+\frac{\gamma}{3}}\right]u\right\|_{L^2}^2 \lesssim \left\| \langle y\rangle^{1+\frac{5}{6}\gamma} u\right\|_{L^2}^2 \le \varepsilon \left\| \langle y\rangle^{2+\frac{5}{6}\gamma} u\right\|_{L^2}^2 + C_\varepsilon \left\|u\right\|_{L^2}^2,$$

then the desired estimate (3.5) follows from the above inequalities and (3.7), completing the proof of Lemma 3.4.

Proof of Proposition 3.1 Let $\rho \in C^1\left(\mathbb{R}^{2n}\right)$ be a real-valued function given by

$$\rho = \rho(x,y) = \frac{\langle y\rangle^{\frac{\gamma}{3}} \partial_x V(x)\cdot y}{\langle \partial_x V(x)\rangle^{4/3}} \phi(x,y)$$

with

$$\phi = \chi\left(\frac{\langle y\rangle^{2+2\gamma/3}}{\langle \partial_x V(x)\rangle^{2/3}}\right),$$

where $\chi \in C_0^\infty(\mathbb{R};\ [0,1])$ such that $\chi = 1$ in $[-1,1]$ and $\operatorname{supp}\chi \subset [-2,2]$. So we have

$$\frac{\langle y\rangle^{2+2\gamma/3}}{\langle \partial_x V(x)\rangle^{2/3}} \le 2, \quad \frac{\langle y\rangle^{\gamma/3}}{\langle \partial_x V(x)\rangle^{2/3}} \le 2. \tag{3.8}$$

And it is easy to verify that $|\rho| \le 1$.

Using the notation $Q = y\cdot D_x - \partial_x V(x)\cdot D_y$,

$$\begin{aligned}
\operatorname{Re}\left(\mathcal{L}u,\ \rho u\right)_{L^2} =& \operatorname{Re}\left(iQu,\ \rho u\right)_{L^2} + \left(\left(B(y)D_y\right)^* \cdot B(y)D_y u,\ \rho u\right)_{L^2} + \left(p(y)u,\ \rho u\right)_{L^2} \\
&+ \left(q(y)u,\ \rho u\right)_{L^2} + \left(\mu^{\frac{1}{2}} m(y)u,\ \rho u\right)_{L^2} + \left(\left(D_y\wedge b\right)\cdot\left(v\wedge D_y\right)u,\ \rho u\right)_{L^2} \\
\gtrsim& \operatorname{Re}\left(iQu,\ \rho u\right)_{L^2} - \left|\left(\mathcal{L}u,\ u\right)_{L^2}\right|,
\end{aligned}$$

which along with yields $\operatorname{Re}(iQu,\ \rho u)_{L^2} \lesssim |(\mathcal{L}u,\ u)_{L^2}| + |(\mathcal{L}u,\ \rho u)_{L^2}|$. Next, we want to give a lower bound for the term on the left side. Direct computation shows that

$$\operatorname{Re}(iQu,\ \rho u)_{L^2} = \sum_{j=1}^{3} (\mathcal{A}_j u,\ u)_{L^2} \tag{3.9}$$

with $\mathcal{A}_j$ given by

$$\begin{aligned}
&\mathcal{A}_1 = \frac{\langle y\rangle^{\frac{\gamma}{3}} |\partial_x V(x)|^2}{\langle \partial_x V(x)\rangle^{4/3}}\phi,\\
&\mathcal{A}_2 = \langle \partial_x V(x)\rangle^{-4/3} \big(\partial_x V(x)\cdot y\big)\partial_x V(x)\cdot\partial_y \Big[\langle y\rangle^{\frac{\gamma}{3}}\phi(x,y)\Big],\\
&\mathcal{A}_3 = \langle y\rangle^{\frac{\gamma}{3}} y\cdot\partial_x\big(\langle \partial_x V(x)\rangle^{-4/3}\big(\partial_x V(x)\cdot y\phi(x,y)\big)\big).
\end{aligned}$$

We will proceed to treat the above three terms. First one has

$$\begin{aligned}
\mathcal{A}_1 &= \langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3}\phi(x,y) - \frac{\langle y\rangle^{\gamma/3}}{\langle \partial_x V(x)\rangle^{4/3}}\phi(x,y)\\
&= \langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3} - \langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3}\big(1-\phi(x,y)\big) - \frac{\langle y\rangle^{\gamma/3}}{\langle \partial_x V(x)\rangle^{2/3}}\phi(x,y)\\
&\geq \langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3} - \langle y\rangle^{2+\gamma}
\end{aligned}$$

from which it follows that

$$(\mathcal{A}_1 u,\ u)_{L^2} \geq \Big(\langle y\rangle^{\gamma/3}\langle \partial_x V(x)\rangle^{2/3} u, u\Big)_{L^2} - \big\|\langle y\rangle^{1+\gamma/2} u\big\|_{L^2}^2. \tag{3.10}$$

Here we used (3.8) in last inequality. As for the term $\mathcal{A}_2$ we make use of the relation

$$\forall\,\sigma\in\mathbb{R},\quad \partial_x V(x)\partial_y\big(\langle y\rangle^{\sigma}\big) = \sigma\langle y\rangle^{\sigma-2}\partial_x V(x)\cdot y$$

to compute

$$\begin{aligned}
\mathcal{A}_2 &= \langle \partial_x V(x)\rangle^{-\frac{4}{3}} |\partial_x V(x)\cdot y|^2 \left[\frac{\gamma}{3}\langle y\rangle^{\frac{\gamma}{3}-2}\phi + \frac{(2+\gamma)\langle y\rangle^{\gamma}}{\langle \partial_x V(x)\rangle^{4/3}}\chi'\left(\frac{\langle y\rangle^{2+2\gamma/3}}{\langle \partial_x V(x)\rangle^{2/3}}\right)\right]\\
&\gtrsim -\langle \partial_x V(x)\rangle^{-4/3}|\partial_x V(x)\cdot y|^2\langle \partial_x V(x)\rangle^{-4/3}\langle y\rangle^{\gamma}\\
&\gtrsim -\langle y\rangle^{2+\gamma},
\end{aligned}$$

the first inequality using the fact that

$$\frac{\gamma}{3}\langle y\rangle^{\frac{\gamma}{3}-2}\phi + \frac{(2+\gamma)\langle y\rangle^{\gamma}}{\langle \partial_x V(x)\rangle^{2/3}}\chi'\left(\frac{\langle y\rangle^{2+2\gamma/3}}{\langle \partial_x V(x)\rangle^{2/3}}\right) > 0,\quad \gamma\in[0,1].$$

As a result, we conclude

$$(\mathcal{A}_2 u,\ u)_{L^2} \geq -\Big(\langle y\rangle^{2+\gamma} u,\ u\Big)_{L^2} = \big\|\langle y\rangle^{1+\gamma/2} u\big\|_{L^2}^2. \tag{3.11}$$

For the term $\mathcal{A}_3$, using (1.8) gives

$$
\begin{aligned}
\mathcal{A}_3 &= \langle y\rangle^{\frac{\gamma}{3}}\, y\cdot\partial_x\Big(\langle\partial_x V(x)\rangle^{-4/3}\left(\partial_x V(x)\cdot y\phi(x,y)\right)\Big)\\
&= \langle y\rangle^{\frac{\gamma}{3}}\, y\cdot\frac{[\partial_x^2 V(x)\cdot y\phi(x,y)+\partial_x V(x)\cdot y\partial_x\phi(x,y)]\,\langle\partial_x V(x)\rangle^{-4/3}}{\langle\partial_x^2 V(x)\rangle^{8/3}}\\
&\quad+\langle y\rangle^{\frac{\gamma}{3}}\, y\cdot\frac{-\frac{4}{3}\partial_x V(x)\cdot y\phi(x,y)\,\langle\partial_x V(x)\rangle^{-4/3}\,\partial_x^2 V(x)}{\langle\partial_x^2 V(x)\rangle^{8/3}}\\
&\ge -\langle y\rangle^{\frac{\gamma}{3}+1}\,\frac{\langle\partial_x V(x)\rangle^2\,|y|\,\phi(x,y)+\langle\partial_x V(x)\rangle\,|y|\,\partial_x\phi(x,y)+\langle\partial_x V(x)\rangle\,|y|\,\phi(x,y)}{\langle\partial_x^2 V(x)\rangle^{8/3}}\\
&\ge -\langle y\rangle^{\frac{\gamma}{3}+2}\,\frac{\langle\partial_x V(x)\rangle^2\,\phi(x,y)+\langle\partial_x V(x)\rangle\,\partial_x\phi(x,y)}{\langle\partial_x^2 V(x)\rangle^{8/3}}\\
&\ge -\langle y\rangle^{2+\gamma/3},
\end{aligned}
\tag{3.12}
$$

and thus

$$
(\mathcal{A}_3 u,\ u)_{L^2}\ge -\Big(\langle y\rangle^{2+\gamma}\,u,\ u\Big)_{L^2}=-\big\|\,\langle y\rangle^{1+\gamma/2}\,u\big\|_{L^2}^2,
$$

this along with (3.9), (3.10) and (3.11) shows that

$$
\begin{aligned}
\Big(\langle y\rangle^{\gamma/3}\,\langle\partial_x V(x)\rangle^{2/3}\,u,u\Big)_{L^2} &\lesssim \big\|\,\langle y\rangle^{1+\gamma/2}\,u\big\|_{L^2}^2+|(\mathcal{L}u,\ u)_{L^2}|+|(\mathcal{L}u,\ \rho u)_{L^2}|\\
&\lesssim |(\mathcal{L}u,\ u)_{L^2}|+|(\mathcal{L}u,\ \rho u)_{L^2}|\,.
\end{aligned}
$$

Now for any $u\in C_0^\infty(\mathbb{R}^{2n})$, we use the above estimate to the function $\langle\partial_x V(x)\rangle^{1/3}\,u$; this gives

$$
\Big(\langle y\rangle^{\gamma/3}\,\langle\partial_x V(x)\rangle^{\frac{4}{3}}\,u,\ u\Big)_{L^2}\quad\lesssim\quad\big\|\,\langle\partial_x V(x)\rangle^{-1/3}\,\mathcal{L}\,\langle\partial_x V(x)\rangle^{1/3}\,u\big\|_{L^2}\big\|\,\langle\partial_x V(x)\rangle\,u\big\|_{L^2},
$$

which, together with the fact that $\gamma\ge 0$, implies

$$
\begin{aligned}
\big\|\,\langle y\rangle^{\gamma/6}\,\langle\partial_x V(x)\rangle^{\frac{2}{3}}\,u\big\|_{L^2}^2 &\lesssim \big\|\,\langle\partial_x V(x)\rangle^{-1/3}\,\mathcal{L}\,\langle\partial_x V(x)\rangle^{1/3}\,u\big\|_{L^2}^2\\
&\lesssim \big\|\mathcal{L}u\big\|_{L^2}^2+\big\|\,\langle\partial_x V(x)\rangle^{-1/3}\,\big[\mathcal{L},\ \langle\partial_x V(x)\rangle^{1/3}\,\big]u\big\|_{L^2}^2.
\end{aligned}
$$

Moreover in view of (1.8), we have

$$
\big\|\,\langle\partial_x V(x)\rangle^{-1/3}\,\big[\mathcal{L},\ \langle\partial_x V(x)\rangle^{1/3}\,\big]u\big\|_{L^2}^2\lesssim\big\|\,\langle y\rangle\,u\big\|_{L^2}^2\lesssim\big\|\mathcal{L}u\big\|_{L^2}^2+\big\|u\big\|_{L^2}^2.
$$

Then the desired inequality (3.1) follows, completing the proof of Proposition 3.1.

4 Hypoelliptic Estimates for the Operator with Parameters

In this section, we always consider $X=(x,\xi)\in\mathbb{R}^6$ as parameters, and study the operator acting on the velocity variable y,

$$
\mathcal{L}_X=iQ_X+\big(B(y)D_y\big)^*\cdot B(y)D_y+p(y)+q(y)+(D_y\wedge b)\cdot(y\wedge D_y)+\mu^{\frac{1}{2}}m(y),\tag{4.1}
$$

where $Q_X = y \cdot \xi - \partial_x V(x) \cdot D_y$ and $B(y)$ is the matrix given in (1.6).

Notations Throughout this section, we will use $\|\cdot\|_{L^2}$ and $\left(\cdot,\, \cdot\right)_{L^2}$ to denote respectively the norm and inner product in the space $L^2(\mathbb{R}_y^3)$. Given a symbol p, we use p^{Wick} and p^w to denote the Wick and Weyl quantization of p in (y, η).

The main result of this section is the following proposition.

Proposition 4.1 Let λ be defined by

$$\lambda = \Big(1 + |\partial_x V \wedge \eta + y \wedge \xi|^2 + |\partial_x V(x)|^2 + |\xi|^2 + |y|^6 + |\eta|^6 + \langle \partial_x V(x) \wedge \xi \rangle^{6/5}\Big)^{\frac{1}{2}}, \tag{4.2}$$

then the following estimate

$$\begin{aligned} &\Big(\langle \partial_x V(x)\rangle^{4/3} + \langle \xi \rangle^{4/3}\Big) \|u\|_{L^2}^2 + \left\| \langle y \rangle^{\gamma/2} |D_y|^2 u\right\|_{L^2}^2 + \left\| \langle y \rangle^{\gamma/2} |y \wedge D_y|^2 u\right\|_{L^2}^2 \\ &+ \left\| \left(\lambda^{2/3}\right)^w u\right\|_{L^2}^2 \lesssim \|\mathcal{L}_X u\|_{L^2}^2 + \|u\|_{L^2}^2 \end{aligned} \tag{4.3}$$

holds for all $u \in \mathcal{S}(\mathbb{R}_y^3)$, uniformly with respect to X.

We would make use of the multiplier method introduced in [4], to show the above proposition through the following subsections.

Before the proof of Proposition 4.1, we list some lemmas.

Lemma 4.2 Let λ be defined in (4.2), then

$$\forall\, \sigma \in \mathbb{R}, \quad \lambda^\sigma \in S(\lambda^\sigma, |dy|^2 + |d\eta|^2) \tag{4.4}$$

uniformly with respect to X. Moreover if $\sigma \leq \frac{1}{3}$ then

$$\forall\ |\alpha| + |\beta| \geq 1, \quad \left|\partial_y^\alpha \partial_\eta^\beta \left(\lambda^\sigma\right)\right| \lesssim \langle \partial_x V(x)\rangle^\sigma + \langle \xi \rangle^\sigma, \tag{4.5}$$

and thus

$$\left(\lambda^\sigma\right)^{\mathrm{Wick}} = \left(\lambda^\sigma\right)^w + \left(\, \langle \partial_x V(x)\rangle^\sigma + \langle \xi \rangle^\sigma\, \right) r^w \tag{4.6}$$

with $r \in \left(1, |dy|^2 + |d\eta|^2\right)$ uniformly with respect to X.

Proof By direct verification, we see that for all $(y, \eta) \in \mathbb{R}^{2n}$ and all $\alpha, \beta \in \mathbb{Z}_+^n$, one has

$$\forall\, (y, \eta) \in \mathbb{R}^{2n},\ \forall\, \alpha, \beta \in \mathbb{Z}_+^n, \quad \left|\partial_y^\alpha \partial_\eta^\beta \left(\lambda(y, \eta)^2\right)\right| \leq \lambda(y, \eta)^2,$$

which implies (4.4). Moreover note that for $\sigma \leq \frac{1}{3}$,

$$\forall\ |\alpha| + |\beta| \geq 1, \quad \left|\partial_y^\alpha \partial_\eta^\beta \left(\lambda(y, \eta)^{1+\sigma}\right)\right| \leq \left(\, \langle \partial_x V(x)\rangle^\sigma + \langle \xi \rangle^\sigma\, \right) \lambda(y, \eta)$$

and thus

$$\forall\, \sigma \in \mathbb{R}, \quad \left|\partial_y^\alpha \partial_\eta^\beta \left(\lambda(y, \eta)^\sigma\right)\right| \lesssim \lambda^{\sigma - \frac{1}{3}} \left(\, \langle \partial_x V(x)\rangle + \langle \xi \rangle\, \right),$$

then we get (4.5) if $\sigma \leq \frac{1}{3}$, and thus (4.6) in view of (2.4), completing the proof of Lemma 4.2.

Lemma 4.3 Let λ be given in (4.2), then for all $u \in \mathcal{S}(\mathbb{R}^3)$, one has

$$\left\| \langle y\rangle^{\gamma/2} |D_y|^2 u\right\|_{L^2}^2 + \left\| \langle y\rangle^{\gamma/2} |y\wedge D_y|^2 u\right\|_{L^2}^2 \lesssim \left\|\mathcal{L}_X u\right\|_{L^2}^2 + \left\|\Phi^{2/3} u\right\|_{L^2}^2 + \left\|\left(\lambda^{2/3}\right)^w u\right\|_{L^2}^2, \tag{4.7}$$

where Φ is defined by

$$\Phi = \Phi(X) = \left(1 + |\partial_x V(x)|^2 + |\xi|^2\right)^{1/2}. \tag{4.8}$$

Proof Similar to (3.2), we have, for any $u \in, \mathcal{S}(\mathbb{R}_y^3)$,

$$\left\| \langle y\rangle^{1+\gamma/2} u\right\|_{L^2}^2 + \left\| \langle y\rangle^{\gamma/2} D_y u\right\|_{L^2}^2 + \left\| \langle y\rangle^{\gamma/2} \left(y\wedge D_y\right) u\right\|_{L^2}^2 \lesssim \operatorname{Re}\left(\mathcal{L}_X u, u\right)_{L^2}. \tag{4.9}$$

Using the above inequality to $D_{y_j} u$ gives

$$\begin{aligned}\sum_{j,k=1}^n \left\| \langle y\rangle^{\gamma/2} D_{y_k}\cdot D_{y_j} u\right\|_{L^2}^2 \quad &\lesssim \quad \sum_{j=1}^n \left|\left(\mathcal{L}_X D_{y_j} u,\ D_{y_j} u\right)_{L^2}\right| \\ &\lesssim \quad \left|\left(\mathcal{L}_X u,\ D_y\cdot D_y u\right)_{L^2}\right| + \sum_{j=1}^n \left|\left(\left[\mathcal{L}_X,\ D_{y_j}\right] u,\ D_{y_j} u\right)_{L^2}\right|,\end{aligned}$$

which with the fact that $\gamma \geq 0$ implies

$$\sum_{j,k=1}^n \left\| \langle y\rangle^{\gamma/2} D_{y_k}\cdot D_{y_j} u\right\|_{L^2}^2 \lesssim \left\|\mathcal{L}_X u\right\|_{L^2}^2 + \sum_{j=1}^n \left|\left(\left[\mathcal{L}_X,\ D_{y_j}\right] u,\ D_{y_j} u\right)_{L^2}\right|. \tag{4.10}$$

So we only need to handle the last term in the above inequality. Direct verification shows

$$\begin{aligned}&\left[\mathcal{L}_X,\ D_{y_j}\right] \\ =&\xi_j + \left(\left(D_{y_j} B(y)\right) D_y\right)^* \cdot B(y) D_y + \left(B(y) D_y\right)^* \cdot \left(D_{y_j} B(y)\right) D_y + \left(D_{y_j} p(y)\right) + \left(D_{y_j} q(y)\right) \\ &+ D_{y_j}(D_y\wedge b)\cdot(y\wedge D_y) + (D_y\wedge b)\cdot D_{y_j}(y\wedge D_y) + D_{y_j}(\mu^{\frac12} m(y)).\end{aligned}$$

This gives

$$\sum_{j=1}^n \left|\left(\left[\mathcal{L},\ D_{y_j}\right] u,\ D_{y_j} u\right)_{L^2}\right| \leq \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 + \mathcal{B}_5 + \mathcal{B}_6 \tag{4.11}$$

with

$$\mathcal{B}_1 = \sum_{j=1}^n \left|\left(\xi_j u,\ D_{y_j} u\right)_{L^2}\right|,$$

$$\mathcal{B}_2 = \sum_{j=1}^n \left(\left|\left(B(y) D_y u,\ \left(D_{y_j} B\right) D_y D_{y_j} u\right)_{L^2}\right| + \left|\left(\left(D_{y_j} B\right) D_y u,\ B(y) D_y D_{y_j} u\right)_{L^2}\right|\right),$$

$$\mathcal{B}_3 = \sum_{j=1}^n \left|\left(\left(D_{y_j} p(y)\right) u,\ D_{y_j} u\right)_{L^2}\right|,\ \ \mathcal{B}_4 = \sum_{j=1}^n \left|\left(\left(D_{y_j} q(y)\right) u,\ D_{y_j} u\right)_{L^2}\right|,$$

$$\mathcal{B}_5 = \sum_{j=1}^n \left(\left|\left((y\wedge D_y) u,\ D_{y_j}(D_y\wedge b) u\right)_{L^2}\right| + \left|\left(D_{y_j}(y\wedge D_y) u,\ (D_y\wedge b) u\right)_{L^2}\right|\right),$$

$$\mathcal{B}_6 = \sum_{j=1}^n \left|\left(D_{y_j}(\mu^{\frac12} m(y)) u,\ D_{y_j} u\right)_{L^2}\right|.$$

By Parseval's theorem, we may write, denoting by $\hat{u}$ the Fourier transform with respect to y, $\left|\left(\xi_j u,\ D_{y_j} u\right)_{L^2}\right| = \left|(\xi_j \hat{u},\ \eta_j \hat{u})_{L^2(\mathbb{R}^6_\eta)}\right|$, and hence

$$\mathcal{B}_1 \le \varepsilon \big\| D_y \cdot D_y u \big\|^2_{L^2} + C_\varepsilon \big\| \langle \xi \rangle^{2/3} u \big\|^2_{L^2},$$

due to the inequality $|\xi_j \eta_j| \le \varepsilon |\eta|^4 + C_\varepsilon \langle \xi \rangle^{4/3}$. From (1.7) it follows that

$$\begin{aligned}
\mathcal{B}_2 + \mathcal{B}_5 \le & \varepsilon \sum_{j,k=1}^{n} \big\| \langle y \rangle^{\gamma/2} D_{y_k} D_{y_j} u \big\|^2_{L^2} + C_\varepsilon \big\| \langle y \rangle^{1+\gamma/2} D_y u \big\|^2_{L^2} \\
\le & \varepsilon \sum_{j,k=1}^{n} \big\| \langle y \rangle^{\gamma/2} D_{y_k} D_{y_j} u \big\|^2_{L^2} + C_\varepsilon \big(\big\| P_X u \big\|^2_{L^2} + \big\| \langle \partial_x V(x) \rangle^{2/3} u \big\|^2_{L^2} \big),
\end{aligned}$$

the last inequality using Lemma 3.4.

$$\mathcal{B}_3 + \mathcal{B}_4 + \mathcal{B}_6 \le \varepsilon \big\| \langle y \rangle^{1+\frac{\gamma}{2}} D_y u \big\|^2_{L^2} + C_\varepsilon \big\| D_y u \big\|^2_{L^2} \le C_\varepsilon \big(\big\| P_X u \big\|^2_{L^2} + \big\| \langle \partial_x V(x) \rangle^{2/3} u \big\|^2_{L^2} \big).$$

Due to the arbitrariness of the number ε, the above inequalities along with (4.10) and (4.11) gives the desired upper bound for the first term on the left side of (4.7).

It remains to treat the second term. In the following discussion, we use the notation

$$T = (T_1, \cdots, T_n) = y \wedge D_y, \quad A = (A_1, A_2, A_3) = y \wedge \xi + \partial_x V(x) \wedge D_y.$$

From (4.9), it follows that

$$\begin{aligned}
& \sum_{j,k=1}^{n} \big(\big\| \langle y \rangle^{\gamma/2} T_k \cdot T_j u \big\|^2_{L^2} + \big\| D_{y_k} \cdot T_j u \big\|^2_{L^2} + \big\| y_k \cdot T_j u \big\|^2_{L^2} \big) \\
\le \ & \sum_{j=1}^{n} \big| (\mathcal{L}_X T_j u,\ T_j u)_{L^2} \big| \le |(\mathcal{L}_X u,\ T \cdot T u)_{L^2}| + \big| \big(\big[\mathcal{L}_X,\ T \big] u,\ T u \big)_{L^2} \big|,
\end{aligned}$$

which with the fact that $\gamma \ge 0$ implies

$$\begin{aligned}
& \sum_{j,k=1}^{n} \big(\big\| \langle y \rangle^{\gamma/2} T_k \cdot T_j u \big\|^2_{L^2} + \big\| D_{y_k} \cdot T_j u \big\|^2_{L^2} + \big\| y_k \cdot T_j u \big\|^2_{L^2} \big) \\
\lesssim & \big\| \mathcal{L}_X u \big\|^2_{L^2} + \big| \big(\big[\mathcal{L}_X,\ T \big] u,\ T u \big)_{L^2} \big|.
\end{aligned} \tag{4.12}$$

In order to handle the last term in the above inequality, we write

$$\begin{aligned}
\big[\mathcal{L},\ T_j \big] = & - A_j + \big[D_y,\ T_j \big] \cdot \nu(y) D_y + D_y \cdot \nu(y) \big[D_y,\ T_j \big] + D_y \cdot \big(T_j \nu(y) \big) D_y \\
& + \big[T,\ T_j \big] \cdot \mu(y) T + T \cdot (T_j \mu(y)) T + T \cdot \mu(y) \big[T,\ T_j \big] + \big(T_j p(y) \big) \\
& + \big[(D_y \wedge b),\ T_j \big] \cdot T + (D_y \wedge b) \cdot [T_j, T] + \big(T_j q(y) \big) + T_j \big(\mu^{\frac{1}{2}} m(y) \big).
\end{aligned}$$

This gives

$$\big| \big(\big[\mathcal{L}_X,\ T \big] u,\ T u \big)_{L^2} \big| \le \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4 + \mathcal{N}_5 \tag{4.13}$$

with

$$\mathcal{N}_1 = \left|(Au,\ Tu)_{L^2}\right|,$$

$$\mathcal{N}_2 = \sum_{j=1}^{n}\left|\Big(\big[D_y,\ T_j\big]\cdot\nu(y)D_yu + D_y\cdot\nu(y)\big[D_y,\ T_j\big]u + D_y\cdot\big(T_j\nu(y)\big)D_yu,\ T_ju\Big)_{L^2}\right|,$$

$$\mathcal{N}_3 = \sum_{j=1}^{n}\left|\Big(\big[T,\ T_j\big]\cdot\mu(y)Tu + T\cdot(T_j\mu(y))Tu + T\cdot\mu(y)\big[T,\ T_j\big]u,\ T_ju\Big)_{L^2}\right|,$$

$$\mathcal{N}_4 = \sum_{j=1}^{n}\left|\Big((\big[(D_y\wedge b),\ T_j\big]\cdot T + (D_y\wedge b)\cdot[T_j, T])u,\ T_ju\Big)_{L^2}\right|,$$

$$\mathcal{N}_5 = \sum_{j=1}^{n}\Big(\left|\big(\big(T_jp(y)\big)u,\ D_{y_j}u\big)_{L^2}\right| + \left|\big(\big(T_jq(y)\big)u,\ D_{y_j}u\big)_{L^2}\right|_{L^2}\Big) + \left|\Big(\big(T_j\mu^{\frac{1}{2}}m(y)\big)u,\ D_{y_j}u\Big)_{L^2}\right|.$$

Next we proceed to treat the above four terms. For the term $\mathcal{N}_1$ one has, with λ defined in (4.2),

$$\begin{aligned}(Au,\ Tu)_{L^2} &\le \varepsilon\big\|\big(\lambda^{1/3}\big)^wTu\big\|_{L^2}^2 + C_\varepsilon\big\|\big(\lambda^{-1/3}\big)^wAu\big\|_{L^2}^2\\ &\le \varepsilon\big\|\big(\lambda^{1/3}\big)^wTu\big\|_{L^2}^2 + C_\varepsilon\big\|\big(\lambda^{2/3}\big)^wu\big\|_{L^2}^2,\end{aligned}$$

the last inequality holding because $\big(\lambda^{-1/3}\big)^wA\big(\lambda^{-2/3}\big)^w \in \mathrm{Op}\big(S(1,\ |dy|^2+|d\eta|^2)\big)$.

On the other hand,

$$\begin{aligned}\big\|\big(\lambda^{1/3}\big)^wTu\big\|_{L^2}^2 &\lesssim \left|\big(\big(\lambda^{2/3}\big)^wTu,\ Tu\big)_{L^2}\right|\\ &\lesssim \left|\big(\big(\lambda^{2/3}\big)^wu,\ T\cdot Tu\big)_{L^2}\right| + \left|\big(\big[\big(\lambda^{2/3}\big)^w,\ T\big]u,\ Tu\big)_{L^2}\right|\\ &\lesssim \varepsilon\big\|T\cdot Tu\big\|_{L^2} + C_\varepsilon\big\|\big(\lambda^{2/3}\big)^wu\big\|_{L^2} + \left|\big(\big[\big(\lambda^{2/3}\big)^w,\ T\big]u,\ Tu\big)_{L^2}\right|.\end{aligned}$$

Observing (4.4), symbolic calculus give that

$$\big[\big(\lambda^{2/3}\big)^w,\ T\big] = \big[\big(\lambda^{2/3}\big)^w,\ y\wedge D_y\big] = D_yd_1^w + yd_2^w + d_3^w$$

with d_j, $1\le j\le 3$, belonging to $S\big(\lambda^{2/3}, |dy|^2+|d\eta|^2\big)$ uniformly with respect to X. This shows

$$\left|\big(\big[\big(\lambda^{2/3}\big)^w,\ T\big]u,\ Tu\big)_{L^2}\right| \lesssim \varepsilon\sum_{j,k=1}^{n}\Big(\big\|D_{y_k}\cdot T_ju\big\|_{L^2}^2 + \big\|y_k\cdot T_ju\big\|_{L^2}^2\Big) + C_\varepsilon\big\|\big(\lambda^{2/3}\big)^wu\big\|_{L^2}^2.$$

Combining the above inequalities, we have

$$\mathcal{N}_1 \lesssim \varepsilon\sum_{j,k=1}^{n}\Big(\big\|\langle y\rangle^{\gamma/2}T_k\cdot T_ju\big\|_{L^2}^2 + \big\|D_{y_k}\cdot T_ju\big\|_{L^2}^2 + \big\|y_k\cdot T_ju\big\|_{L^2}^2\Big) + C_\varepsilon\big\|\big(\lambda^{2/3}\big)^wu\big\|_{L^2}^2.$$

Direct verification shows

$$\big[T_j,\ D_{y_k}\big] = \sum_\ell a_{j,k}^\ell D_{y_\ell},\quad \big[T_1,\ T_2\big] = T_3,\quad \big[T_1,\ T_3\big] = T_2,\quad \big[T_2,\ T_3\big] = T_1$$

with $a^{\ell}_{j,k} \in \{0,-1,+1\}$, and thus

$$\begin{aligned}
\mathcal{N}_2+\mathcal{N}_3 \lesssim & \varepsilon \sum_{j,k=1}^{n} \Big(\big\| \langle y\rangle^{\gamma/2} T_k \cdot T_j u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} D_{y_k} \cdot T_j u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} y_k \cdot T_j u \big\|_{L^2}^2 \Big) \\
& + C_\varepsilon \big(\big\| \langle y\rangle^{\gamma/2} T u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} D_y u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} y u \big\|_{L^2}^2 \big) \\
\lesssim & \varepsilon \sum_{j,k=1}^{n} \Big(\big\| \langle y\rangle^{\gamma/2} T_k \cdot T_j u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} D_{y_k} \cdot T_j u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} y_k \cdot T_j u \big\|_{L^2}^2 \Big) \\
& + C_\varepsilon \big(\|\mathcal{L} u\|_{L^2}^2 + \|u\|_{L^2}^2 \big), \\
\mathcal{N}_4 \lesssim & \sum_{j,k=1}^{n} \big\| \langle y\rangle^{\gamma-1} T_k \cdot T_j u \big\|_{L^2}^2 \lesssim \varepsilon \sum_{j,k=1}^{n} \big\| \langle y\rangle^{\gamma/2} T_k \cdot T_j u \big\|_{L^2}^2,
\end{aligned}$$

the last inequality using (3.2). It remains to treat $\mathcal{N}_5$, and by (1.2) and (3.2), we have

$$\mathcal{N}_5 \lesssim \big\| \langle y\rangle^{1+\gamma/2} u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} D_y u \big\|_{L^2}^2 \lesssim \big(\|\mathcal{L}_X u\|_{L^2}^2 + \|u\|_{L^2}^2 \big).$$

Combining the above estimates, we conclude

$$\begin{aligned}
& \mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_4+\mathcal{N}_4+\mathcal{N}_5 \\
\lesssim & \varepsilon \sum_{j,k=1}^{n} \Big(\big\| \langle y\rangle^{\gamma/2} T_k \cdot T_j u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} D_{y_k} \cdot T_j u \big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} y_k \cdot T_j u \big\|_{L^2}^2 \Big) \\
& + C_\varepsilon \Big(\|\mathcal{L}_X u\|_{L^2}^2 + \big\| \big(\lambda^{2/3}\big)^w u \big\|_{L^2}^2 + \|u\|_{L^2}^2 \Big).
\end{aligned}$$

This along with (4.12) and (4.13) yields the desired upper bound for $\big\| \langle y\rangle^{\gamma/2} |y\wedge D_y|^2 \big\|_{L^2}$, letting ε small enough. The proof of Lemma 4.3 is thus completed.

Lemma 4.4 Let $g \in S(1, |dy|^2+|d\eta|^2)$ uniformly with respect to X, and let λ be defined in (4.2). Then for any $\varepsilon>0$, there exists a constant C_ε such that

$$\begin{aligned}
& \big(\mathcal{L}_X \big(\lambda^{1/3}\big)^w u,\ g^w \big(\lambda^{1/3}\big)^w u \big)_{L^2} \\
\lesssim & \varepsilon \big\| \big(\lambda^{2/3}\big)^w u \big\|_{L^2}^2 + C_\varepsilon \Big\{ \|\mathcal{L}_X u\|_{L^2}^2 + \big\|\Phi^{2/3} u\big\|_{L^2}^2 + \big\| \langle \partial_x V(x)\wedge \xi\rangle^{2/5} u \big\|_{L^2}^2 \Big\},
\end{aligned} \tag{4.14}$$

where Φ is given in (4.8).

Proof As a preliminary step we firstly show that for any $\varepsilon>0$, there exists a constant C_ε such that

$$\begin{aligned}
& \big(\big[\mathcal{L}_X,\ \big(\lambda^{1/3}\big)^w\big] u,\ d^w \big(\lambda^{1/3}\big)^w u \big)_{L^2} \\
\lesssim & \varepsilon \big(\mathcal{L}_X \big(\lambda^{1/3}\big)^w u,\ \big(\lambda^{1/3}\big)^w u \big)_{L^2} + C_\varepsilon \Big\{ \big\|\Phi^{2/3} u\big\|_{L^2}^2 + \big\| \langle \partial_x V(x)\wedge \xi\rangle^{2/5} u \big\|_{L^2}^2 \Big\},
\end{aligned} \tag{4.15}$$

where d is an arbitrary symbol belonging to $S(1, |dy|^2+|d\eta|^2)$ uniformly with respect to X. Observing (1.3) and (4.5), symbolic calculus (see for instance Theorem 2.3.8 in [11]) give that the symbols of the following commutators $\big[\nu(y),\ \big(\lambda^{1/3}\big)^w\big]$, $\big[\mu(y),\ \big(\lambda^{1/3}\big)^w\big]$, belong to $S\big(\Phi^{1/3}, |dy|^2+|d\eta|^2\big)$ uniformly with respect to X. As a result, using the notation

$$\begin{aligned}
\mathcal{Z}_1 &= \big(D_y\cdot\big[\nu(y),\ \big(\lambda^{1/3}\big)^w\big]D_yu,\ d^w\big(\lambda^{1/3}\big)^w u\big)_{L^2},\\
\mathcal{Z}_2 &= \big(\big(y\wedge D_y\big)\cdot\big[\mu(y),\ \big(\lambda^{1/3}\big)^w\big]\big(y\wedge D_y\big)u,\ d^w\big(\lambda^{1/3}\big)^w u\big)_{L^2},
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{Z}_1+\mathcal{Z}_2 \leq& \varepsilon\big\|\left\langle D_y\right\rangle d^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2+\varepsilon\big\|\big(y\wedge D_y\big)d^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2\\
&+C_\varepsilon\big\|\left\langle D_y\right\rangle\Phi^{1/3}u\big\|_{L^2}^2+C_\varepsilon\big\|\big(y\wedge D_y\big)\Phi^{1/3}u\big\|_{L^2}^2\\
\leq&\varepsilon\big\|\big(\left\langle D_y\right\rangle+\left\langle y\right\rangle\big)\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2+\varepsilon\big\|\big(y\wedge D_y\big)\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2\\
&+C_\varepsilon\Phi^{2/3}\big\|\left\langle D_y\right\rangle u\big\|_{L^2}^2+C_\varepsilon\Phi^{2/3}\big\|\big(y\wedge D_y\big)u\big\|_{L^2}^2.
\end{aligned}$$

The last inequality holding because

$$\big[D_y,\ d^w\big],\quad \big[y\wedge D_y,\ d^w\big]\big(\big(1+|y|^2+|\eta|^2\big)^{-1/2}\big)^w\in\mathrm{Op}\big(S\big(1,|dy|^2+|d\eta|^2\big)\big),$$

since $d\in S\big(1,|dy|^2+|d\eta|^2\big)$ uniformly with respect to X. Moreover using (4.9) gives

$$\mathcal{Z}_1+\mathcal{Z}_2\leq\varepsilon\left(\mathcal{L}_X\big(\lambda^{1/3}\big)^w u,\ \big(\lambda^{1/3}\big)^w u\right)_{L^2}+C_\varepsilon\left\{\big\|\mathcal{L}_X u\big\|_{L^2}^2+\big\|\Phi^{2/3}u\big\|_{L^2}^2\right\}. \tag{4.16}$$

Denote

$$\begin{aligned}
\mathcal{Z}_3 =& \big(\big[D_y,\ \big(\lambda^{1/3}\big)^w\big]\cdot\mu(y)D_yu,\ d^w\big(\lambda^{1/3}\big)^w u\big)_{L^2}\\
&+\big(D_y\cdot\mu(y)\big[D_y,\ \big(\lambda^{1/3}\big)^w\big]u,\ d^w\big(\lambda^{1/3}\big)^w u\big)_{L^2},\\
\mathcal{Z}_4 =& \big(\big[y\wedge D_y,\ \big(\lambda^{1/3}\big)^w\big]\cdot\mu(y)\big(y\wedge D_y\big)u,\ d^w\big(\lambda^{1/3}\big)^w u\big)_{L^2}\\
&+\big(\big(y\wedge D_y\big)\cdot\mu(y)\big[y\wedge D_y,\ \lambda^{1/3}\big]^w u,\ a^w\big(\lambda^{1/3}\big)^w u\big)_{L^2},\\
\mathcal{Z}_5 =& \big(\big[b\wedge D_y,\ \big(\lambda^{1/3}\big)^w\big]\cdot\big(y\wedge D_y\big)u,\ d^w\big(\lambda^{1/3}\big)^w u\big)_{L^2}\\
&+\big(\big(b\wedge D_y\big)\cdot\big[y\wedge D_y,\ \big(\lambda^{1/3}\big)^w\big]u,\ d^w\big(\lambda^{1/3}\big)^w u\big)_{L^2}.
\end{aligned}$$

Observing (4.5), symbolic calculus give that

$$\begin{aligned}
\big[D_y,\ \big(\lambda^{1/3}\big)^w\big] &= f_1^w,\\
\big[\big(y\wedge D_y\big),\ \big(\lambda^{1/3}\big)^w\big] &= f_2^wD_y+f_3^wy+f_4^w,\\
\big[\big(b\wedge D_y\big),\ \big(\lambda^{1/3}\big)^w\big] &= f_5^wD_y+f_6^wy+f_7^w
\end{aligned}$$

with f_j, $1\leq j\leq 7$, belonging to $S\big(\Phi^{1/3},|dy|^2+|d\eta|^2\big)$ uniformly with respect to ε and X. It then follows that

$$\begin{aligned}
\mathcal{Z}_3+\mathcal{Z}_4+\mathcal{Z}_5\leq&\varepsilon\big\|\left\langle D_y\right\rangle d^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2+\varepsilon\big\|\left\langle y\right\rangle^{1+\gamma/2}d^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2\\
&+\varepsilon\big\|\left\langle y\right\rangle^{\gamma/2}\left\langle D_y\right\rangle d^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2+\varepsilon\big\|\left\langle y\right\rangle^{\gamma/2}\left\langle y\wedge D_y\right\rangle d^w\big(\lambda^{1/3}\big)^w u\big\|_{L^2}^2\\
&+C_\varepsilon\big\|\left\langle y\right\rangle^{\gamma/2}\big(\left\langle y\right\rangle+\left\langle D_y\right\rangle\big)\Phi^{1/3}u\big\|_{L^2}^2+C_\varepsilon\big\|\left\langle y\right\rangle^{\gamma/2}\big(y\wedge D_y\big)\Phi^{1/3}u\big\|_{L^2}^2.
\end{aligned}$$

Using similar arguments as the treatment of $\mathcal{Z}_1$ and $\mathcal{Z}_2$, we conclude

$$\mathcal{Z}_3+\mathcal{Z}_4+\mathcal{Z}_5\leq\varepsilon\left(\mathcal{L}_X\big(\lambda^{1/3}\big)^wu,\ \big(\lambda^{1/3}\big)^wu\right)_{L^2}+C_\varepsilon\left\{\left\|\mathcal{L}_Xu\right\|_{L^2}^2+\left\|\Phi^{2/3}u\right\|_{L^2}^2\right\}.$$

This along with (4.16) gives

$$\begin{aligned}&\Big(\big[\big(B(y)D_y\big)^*B(y)D_y,\ \big(\lambda^{1/3}\big)^{\mathrm{Wick}}\big]u,\ a^w\big(\lambda^{1/3}\big)^{\mathrm{Wick}}u\Big)_{L^2}\\&\lesssim\varepsilon\ \left(\mathcal{L}_X\big(\lambda^{1/3}\big)^wu,\ \big(\lambda^{1/3}\big)^wu\right)_{L^2}+C_\varepsilon\left\{\left\|\mathcal{L}_Xu\right\|_{L^2}^2+\left\|\Phi^{2/3}u\right\|_{L^2}^2\right\},\end{aligned}\tag{4.17}$$

since

$$\Big(\big[\big(B(y)D_y\big)^*B(y)D_y,\ \big(\lambda^{1/3}\big)^{\mathrm{Wick}}\big]u,\ d^w\big(\lambda^{1/3}\big)^{\mathrm{Wick}}u\Big)_{L^2}=\sum_{1\leq j\leq5}\mathcal{Z}_j.$$

Moreover, we have

$$\begin{aligned}&\Big(\big[p(y)+q(y)+\mu^{\frac12}m(y),\ \big(\lambda^{1/3}\big)^{\mathrm{Wick}}\big]u,\ d^w\big(\lambda^{1/3}\big)^{\mathrm{Wick}}u\Big)_{L^2}\\&\lesssim\varepsilon\left(\mathcal{L}_X\big(\lambda^{1/3}\big)^wu,\ \big(\lambda^{1/3}\big)^wu\right)_{L^2}+C_\varepsilon\left\|\mathcal{L}_Xu\right\|_{L^2}^2+\Phi^{4/3}\left\|u\right\|_{L^2}^2,\end{aligned}\tag{4.18}$$

which can be deduced similarly as above, since by (1.3),

$$\big[p(y)+q(y)+\mu^{\frac12}m(y),\ \big(\lambda^{1/3}\big)^w\big]\in\mathrm{Op}\big(S\big(\left\langle y\right\rangle^{1+\gamma}\Phi^{1/3},|dy|^2+|d\eta|^2\big)\big)$$

uniformly with respect to X. Next we will treat the commutator $\big[iQ_X,\ \big(\lambda^{1/3}\big)^w\big]$, whose symbol is

$$-\frac{\lambda^{\frac13-2}}{3}\Big[\big(\partial_xV(x)\wedge\eta+y\wedge\xi\big)\cdot\big(\partial_xV(x)\wedge\xi\big)-6\left|\eta\right|^4\xi\cdot\eta\Big].$$

In view of (4.4) and (4.2), one could verify that the above symbol belongs to

$$S\left(\left\langle\partial_xV(x)\wedge\xi\right\rangle^{4/5}\lambda^{-1/3}+\left\langle\xi\right\rangle^{2/3}\lambda^{-/3},\ |dy|^2+|d\eta|^2\right)$$

uniformly with respect to X. As a result, observing $\lambda^{1/3}\in S(\lambda^{1/3},\ |dy|^2+|d\eta|^2)$ uniformly with respect to X, we have

$$\big(\lambda^{1/3}\big)^wd^w\big[iQ_X,\ \big(\lambda^{1/3}\big)^w\big]\in\mathrm{Op}\big(S\big(\left\langle\partial_xV(x)\wedge\xi\right\rangle^{4/5}+\left\langle\xi\right\rangle^{2/3},\ |dy|^2+|d\eta|^2\big)\big)$$

uniformly with respect to X, which implies

$$\big(\big[iQ_X,\ \big(\lambda^{1/3}\big)^w\big]u,\ d^w\big(\lambda^{1/3}\big)^wu\big)_{L^2}\lesssim\left\|\left\langle\partial_xV(x)\wedge\xi\right\rangle^{2/5}u\right\|_{L^2}^2+\left\|\left\langle\xi\right\rangle^{2/3}u\right\|_{L^2}^2.$$

This along with (4.17) and (4.18) gives (4.15), since

$$\begin{aligned}\big[\mathcal{L}_X,\ \big(\lambda^{1/3}\big)^w\big]=&\big[iQ_X,\ \big(\lambda^{1/3}\big)^w\big]+\big[\big(B(y)D_y\big)^*B(y)D_y+(D_y\wedge b)\cdot(y\wedge D_y),\ \big(\lambda^{1/3}\big)^w\big]\\&+\big[p(y)+q(y)+\mu^{\frac12}m(y),\ \big(\lambda^{1/3}\big)^w\big].\end{aligned}$$

Next we prove (4.14). The relation

$$
\begin{aligned}
&\mathrm{Re}\left(\mathcal{L}_X\left(\lambda^{1/3}\right)^w u,\ \left(\lambda^{1/3}\right)^w u\right)_{L^2}+\mathrm{Re}\left(\mathcal{L}_X\left(\lambda^{1/3}\right)^w u,\ g^w\left(\lambda^{1/3}\right)^w u\right)_{L^2}\\
=&\mathrm{Re}\left(\mathcal{L}_X u,\ \left(\lambda^{1/3}\right)^w\left(\mathrm{Id}+g^w\right)\left(\lambda^{1/3}\right)^w u\right)_{L^2}+\mathrm{Re}\left(\left[\mathcal{L}_X,\ \left(\lambda^{1/3}\right)^w\right]u,\ \left(\mathrm{Id}+g^w\right)\left(\lambda^{1/3}\right)^w u\right)_{L^2}
\end{aligned}
$$

gives, with $\tilde{\varepsilon}>0$ arbitrary,

$$
\begin{aligned}
&\mathrm{Re}\left(\mathcal{L}_X\left(\lambda^{1/3}\right)^w u,\ \left(\lambda^{1/3}\right)^w u\right)_{L^2}+\mathrm{Re}\left(\mathcal{L}_X\left(\lambda^{1/3}\right)^w u,\ g^w\left(\lambda^{1/3}\right)^w u\right)_{L^2}\\
\lesssim\ &\tilde{\varepsilon}\left\|\left(\lambda^{2/3}\right)^w\right\|_{L^2}^2+C_{\tilde{\varepsilon}}\left\|\mathcal{L}_X u\right\|_{L^2}^2+\mathrm{Re}\left(\left[\mathcal{L}_X,\ \left(\lambda^{1/3}\right)^w\right]u,\ \left(\mathrm{Id}+g^w\right)\left(\lambda^{1/3}\right)^w u\right)_{L^2}.
\end{aligned}
$$

We could apply (4.15) with $d=1+g$ to estimate the last term in the above inequality; this gives, with $\varepsilon>0$ arbitrary,

$$
\begin{aligned}
&\mathrm{Re}\left(\mathcal{L}_X\left(\lambda^{1/3}\right)^w u,\ \left(\lambda^{1/3}\right)^w u\right)_{L^2}+\mathrm{Re}\left(\mathcal{L}_X\left(\lambda^{1/3}\right)^w u,\ g^w\left(\lambda^{1/3}\right)^w u\right)_{L^2}\\
\lesssim\ &\varepsilon\left(\mathcal{L}_X\left(\lambda^{1/3}\right)^w u,\ \left(\lambda^{1/3}\right)^w u\right)_{L^2}+\tilde{\varepsilon}\left\|\left(\lambda^{2/3}\right)^w\right\|_{L^2}^2+C_{\varepsilon,\tilde{\varepsilon}}\left\|\mathcal{L}_X u\right\|_{L^2}^2+\left\|\Phi^{2/3}u\right\|_{L^2}^2\\
&+\left\|\left\langle\partial_x V(x)\wedge\xi\right\rangle^{2/5}u\right\|_{L^2}^2.
\end{aligned}
$$

Let ε small enough yields the desired estimate (4.14). The proof is thus completed.

4.1 Proof of Proposition 4.1

In what follows, let h_N, with N a large integer, be a symbol defined by

$$
h_N=h_N(y,\eta)=\frac{\partial_x V(x)\cdot y+\xi\cdot\eta+\left(\partial_x V\wedge\eta+y\wedge\xi\right)\cdot\left(y\wedge\eta\right)}{\lambda_N^{4/3}}\psi_N(y,\eta), \tag{4.19}
$$

where

$$
\lambda_N=\Big(1+\left|\partial_x V(x)\wedge\eta+y\wedge\xi\right|^2+\left|\partial_x V(x)\right|^2+|\xi|^2+N^{-1}\left\langle\partial_x V(x)\wedge\xi\right\rangle^{6/5}\Big)^{\frac{1}{2}} \tag{4.20}
$$

and

$$
\psi_N(y,\eta)\ =\ \chi\left(\frac{\left(|y\wedge\eta|^2+|y|^{2+\gamma}+|\eta|^2\right)N^2}{\lambda_N^{2/3}}\right) \tag{4.21}
$$

with $\chi\in C_0^\infty(\mathbb{R};\ [0,1])$ such that $\chi=1$ in $[-1,1]$ and $\operatorname{supp}\chi\subset[-2,2]$.

Lemma 4.5 Let λ_N be given in (4.20). Then

$$
\forall\,\sigma\in\mathbb{R},\quad \lambda_N^\sigma\in S(\lambda_N^\sigma,|dy|^2+|d\eta|^2) \tag{4.22}
$$

uniformly with respect to X. Moreover, if $\sigma\leq 1$, then

$$
\forall\ |\alpha|+|\beta|\geq 1,\quad \left|\partial_y^\alpha\partial_\eta^\beta\left(\lambda_N^\sigma\right)\right|\lesssim\left\langle\partial_x V(x)\right\rangle^\sigma+\langle\xi\rangle^\sigma. \tag{4.23}
$$

Proof The proof is the same as Lemma 4.2.

Lemma 4.6 The symbol h_N given in (4.19) belongs to $S(1, |dy|^2 + |d\eta|^2)$ uniformly with respect to X.

Proof straightforward calculatation to get

$$\begin{aligned}&\left|\partial_x V(x)\cdot y+\xi\cdot\eta+\big(\partial_x V\wedge\eta+y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\right|\\ \le&\Big(|\partial_x V(x)|^{4/3}+|\partial_x V\wedge\eta+y\wedge\xi|^{4/3}+|\xi|^{4/3}\Big)+\big(\,|y\wedge\eta|^4+|y|^4+|\eta|^4\,\big),\end{aligned}$$

combined with the following inequality $\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big)\le\lambda_N^{2/3}$ in suppψ_N give the desired lemma.

Lemma 4.7 Let λ_N and ψ_N be given in (4.20) and (4.21). Then for any $\sigma\in\mathbb{R}$, the following two inequalities

$$\left|\big(\xi\cdot\partial_\eta+\partial_x V(x)\cdot\partial_y\big)\lambda_N^\sigma\right|\lesssim N\lambda_N^{\sigma+\frac{2}{3}} \tag{4.24}$$

and

$$\left|\big(\xi\cdot\partial_\eta+\partial_x V(x)\cdot\partial_y\big)\psi_N\right|\lesssim N^3\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big) \tag{4.25}$$

hold uniformly with respect to (x,ξ).

Proof Using the inequality $\langle\partial_x V\wedge\xi\rangle\le N^{5/6}\lambda_N^{5/3}$ due to (4.20), we can caiculate that for any $\sigma\in\mathbb{R}$ one has

$$\begin{aligned}\left|\xi\cdot\partial_\eta\big(\lambda_N^\sigma\big)\right|&\le\left|\frac{\sigma}{2}\right|\,|\partial_x V\wedge\eta+y\wedge\xi|\,|\partial_x V(x)\wedge\xi|\,\lambda_N^{\sigma-2}\lesssim\lambda_N^{\frac{2}{3}+\sigma},\\ \left|\partial_x V(x)\cdot\partial_y\big(\lambda_N^\sigma\big)\right|&\le\left|\frac{\sigma}{2}\right|\,|\partial_x V\wedge\eta+y\wedge\xi|\,|\partial_x V(x)\wedge\xi|\,\lambda_N^{\sigma-2}\lesssim\lambda_N^{\frac{2}{3}+\sigma}.\end{aligned}$$

And thus (4.24) follows. In order to show (4.25), we write $\left|\big(\xi\cdot\partial_\eta+\partial_x V(x)\cdot\partial_y\big)\psi_N\right|=\big(\mathcal{K}_1+\mathcal{K}_2\big)$ with

$$\begin{aligned}\mathcal{K}_1&=N^2\left|\lambda_N^{-\frac{2}{3}}\big(\xi\cdot\partial_\eta+\partial_x V(x)\cdot\partial_y\big)\big[\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big]\chi'\right.\\ &\qquad\left.\Big((|y\wedge\eta|^2+|y|^2+|\eta|^2)N^2\lambda_N^{-2/3}\Big)\right|,\\ \mathcal{K}_2&=N^2\left|\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big)\big[\big(\xi\cdot\partial_\eta+\partial_x V(x)\cdot\partial_y\big)\lambda_N^{-2/3}\big]\chi'\right.\\ &\qquad\left.\Big((|y\wedge\eta|^2+|y|^2+|\eta|^2)N^2\lambda_N^{-2/3}\Big)\right|.\end{aligned}$$

Using (4.24) shows $\mathcal{K}_2\lesssim N^3\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big)$. Moreover direct computation gives

$$\mathcal{K}_1\lesssim N^2\lambda_N^{1/3}\big(\,|y\wedge\eta|+|y|+|\eta|\,\big)\chi'\Big((|y\wedge\eta|^2+|y|^2+|\eta|^2)N^2\lambda_N^{-2/3}\Big)\lesssim N^3\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big),$$

the last inequality following from the fact that $\lambda_N^{2/3}\lesssim\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big)N^2$ on the support of the function $\chi'\Big((|y\wedge\eta|^2+|y|^2+|\eta|^2)N^2\lambda_N^{-2/3}\Big)$, then the above inequalities yield the desired inequality (4.25). The proof of Lemma is thus completed.

The rest of this section is occupied by

Proof of Proposition 4.1 Since the proof is quite long, we divide it into three steps.

Step I Let N be a large integer to be determined later and $H = h_N^{\mathrm{Wick}}$ be the Wick quantization of the symbol h_N given in (4.19). To simplify the notation we will use C_N to denote different suitable constants which depend only on N. In the following discussion, let $u \in \mathcal{S}(\mathbb{R}_y^3)$. By (2.4) and Lemma 4.6, we can find a symbol $\tilde{h}_N$ such that $H = \tilde{h}_N^w$ with $\tilde{h}_N \in S(1, |dy|^2 + |d\eta|^2)$ uniformly with respect to X. Then using Lemma 3.3, one gives

$$\left|\Big(\big(B(y)D_y\big)^* B(y)D_y u,\ Hu\Big)_{L^2} + (pu,\ Hu)_{L^2}\right| \lesssim \mathrm{Re}\ (\mathcal{L}_X u,\ u)_{L^2}.$$

This together with the relation

$$\begin{aligned}&\mathrm{Re}\ (iQ_X u,\ H\,u)_{L^2}\\ =&\mathrm{Re}\ (\mathcal{L}_X u,\ H\,u)_{L^2} - \mathrm{Re}\ \Big(\big(B(y)D_y\big)^* B(y)D_y u,\ Hu\Big)_{L^2} - \mathrm{Re}\ (pu,\ Hu)_{L^2}\\ &- \mathrm{Re}\ \big(\big(b\wedge D_y\big)\cdot(y\wedge D_y)u,\ Hu\big)_{L^2} - \mathrm{Re}\ (qu,\ Hu)_{L^2} - \mathrm{Re}\ \Big(\mu^{\frac12} m(y)u,\ Hu\Big)_{L^2}\end{aligned}$$

yields

$$\mathrm{Re}\ (iQ_X u,\ H\,u)_{L^2} \lesssim |(\mathcal{L}_X u,\ u)_{L^2}| + |(\mathcal{L}_X u, Hu)_{L^2}|. \tag{4.26}$$

Next we will give a lower bound for the term on the left side. Observe the symbol of Q_X is a first order polynomial in y, η. Then $iQ_X = i\big(y\cdot\xi - \partial_x V(x)\cdot\eta\big)^{\mathrm{Wick}}$, and hence

$$\mathrm{Re}\ (iQ_X u,\ H\,u)_{L^2} = \frac{1}{4\pi}\Big(\big\{h,\ y\cdot\xi - \partial_x V(x)\cdot\eta,\big\}^{\mathrm{Wick}} u,\ u\Big)_{L^2}, \tag{4.27}$$

where $\{\cdot,\ \cdot\}$ is the Poisson bracket defined in (2.5). Direct calculus shows

$$\begin{aligned}&\big\{h,\ y\cdot\xi - \partial_x V(x)\cdot\eta\big\}\\ =&\frac{|\partial_x V(x)\wedge\eta + y\wedge\xi|^2 + |\partial_x V(x)|^2 + |\xi|^2 + 2\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)}{\lambda_N^{4/3}}\psi_N\\ &+\big[\partial_x V(x)\cdot y + \xi\cdot\eta + \big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big]\big[\big(\xi\cdot\partial_\eta + \partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\\ =&\lambda_N^{2/3}\psi_N - \frac{1+N^{-1}\left\langle\partial_x V(x)\wedge y\right\rangle^{6/5}}{\lambda_N^{4/3}}\psi_N + \frac{2\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)}{\lambda_N^{4/3}}\psi_N\\ &+\big[\partial_x V(x)\cdot y + \xi\cdot\eta + \big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big]\big[\big(\xi\cdot\partial_\eta + \partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\\ \geq&\lambda_N^{2/3} - \lambda_N^{2/3}(1-\psi_N) - \frac{1+N^{-1}\left\langle\partial_x V\wedge\xi\right\rangle^{6/5}}{\lambda_N^{4/3}} - \frac{2\left|\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)\right|}{\lambda_N^{4/3}}\psi_N\\ &-\Big|\big[\partial_x V(x)\cdot y + \xi\cdot\eta + \big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big]\big[\big(\xi\cdot\partial_\eta + \partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\Big|\\ \geq&\lambda_N^{2/3} - N^2\big(|y\wedge\eta|^2 + |y|^2 + |\eta|^2\big) - \frac{1+N^{-1}\left\langle\partial_x V\wedge\xi\right\rangle^{6/5}}{\lambda_N^{4/3}} - \frac{2\left|\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)\right|}{\lambda_N^{4/3}}\psi_N\\ &-\Big|\big[\partial_x V(x)\cdot y + \xi\cdot\eta + \big(\partial_x V\wedge\eta + y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big]\big[\big(\xi\cdot\partial_\eta + \partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\Big|,\end{aligned}$$

the last inequality holding because $\lambda_N^{2/3} \leq N^2\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big)$ on the support of $1-\psi_N$. Due to the positivity of the Wick quantization, the above inequalities, along with (4.26), (4.27) and the estimate

$$\Big(\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big)^{\rm Wick}u,\ u\Big)_{L^2} \lesssim |(\mathcal{L}_X u,\ u)_{L^2}| + \big\|u\big\|_{L^2}^2 \tag{4.28}$$

due to (4.9), yield

$$\Big(\big(\lambda_N^{2/3}\big)^{\rm Wick}u,\ u\Big)_{L^2R_y^3)} \lesssim \sum_{j=1}^{3}\big(R_j^{\rm Wick}u,\ u\big)_{L^2} + C_N\,|(\mathcal{L}_X u,\ u)_{L^2}| + |(\mathcal{L}_X u,\ Hu)_{L^2}|\,, \tag{4.29}$$

where R_j are given by

$$\begin{aligned}
R_1 &= \frac{1+N^{-1}\left<\partial_x V\wedge\xi\right>^{6/5}}{\lambda_N^{4/3}},\ R_2=\frac{2\left|\big(\partial_x V\wedge\xi\big)\cdot\big(y\wedge\eta\big)\right|}{\lambda_N^{4/3}}\psi_N,\\
R_3 &= \Big|\big[\partial_x V(x)\cdot y+\xi\cdot\eta+\big(\partial_x V\wedge\eta+y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big]\big[\big(\xi\cdot\partial_\eta+\partial_x V\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\Big|.
\end{aligned}$$

Step II In this step, we will treat the above terms R_j, and show that there exists a symbol q, belonging to $S(1,|dy|^2+|d\eta|^2)$ uniformly with respect to X, such that

$$\begin{aligned}
&\sum_{j=1}^{3}\big(R_j^{\rm Wick}u,\ u\big)_{L^2} \qquad\qquad (4.30)\\
\lesssim\ & N^{-1/3}\Big(\big(\lambda_N^{2/3}\big)^{\rm Wick}u,\ u\Big)_{L^2} + C_N\Big\{|(\mathcal{L}_X u,\ u)_{L^2}| + \big|\big(\mathcal{L}_X u,\ q^{\rm Wick}u\big)_{L^2}\big| + \big\|u\big\|_{L^2}^2\Big\}. \qquad (4.31)
\end{aligned}$$

For this purpose, we define q by

$$q(y,\eta)=q_X(y,\eta)=\frac{\big(\partial_x V(x)\wedge\xi\big)\cdot\big(\partial_x V(x)\wedge\eta+y\wedge\xi\big)}{\left<\partial_x V(x)\wedge\xi\right>^{8/5}}\varphi(y,\eta)$$

with

$$\varphi(y,\eta)=\chi\left(\frac{|\partial_x V(x)\wedge\eta+y\wedge\xi|^2+|\partial_x V(x)|^2+|\xi|^2}{\left<\partial_x V(x)\wedge\xi\right>^{6/5}}\right).$$

Then one can verify that $q\in S(1,|dy|^2+|d\eta|^2)$ uniformly with respect to (x,ξ). Thus by (3.4), we conclude

$$\big(iQ_X u,\ q^{\rm Wick}u\big)_{L^2} \lesssim |(\mathcal{L}_X u,\ u)_{L^2}| + \big|\big(\mathcal{L}_X u,\ q^{\rm Wick}u\big)_{L^2}\big|\,. \tag{4.32}$$

On the other hand, it is just a direct computation of the Poisson bracke to see that

$$\begin{aligned}
\big(iQ_X u,\ q^{\rm Wick}u\big)_{L^2} &= \frac{1}{4\pi}\Big(\big\{q(y,\eta),\ y\cdot\xi-\partial_x V(x)\cdot\eta\big\}^{\rm Wick}u,\ u\Big)_{L^2}\\
&= \frac{1}{4\pi}\big(R_{1,1}^{\rm Wick}u,\ u\big)_{L^2}+\frac{1}{4\pi}\big(R_{1,2}^{\rm Wick}u,\ u\big)_{L^2}
\end{aligned} \tag{4.33}$$

with

$$R_{1,1}=\frac{2\left|\partial_xV(x)\wedge\xi\right|^2}{\langle\partial_xV(x)\wedge\xi\rangle^{8/5}}\varphi,$$
$$R_{1,2}=\frac{\big(\partial_xV(x)\wedge\xi\big)\cdot(\partial_xV(x)\wedge\eta+y\wedge\xi)}{\langle\partial_xV(x)\wedge\xi\rangle^{8/5}}\big[\big(\xi\cdot\partial_\eta+\partial_xV(x)\cdot\partial_y\big)\varphi(y,\eta)\big].$$

Moreover, we have $R_{1,2}\lesssim\big(\left|\partial_xV(x)\wedge\eta+y\wedge\xi\right|^2+\left|\partial_xV(x)\right|^2+|\xi|^2\big)^{1/3}\lesssim\lambda_N^{2/3}$, and

$$\begin{aligned}R_{1,1}&=\langle\partial_xV(x)\wedge\xi\rangle^{2/5}-\frac{1}{\langle\partial_xV(x)\wedge\xi\rangle^{8/5}}\varphi-\langle\partial_xV(x)\wedge\xi\rangle^{2/5}\big(1-\varphi\big)\\&\geq\langle\partial_xV(x)\wedge\xi\rangle^{2/5}-\frac{1}{\langle\partial_xV(x)\wedge\xi\rangle^{8/5}}-\big(\left|\partial_xV(x)\wedge\eta+y\wedge\xi\right|^2+\left|\partial_xV(x)\right|^2+|\xi|^2\big)^{1/3},\end{aligned}$$

where the last inequality holds because

$$\langle\partial_xV(x)\wedge\xi\rangle^{2/5}\leq\big(\left|\partial_xV(x)\wedge\eta+y\wedge\xi\right|^2+\left|\partial_xV(x)\right|^2+|\xi|^2\big)^{1/3}$$

on the support of $1-\varphi$. These inequalities, combining (4.33) and (4.32), yield

$$\begin{aligned}&\Big(\big(\langle\partial_xV(x)\wedge\xi\rangle^{2/5}\big)^{\mathrm{Wick}}u,\,u\Big)_{L^2}\\\lesssim\;&\Big(\big(\lambda_N^{2/3}\big)^{\mathrm{Wick}}u,\,u\Big)_{L^2}+\left|(\mathcal{L}_Xu,\,u)_{L^2}\right|+\left|\big(\mathcal{L}_Xu,\,q^{\mathrm{Wick}}u\big)_{L^2}\right|+\left\|u\right\|_{L^2}^2.\end{aligned}$$

Consequently, observing that

$$R_1=\frac{1+N^{-1}\langle\partial_xV\wedge\xi\rangle^{6/5}}{\lambda_N^{4/3}}\leq N^{-1/3}\langle\partial_xV(x)\wedge\xi\rangle^{2/5}+1$$

and

$$R_2=\frac{\left|\big(\partial_xV\wedge\xi\big)\cdot(y\wedge\eta)\right|}{\lambda_N^{4/3}}\psi_N\lesssim\frac{N^{-1}\langle\partial_xV\wedge\xi\rangle}{\lambda}\frac{N\left|y\wedge\eta\right|}{\lambda_N^{1/3}}\psi_N\leq N^{-1/2}\langle\partial_xV(x)\wedge\xi\rangle^{2/5},$$

we get the desired upper bound for the terms R_1 and R_2. It remains to handle R_3. By virtue of (4.24) and (4.25), we compute

$$\begin{aligned}R_3\lesssim&\big[\partial_xV(x)\cdot y+\xi\cdot\eta+\big(\partial_xV\wedge\eta+y\wedge\xi\big)\cdot\big(y\wedge\eta\big)\big]\big[\big(\xi\cdot\partial_\eta+\partial_xV\cdot\partial_y\big)\big(\lambda_N^{-4/3}\psi_N\big)\big]\\\lesssim&\frac{\partial_xV(x)\cdot y+\xi\cdot\eta+\big(\partial_xV\wedge\eta+y\wedge\xi\big)\cdot\big(y\wedge\eta\big)}{\lambda_N^{\frac{2}{3}}}\psi_N\\&+\frac{\partial_xV(x)\cdot y+\xi\cdot\eta+\big(\partial_xV\wedge\eta+y\wedge\xi\big)\cdot\big(y\wedge\eta\big)}{\lambda_N^{\frac{4}{3}}}\psi_N'\big(\left|y\wedge\eta\right|^2+|y|^2+|\eta|^2\big)\\\lesssim&N\lambda_N^{1/3}(|y\wedge\eta|+|y|+|\xi|)+N^2\big(\left|y\wedge\eta\right|^2+|y|^2+|\eta|^2\big)\\\lesssim&N^{-1}\lambda_N^{2/3}+C_N\big(\left|y\wedge\eta\right|^2+|y|^2+|\eta|^2\big).\end{aligned}$$

The forth inequlity result from

$$\lambda_N^{2/3} \le N^2\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big) \quad \text{in supp}\psi_N \text{ or supp}\psi'_N.$$

As a result, the positivity of Wick quantization gives

$$\begin{aligned}\big(R_3^{\mathrm{Wick}}u,\ u\big)_{L^2} \lesssim & N^{-1}\left(\big(\lambda_N^{2/3}\big)^{\mathrm{Wick}}u,\ u\right)_{L^2}+C_N\left(\big(\,|y\wedge\eta|^2+|y|^2+|\eta|^2\,\big)^{\mathrm{Wick}}u,\ u\right)_{L^2}\\ \lesssim & N^{-1}\left(\big(\lambda_N^{2/3}\big)^{\mathrm{Wick}}u,\ u\right)_{L^2}+C_N\left\{|(\mathcal{L}_Xu,\ u)_{L^2}|+\|u\|_{L^2}^2\right\}.\end{aligned}$$

Thus the desired estimate (4.30) follows.

Step III Now, we proceed the proof of Proposition 4.1. From (4.29) and (4.30), it follows that there exists a symbol $p\in S(1,|dy|^2+|d\eta|^2)$ uniformly with respect to X, such that

$$\begin{aligned}&\left(\big(\lambda_N^{2/3}\big)^{\mathrm{Wick}}u,u\right)_{L^2}\\ \lesssim\ & N^{-1/3}\left(\big(\lambda_N^{2/3}\big)^{\mathrm{Wick}}u,\ u\right)_{L^2}+C_N\left\{|(\mathcal{L}_Xu,\ u)_{L^2}|+\left|\big(\mathcal{L}_Xu,\ p^{\mathrm{Wick}}u\big)_{L^2}\right|+\|u\|_{L^2}^2\right\},\end{aligned}$$

which allows us to choose an integer N_0 large enough, such that

$$\left(\big(\lambda_{N_0}^{2/3}\big)^{\mathrm{Wick}}u,\ u\right)_{L^2}\lesssim C_{N_0}\left\{|(\mathcal{L}_Xu,\ u)_{L^2}|+\left|\big(\mathcal{L}_Xu,\ p^{\mathrm{Wick}}u\big)_{L^2}\right|+\|u\|_{L^2}^2\right\}.$$

Consequently, observing that $\lambda^{2/3}\lesssim\lambda_{N_0}^{2/3}+|y|^2+|\eta|^2$ with λ defined in (4.2), we get, combining (4.28),

$$\left(\big(\lambda^{2/3}\big)^{\mathrm{Wick}}u,\ u\right)_{L^2}\lesssim|(\mathcal{L}_Xu,\ u)_{L^2}|+\left|\big(\mathcal{L}_Xu,\ p^{\mathrm{Wick}}u\big)_{L^2}\right|+\|u\|_{L^2}^2. \tag{4.34}$$

Since $\langle\partial_xV(x)\rangle^{2/3}+\langle\xi\rangle^{2/3}\le\lambda^{2/3}$, the above inequality yields

$$\left(\Big(\langle\partial_xV(x)\rangle^{2/3}+\langle\xi\rangle^{2/3}\Big)u,\ u\right)_{L^2}\lesssim|(\mathcal{L}_Xu,\ u)_{L^2}|+\left|\big(\mathcal{L}_Xu,\ p^{\mathrm{Wick}}u\big)_{L^2}\right|+\|u\|_{L^2}^2.$$

Since $p\in S(1,|dy|^2+|d\eta|^2)$ uniformly with respect to X, then applying the above inequality to the function $\Big(\langle\partial_xV(x)\rangle^{2/3}+\langle\xi\rangle^{2/3}\Big)^{1/2}u$ implies

$$\Big(\langle\partial_xV(x)\rangle^{4/3}+\langle\xi\rangle^{4/3}\Big)\|u\|_{L^2}^2\lesssim\|\mathcal{L}_Xu\|_{L^2}^2+\|u\|_{L^2}^2. \tag{4.35}$$

Similarly, since $\langle\partial_xV(x)\wedge\xi\rangle^{2/5}\le\lambda^{2/3}$, then by virtue of (4.34) we have, repeating the above arguments,

$$\langle\partial_xV(x)\wedge\xi\rangle^{4/5}\|u\|_{L^2}^2\lesssim\|\mathcal{L}_Xu\|_{L^2}^2+\|u\|_{L^2}^2. \tag{4.36}$$

Now, we apply (4.34) to the function $\big(\lambda^{1/3}\big)^wu$, to get

$$\begin{aligned}&\left(\big(\lambda^{2/3}\big)^{\mathrm{Wick}}\big(\lambda^{1/3}\big)^wu,\ \big(\lambda^{1/3}\big)^wu\right)_{L^2}\\ \lesssim&\left|\big(\mathcal{L}_X\big(\lambda^{1/3}\big)^wu,\ \big(\lambda^{1/3}\big)^wu\big)_{L^2}\right|+\left|\big(\mathcal{L}_X\big(\lambda^{1/3}\big)^wu,\ p^{\mathrm{Wick}}\big(\lambda^{1/3}\big)^wu\big)_{L^2}\right|\\ \lesssim&\,\varepsilon\left\|\big(\lambda^{2/3}\big)^wu\right\|_{L^2}^2+C_\varepsilon\left\|\Phi^{2/3}u\right\|_{L^2}^2+C_\varepsilon\left\|\langle\partial_xV(x)\wedge\xi\rangle^{2/5}u\right\|_{L^2}^2,\end{aligned}$$

where the last inequality follows from (4.14). Furthermore, using (4.6) implies

$$\Big(\big(\lambda^{2/3}\big)^{\mathrm{Wick}}\big(\lambda^{1/3}\big)^{w}u,\ \big(\lambda^{1/3}\big)^{w}u\Big)_{L^2}\gtrsim\big\|\big(\lambda^{2/3}\big)^{w}u\big\|_{L^2}^2-\big\|\Phi^{2/3}u\big\|_{L^2}^2.$$

Combining the above inequalities, we have

$$\begin{aligned}\big\|\big(\lambda^{2/3}\big)^{w}u\big\|_{L^2}^2&\lesssim\varepsilon\,\big\|\big(\lambda^{2/3}\big)^{w}u\big\|_{L^2}^2+C_\varepsilon\big\|\Phi^{2/3}u\big\|_{L^2}^2+C_\varepsilon\big\|\left\langle\partial_xV(x)\wedge\xi\right\rangle^{2/5}u\big\|_{L^2}^2\\&\lesssim\varepsilon\,\big\|\big(\lambda^{2/3}\big)^{w}u\big\|_{L^2}^2+C_\varepsilon\big\|\mathcal{L}_Xu\big\|_{L^2}^2+C_\varepsilon\big\|u\big\|_{L^2}^2,\end{aligned}$$

the last inequality following from (4.35) and (4.36). Let the number ε small enough yields $\big\|\big(\lambda^{2/3}\big)^{w}u\big\|_{L^2}\lesssim\big\|\mathcal{L}_Xu\big\|_{L^2}+\big\|u\big\|_{L^2}$. This, along with (4.7) and (4.35), gives the desired estimate (4.3), completing the proof of Proposition 4.1.

5 Proof of Theorem 1.1: Regularity Estimates in All Variables

In this section, we will show the hypoelliptic estimates in spatial and velocity variables for the original operator $\mathcal{L}$.

Proposition 5.1 Let $V(x)$ be a C^2-function satisfying assumption (1.8). Then for any $u\in C_0^\infty\big(\mathbb{R}^{2n}\big)$, one has

$$\big\|\,|D_x|^{2/3}u\big\|_{L^2}+\big\|\left\langle y\right\rangle^{\gamma/2}|D_y|^2u\big\|_{L^2}+\big\|\left\langle y\right\rangle^{\gamma/2}|y\wedge D_y|^2u\big\|_{L^2}\lesssim\big\|\mathcal{L}u\big\|_{L^2}+\big\|u\big\|_{L^2}.\tag{5.1}$$

Proof The proof of is quite similar as that of Proposition 4.1 in [2] and [3]. So we only give a sketch here and refer to [2] and [3] for more detailed discussions. With each fixed $x_\mu\in\mathbb{R}^3$, we associate an operator

$$\begin{aligned}\mathcal{L}_{x_\mu}\ &=\ i\big(y\cdot D_x-\partial_xV(x_\mu)\cdot D_y\big)+\big(B(y)D_y\big)^*\cdot B(y)D_y+p(y)\\&\quad+(D_y\wedge b)\cdot(y\wedge D_y)+q(y)+\mu^{\frac12}m(y).\end{aligned}$$

Let P_{X_μ}, with $X_\mu=(x_\mu,\xi)$, be the operator defined in (4.1), i.e.,

$$\begin{aligned}\mathcal{L}_{X_\mu}\ &=\ i\big(y\cdot\xi-\partial_xV(x_\mu)\cdot D_y\big)+\big(B(y)D_y\big)^*\cdot B(y)D_y+p(y)\\&\quad+(D_y\wedge b)\cdot(y\wedge D_y)+q(y)+\mu^{\frac12}m(y).\end{aligned}$$

Observe $\mathcal{F}_x\mathcal{L}_{x_\mu}=\mathcal{L}_{X_\mu}$, where $\mathcal{F}_x$ stands for the partial Fourier transform in x variable. Suppose V satisfies condition (1.8). Then performing the Fourier transform with respect to x, it follows from (4.3) that $\forall\,u\in C_0^\infty\big(\mathbb{R}^6\big)$,

$$\big\|\left\langle D_x\right\rangle^{2/3}u\big\|_{L^2}+\big\|\left\langle y\right\rangle^{\gamma/2}|D_y|^2u\big\|_{L^2}+\big\|\left\langle y\right\rangle^{\gamma/2}|y\wedge D_y|^2u\big\|_{L^2}\lesssim\big\|\mathcal{L}_{x_\mu}u\big\|_{L^2}+\big\|u\big\|_{L^2}.\tag{5.2}$$

Lemma 4.2 in [1] shows the metric g defined by $g_x=\left\langle\partial_xV(x)\right\rangle^{2/3}|dx|^2,\ \ x\in\mathbb{R}^3$ is slowly varying, i.e., we can find two constants $C_*,r_0>0$ such that if $g_x(x-y)\le r_0^2$, then

$$C_*^{-1}\le\frac{g_x}{g_y}\le C_*.$$

The main feature of a slowly varying metric is that it allows us to introduce some partitions of unity related to the metric (see for instance Lemma 18.4.4 of [8]). Precisely, we could find a constant $r > 0$ and a sequence $x_\mu \in \mathbb{R}^n, \mu \geq 1$, such that the union of the balls

$$\Omega_{\mu,r} = \left\{x \in \mathbb{R}^n; \quad g_{x_\mu}\big(x - x_\mu\big) < r^2\right\}$$

coves the whole space $\mathbb{R}^n$. Moreover there exists a positive integer N_r, depending only on r, such that the intersection of more than N_r balls is always empty. One can choose a family of nonnegative functions $\{\varphi_\mu\}_{\mu\geq 1}$ in $S(1, g)$ such that

$$\operatorname{supp} \varphi_\mu \subset \Omega_{\mu,r}, \quad \sum_{\mu\geq 1} \varphi_\mu^2 = 1 \quad \text{and} \quad \sup_{\mu\geq 1} |\partial_x \varphi_\mu(x)| \lesssim \langle \partial_x V(x)\rangle^{\frac{1}{3}}. \tag{5.3}$$

Repeat the precess in [3], we see

$$\big\| \langle D_x\rangle^{2/3} u\big\|_{L^2}^2 \lesssim \sum_{\mu\geq 1} \big\| \langle D_x\rangle^{2/3} \varphi_\mu u\big\|_{L^2}^2 + \big\|\mathcal{L}u\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2. \tag{5.4}$$

Using the notation

$$R_\mu = -y \cdot \partial_x \varphi_\mu(x) - \varphi_\mu\big(\partial_x V(x) - \partial_x V(x_\mu)\big) \cdot \partial_y,$$

we may write $\varphi_\mu \mathcal{L}u = \mathcal{L}_{x_\mu} \varphi_\mu u + R_\mu u$, then

$$\sum_{\mu\geq 1} \big\|\mathcal{L}_{x_\mu} \varphi_\mu u\big\|_{L^2}^2 \leq 2\sum_{\mu\geq 1} \big(\big\|\varphi_\mu \mathcal{L}u\big\|_{L^2}^2 + \big\|R_\mu u\big\|_{L^2}^2\big) \leq 2\big\|\mathcal{L}u\big\|_{L^2}^2 + 2\sum_{\mu\geq 1} \big\|R_\mu u\big\|_{L^2}^2.$$

On the other hand, by Lemma 4.9 in [1], we have $\sum\limits_{\mu\geq 1} \big\|R_\mu u\big\|_{L^2}^2 \lesssim \big\|\mathcal{L}u\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2$. The above two inequalities yield

$$\forall\, u \in C_0^\infty\big(\mathbb{R}^{2n}\big), \quad \sum_{\mu\geq 1} \big\|\mathcal{L}_{x_\mu} \varphi_\mu u\big\|_{L^2}^2 \lesssim \big\|\mathcal{L}u\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2.$$

Using (5.4) and (5.2), we have

$$\begin{aligned} \big\| \langle D_x\rangle^{\frac{2}{3}} u\big\|_{L^2}^2 &\lesssim \sum_{\mu\geq 1} \big\| \langle D_x\rangle^{\frac{2}{3}} \varphi_\mu u\big\|_{L^2}^2 + \big\|\mathcal{L}u\big\|_{L^2}^2 + C\big\|u\big\|_{L^2}^2 \\ &\lesssim \sum_{\mu\geq 1} \big\|\mathcal{L}_{x_\mu} \varphi_\mu u\big\|_{L^2}^2 + \big\|\mathcal{L}u\big\|_{L^2}^2 + \big\|u\big\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} &\big\| \langle y\rangle^{\gamma/2} |D_y|^2 u\big\|_{L^2}^2 + \big\| \langle y\rangle^{\gamma/2} |y \wedge D_y|^2 u\big\|_{L^2}^2 \\ = &\sum_{\mu\geq 1} \big\| \langle y\rangle^{\gamma/2} |D_y|^2 \varphi_\mu u\big\|_{L^2}^2 + \sum_{\mu\geq 1} \big\| \langle y\rangle^{\gamma/2} |y \wedge D_y|^2 \varphi_\mu u\big\|_{L^2}^2 \\ \lesssim &\sum_{\mu\geq 1} \big\|\mathcal{L}_{x_\mu} \varphi_\mu u\big\|_{L^2}^2 + \sum_{\mu\geq 1} \big\|\varphi_\mu u\big\|_{L^2}^2. \end{aligned}$$

As a result, combining these inequalities gives (5.1). The proof is then completed.

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带有外势朗道算子的全局性亚椭圆估计

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摘要: 本文主要研究了具有外势的朗道型算子的亚椭圆性. 利用傅里叶变换和象征类计算, 在基于一些假设的前提下得到了朗道算子的全局性亚椭圆估计.

关键词: 全局亚椭圆性; 朗道方程; 外势; Wick 量子化

MR(2010)**主题分类号**: 35A25; 47J05 **中图分类号**: O175.3