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COMMUTATORS GENERATED BY LUSIN-AREA INTEGRAL AND LOCAL CAMPANATO FUNCTIONS ON GENERALIZED LOCAL MORREY SPACES

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Abstract: In the paper, the boundedness of the Lusin-area integral and its commutator on the generalized local Morrey spaces are established. Using the pointwise estimate of Lusin-area integral $\mu_{\Omega,S}$ and Hardy inequality, we study the boundedness of the Lusin-area integral $\mu_{\Omega,S}$ on the generalized local Morrey spaces, as well as the boundedness of the commutators generated by $\mu_{\Omega,S}$ and local Campanato functions, which extend the previous results.

Keywords: Lusin-area integral; commutator; local Campanato function; generalized local Morrey space

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1 Introduction

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^s(\mathbb{S}^{n-1})$ $(1 < s \leq \infty)$ be homogeneous of degree zero and satisfy the cancellation condition

$$\int_{S^{n-1}} \Omega(x^{'})d\sigma(x^{'}) = 0, \tag{1.1}$$

where $x^{'} = \frac{x}{|x|}$ for any $x \neq 0$. The Lusin-area integral $\mu_{\Omega,S}$ is defined by

$$\mu_{\Omega,S}(f)(x) = \left(\int\int_{\Gamma(x)} \left|\frac{1}{t}\int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{1.2}$$

where

$$\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\} \text{ and } \mathbb{R}^{n+1}_+ = \mathbb{R} \times (0,\infty).$$

Moreover, let $\vec{b} = (b_1, b_2, \cdots, b_m)$, where $b_i \in L_{\rm loc}(\mathbb{R}^n)$ for $1 \leq i \leq m$. Then the multilinear commutator generated by $\vec{b}$ and $\mu_{\Omega,S}$ can be defined as follows:

$$\mu^{\vec{b}}_{\Omega,S}f(x) = \left(\int\int_{\Gamma(x)} \left|\frac{1}{t}\int_{|y-z|<t} \prod_{i=1}^{m} \frac{\Omega(y-z)}{|y-z|^{n-1}}(b_i(x)-b_i(z))f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}. \tag{1.3}$$

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It is well known that the Lusin-area integral plays an important role in harmonic analysis and PDE (for example, see [1–8]). Therefore, it is a very interesting problem to discuss the boundedness of the Lusin-area integral. In [2], Ding, Fan and Pan studied the weighted L^p boundedness of the area integral $\mu_{\Omega,S}$. In [3], the authors investigated the boundeness of $\mu_{\Omega,S}$ on the weighted Morrey spaces. The commutators generated by $\mu_{\Omega,S}$ attracted much attention too. In [5] and [6], the authors discussed the weighted L^p boundedness and endpoint estimates for the higher order commutators generated by $\mu_{\Omega,S}$ and BMO function, respectively. In [8], the authors showed that the commutator generated by $\mu_{\Omega,S}$ and VMO is a compact operator in the Morrey space.

Moreover, the classical Morrey space $M_{p,\lambda}$ were first introduced by Morrey in [9] to study the local behavior of solutions to second order elliptic partial differential equations. And, in [10], the authors introduced the local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi}$, and they also studied the boundedness of the homogeneous singular integrals with rough kernel on these spaces.

Motivated by the works of [2, 3, 5, 8, 10, 13], we are going to consider the boundedness of $\mu_{\Omega,S}$ on the local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi}$, as well as the boundedness of the commutators generated by $\mu_{\Omega,S}$ and local Campanato functions.

2 Some Definitions and Lemmas

Definition 2.1 [10] Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and $1\le p\le\infty$. For any fixed $x_0\in\mathbb{R}^n$, a function $f\in L^q_{\rm loc}$ is said to belong to the local Morrey space, if

$$\|f\|_{LM^{\{x_0\}}_{p,\varphi}}=\sup_{r>0}\varphi(x_0,r)^{-1}|B(x_0,r)|^{-\frac{1}{p}}\|f\|_{L^p(B(x_0,r))}<\infty.$$

And we denote

$$LM^{\{x_0\}}_{p,\varphi}\equiv LM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)=\{f\in L^q_{\rm loc}(\mathbb{R}^n):\|f\|_{LM^{\{x_0\}}_{p,\varphi}}<\infty\}.$$

According to this definition, we recover the local Morrey space$LM^{\{x_0\}}_{p,\lambda}$ under the choice $\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}$.

Definition 2.2 [10] Let $1\le q<\infty$ and $0\le\lambda<\frac{1}{n}$. A function $f\in L^q_{\rm loc}(\mathbb{R}^n)$ is said to belong to the space $LC^{\{x_0\}}_{q,\lambda}$ (local Campanato space), if

$$\|f\|_{LC^{\{x_0\}}_{q,\lambda}}=\sup_{r>0}\left(\frac{1}{|B(x_0,r)|^{1+\lambda q}}\int_{B(x,r)}|f(y)-f_{B(x_0,r)}|^q dy\right)^{\frac{1}{q}}<\infty,$$

where

$$f_{B(x_0,r)}=\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}f(y)dy.$$

Define

$$LC^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)=\{f\in L^q_{\rm loc}(\mathbb{R}^n):\|f\|_{LC^{\{x_0\}}_{q,\lambda}}<\infty\}.$$

Remark 2.1 [10] Note that, the central BMO space $CBMO_q(\mathbb{R}^n) = LC_{q,0}^{\{0\}}(\mathbb{R}^n)$ and $CBMO_q^{\{x_0\}}(\mathbb{R}^n) = LC_{q,0}^{\{x_0\}}(\mathbb{R}^n)$. Moreover, imagining that the behavior of $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$, since there is no analogy of the John-Nirenberg inequality of BMO for the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$.

Lemma 2.1 [10] Let $1 < q < \infty$, $0 < r_2 < r_1$ and $b \in LC_{q,\lambda}^{\{x_0\}}, 0 \leq \lambda < \frac{1}{n}$, then

$$\left(\frac{1}{|B(x_0,r_1)|^{1+\lambda q}}\int_{B(x_0,r_1)}|b(x)-b_{B(x_0,r_2)}|^q dx\right)^{\frac{1}{q}} \leq C\left(1+\ln\frac{r_1}{r_2}\right)\|b\|_{LC_{q,\lambda}^{\{x_0\}}}.$$

And from this inequality, we have

$$|b_{B(x_0,r_1)} - b_{B(x_0,r_2)}| \leq C\left(1+\ln\frac{r_1}{r_2}\right)|B(x_0,r_1)|^{\lambda}\|b\|_{LC_{q,\lambda}^{\{x_0\}}}.$$

In this section, we are going to use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^{\infty} g(s)w(s)ds,\ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

Lemma 2.2 [11, 12] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w g(t) \leq C\operatorname*{ess\,sup}_{t>0} v_1(t)g(t) \tag{2.1}$$

holds for some $C > 0$ and all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t)\int_t^{\infty}\frac{w(s)}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)}ds < \infty.$$

Moreover, if $\tilde{C}$ is the minimum value of C in (2.1), then $\tilde{C} = B$.

Lemma 2.3 [2] Suppose that $1 < q, s \leq \infty$ and $\Omega \in L^s(\mathbb{S}^{n-1})$ satisfying (1.1). If q, s and weighted function w satisfy one of the following conditions

(i) $\max\{s', 2\} = \eta < q < \infty$, and $w \in A_{q/\eta}$;

(ii) $2 < q < s$, and $w^{1-(q/2)'} \in A_{q'/s'}$;

(iii) $2 \leq q < \infty$, and $w^{s'} \in A_{q/2}$,

then the operator $\mu_{\Omega,S}$ is bounded on $L^q(w,\mathbb{R}^n)$ space, where $s' = \frac{s}{s-1}$ is the conjugate exponent of s.

Remark 2.2 From Lemma 2.3, it's obvious that when $\Omega \in L^s(\mathbb{S}^{n-1})$ $(1 < s \leq \infty)$ satisfies condition (1.1), the operator $\mu_{\Omega,S}$ is bounded on $L^q(R^n)$ space for $2 \leq q < \infty$.

3 Lusin-Area Integral on Generalized Local Morrey Spaces

Theorem 3.1 Let $\Omega \in L^s(\mathbb{S}^{n-1})$ $(1 < s \le \infty)$ satisfy condition (1.1) and $\max\{2, s'\} < q < \infty$, where $s' = \frac{s}{s-1}$ is the conjugate exponent of s. Then the inequality

$$\|\mu_{\Omega,S}(f)\|_{L^q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L^q(B(x_0,l))} \frac{dl}{l^{\frac{n}{q}+1}}$$

holds for any ball $B(x_0, r)$.

Proof Let $B = B(x_0, r)$. We write $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$. Thus we have

$$\|\mu_{\Omega,S}(f)\|_{L^q(B)} \le \|\mu_{\Omega,S}(f_1)\|_{L^q(B)} + \|\mu_{\Omega,S}(f_2)\|_{L^q(B)}.$$

Since $\mu_{\Omega,S}$ is bounded on $L^q(\mathbb{R}^n)$ space (see Lemma 2.3), then it follows that

$$\|\mu_{\Omega,S} f_1\|_{L^q(B)} \lesssim \|f\|_{L^q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L^q(B(x_0,l))} \frac{dl}{l^{\frac{n}{q}+1}}. \tag{3.1}$$

Our attention will be focused now on $|\mu_{\Omega,S} f_2(x)|$ for $x \in B$,

$$\begin{aligned} |\mu_{\Omega,S} f_2(x)| &\le \left(\int\int_{\Gamma(x)} \left|\frac{1}{t}\int_{(2B)^c \bigcap \{z:|y-z|<t\}} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\ &\le \left(\int\int_{\Gamma(x)} \left|\frac{1}{t}\sum_{j=1}^{\infty}\int_{(2^{j+1}B\setminus 2^jB) \bigcap \{z:|y-z|<t\}} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

Without loss of generality, we can assume that for any $x \in B$, $(y,t) \in \Gamma(x)$ and $z \in 2^{j+1}B \setminus 2^jB$, we have $B(x,t) \bigcap B(z,t) \ne \varnothing$. Thus there exists $y_0 \in B(x,t) \bigcap B(z,t)$, such that

$$2t \ge |x - y_0| + |y_0 - z| \ge |x - z| \ge |z - x_0| - |x - x_0| \ge 2^j r - r \ge 2^{j-1} r.$$

Hence

$$|\mu_{\Omega,S} f_2(x)| \le \left(\int_{2^{j-2}r}^{\infty}\int_{|x-y|<t} \left|\frac{1}{t}\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

When $\Omega \in L^\infty(\mathbb{S}^{n-1})$, it follows from the Hölder's inequality that

$$\begin{aligned} &|\mu_{\Omega,S} f_2(x)| \\ \le\ & \|\Omega\|_{L^\infty(S^{n-1})} \left(\int_{2^{j-2}r}^{\infty}\int_{|x-y|<t} \left[\frac{1}{t}\sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{\frac{n-1}{n}}} \int_{2^{j+1}B\setminus 2^jB} |f(z)|dz\right]^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\ \lesssim\ & \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{\frac{n-1}{n}}} \int_{2^{j+1}B} |f(z)|dz \left(\int_{2^{j-2}r}^{\infty}\int_{|x-y|<t} \frac{dydt}{t^{n+3}}\right)^{\frac{1}{2}} \\ \lesssim\ & \sum_{j=1}^{\infty} (2^{j+1}r)^{-n} \int_{2^{j+1}B} |f(z)|dz \lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-\frac{n}{q}} \|f(z)\|_{L^q(B(x_0,2^{j+1}r))} \\ \lesssim\ & \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} l^{-\frac{n}{q}-1} dl \|f\|_{L^q(B(x_0,2^{j+1}r))} \lesssim \int_{2r}^{\infty} \|f\|_{L^q(B(x_0,l))} \frac{dl}{l^{\frac{n}{q}+1}}. \end{aligned} \tag{3.3}$$

When $\Omega \in L^s(\mathbb{S}^{n-1}), 1 < s < \infty$, it is obvious that

$$\left(\int_{2^{j+1}B} |\Omega(y-z)|^s dz\right)^{\frac{1}{s}} \lesssim \left(\int_0^{2^{j+1}r} l^{n-1} dl \int_{S^{n-1}} |\Omega(u)|^s du\right)^{\frac{1}{s}} \lesssim \|\Omega\|_{L^s(S^{n-1})} |2^{j+1}B|^{\frac{1}{s}}. \tag{3.4}$$

Thus from Hölder's inequality and (3.4), we have

$$\begin{aligned}
&|\mu_{\Omega,S} f_2(x)| \\
&\le \left(\int_{2^{j-2}r}^\infty \int_{|x-y|<t} \left|\frac{1}{t}\sum_{j=1}^\infty \int_{2^{j+1}B\backslash 2^jB} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{2^{j-2}r}^\infty \int_{|x-y|<t} \left[\sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^{\frac{1}{s'}-\frac{1}{n}}} \|\Omega\|_{L^s(S^{n-1})} \left(\int_{2^{j+1}B\backslash 2^jB} |f(z)|^{s'} dz\right)^{\frac{1}{s'}}\right]^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^\infty |2^{j+1}B|^{\frac{1}{s'}-\frac{1}{n}} \left(\int_{2^{j+1}B} |f(z)|^{s'} dz\right)^{\frac{1}{s'}} \left(\int_{2^{j-2}r}^\infty \int_{|x-y|<t} \frac{dydt}{t^{n+3}}\right)^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^\infty (2^{j+1}r)^{-\frac{n}{q}} \|f\|_{L^q(B(x_0,2^{j+1}r))} \\
&\lesssim \int_{2r}^\infty \|f\|_{L^q(B(x_0,l))} \frac{dl}{l^{\frac{n}{q}+1}}.
\end{aligned} \tag{3.5}$$

So

$$\|\mu_{\Omega,S} f_2\|_{L^q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty \|f\|_{L^q(B(x_0,l))} \frac{dl}{l^{\frac{n}{q}+1}}. \tag{3.6}$$

Therefore combining (3.1) and (3.6), we have

$$\|\mu_{\Omega,S} f\|_{L^q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty \|f\|_{L^q(B(x_0,l))} \frac{dl}{l^{\frac{n}{q}+1}}.$$

Thus we complete the proof of Theorem 3.1.

Theorem 3.2 Let $\Omega \in L^s(\mathbb{S}^{n-1})$ $(1 < s \le \infty)$ satisfy condition (1.1) and $\max\{2, s'\} < q < \infty$. Then, if functions φ, $\psi : \mathbb{R}^n \times (0,\infty) \to (0,+\infty)$ satisfy the inequality

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{l<\tau<\infty} \varphi(x,\tau)\tau^{\frac{n}{p}}}{l^{\frac{n}{p}+1}} dl \le C\psi(x_0, r), \tag{3.7}$$

where C does not depend on x and r, the operator $\mu_{\Omega,S}$ is bounded from $LM^{\{x_0\}}_{p,\varphi}$ to $LM^{\{x_0\}}_{p,\psi}$.

Proof Taking $v_1(l) = \varphi(x_0,l)^{-1} l^{-\frac{n}{q}}$, $v_2(l) = \psi(x_0,l)^{-1}$, $g(l) = \|f\|_{L^q(B(x_0,l))}$ and $w(l) = l^{-\frac{n}{q}-1}$, then from Theorem 3.1, we have

$$\operatorname*{ess\,sup}_{l>0} v_2(l) \int_l^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Thus from Lemma 2.2, it follows that

$$\operatorname*{ess\,sup}_{l>0} v_2(l) H_w g(l) \le C \operatorname*{ess\,sup}_{l>0} v_1(l) g(l).$$

Therefore

$$\begin{aligned}\|\mu_{\Omega,S}f\|_{LM^{\{x_0\}}_{q,\psi}} &= \sup_{r>0}\psi(x_0,r)^{-1}|B(x_0,r)|^{-\frac{1}{q}}\|\mu_{\Omega,S}f\|_{L^q(B(x_0,r))}\\ &\lesssim \sup_{r>0}\psi(x_0,r)^{-1}\int_r^\infty \|f\|_{L^q(B(x_0,l))}\frac{dl}{l^{\frac{n}{q}+1}}\\ &\lesssim \sup_{r>0}\varphi(x_0,r)^{-1}r^{-\frac{n}{q}}\|f\|_{L^q(B(x_0,r)} = \|f\|_{LM^{\{x_0\}}_{q,\varphi}}.\end{aligned}$$

Thus we complete the proof of Theorem 3.2.

4 Commutators Generated by Lusin-Area Integral on Generalized Local Morrey Spaces

Theorem 4.1 Let $\Omega \in L^s(\mathbb{S}^{n-1})$ $(1 < s \le \infty)$ satisfy condition (1.1) and $\max\{2, s'\} < q < \infty$. Let $1 < p, q_1, q_2, \cdots, q_m < \infty$, such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$, and $b_i \in LC^{\{x_0\}}_{q_i,\lambda_i}$ for $0 \le \lambda_i < \frac{1}{n}, i = 1, 2, \cdots, m$. Then the inequality

$$\|\mu^{\vec{b}}_{\Omega,S}f\|_{L^q(B(x_0,r))} \lesssim \prod_{i=1}^m \|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} r^{\frac{n}{q}}\int_{2r}^\infty \left(1+\ln\frac{l}{r}\right)^m \|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-n\lambda}}$$

holds for any ball $B(x_0, r)$, where $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$.

Proof Without loss of generality, it is sufficient for us to show that the conclusion holds for $m = 2$.

Let $B = B(x_0, r)$. And we write $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$, $f_2 = f\chi_{(2B)^c}$. Thus we have

$$\|\mu^{(b_1,b_2)}_{\Omega,S}f\|_{L^q(B)} \le \|\mu^{(b_1,b_2)}_{\Omega,S}f_1\|_{L^q(B)} + \|\mu^{(b_1,b_2)}_{\Omega,S}f_2\|_{L^q(B)} =: I + II.$$

Let us estimate I and II, respectively. It is obvious that

$$\begin{aligned}&\|\mu^{(b_1,b_2)}_{\Omega,S}f_1\|_{L^q(B)}\\ =\ &\|(b_1-(b_1)_B)(b_2-(b_2)_B)\mu_{\Omega,S}f_1\|_{L^q(B)} + \|(b_1-(b_1)_B)\mu_{\Omega,S}(b_2-(b_2)_B)f_1\|_{L^q(B)}\\ &+\|(b_2-(b_2)_B)\mu_{\Omega,S}(b_1-(b_1)_B)f_1\|_{L^q(B)} + \|\mu_{\Omega,S}(b_1-(b_1)_B)(b_2-(b_2)_B)f_1\|_{L^q(B)}\\ =:\ &I_1+I_2+I_3+I_4.\end{aligned}$$

From Lemma 2.1, it is easy to see that

$$\|b_i - (b_i)_B\|_{L^{q_i}(B)} \le Cr^{\frac{n}{p_i}+n\lambda_i}\|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}} \text{ for } i = 1, 2. \tag{4.1}$$

Since $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{p}$ and $\max\{2, s'\} < q < \infty$. It is obvious that $\max\{2, s'\} < p < \infty$. Thus using Hölder's inequality, Theorem 3.1 and (4.1), we have

$$\begin{aligned}I_1 &\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(B)}\|\mu_{\Omega,S}f_1\|_{L^p(B)}\\ &\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(B)}r^{\frac{n}{p}}\int_{2r}^\infty \|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1}}\\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{q}}\int_{2r}^\infty\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.\end{aligned} \tag{4.2}$$

Moreover, from Lemma 2.1, it is easy to see that

$$\|b_i-(b_i)_B\|_{L^{p_i}(2B)}\le Cr^{\frac{n}{p_i}+n\lambda_i}\|b_i\|_{LC^{\{x_0\}}_{p_i,\lambda_i}}\ \text{ for } i=1,2. \tag{4.3}$$

And let $\frac{1}{\bar q}=\frac{1}{q_2}+\frac{1}{p}$. Then it is easy to see that $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{\bar q}$ and $\max\{s',2\}<\bar q<\infty$. Then similarly to the estimate of (4.2), we have

$$\begin{aligned}
I_2 &\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|\mu_{\Omega,S}(b_2-(b_2)_B)f_1\|_{L^{\bar q}(B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(2B)}\|f\|_{L^p(2B)}\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.
\end{aligned}$$

Similarly,

$$I_3 \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.$$

Similarly, since $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{p}$. Then, by Lemma 2.3, Hölder's inequality and (4.3), we obtain

$$\begin{aligned}
I_4 &= \|\mu_{\Omega,S}(b_1-(b_1))(b_2-(b_2)_B)f_1\|_{L^q(B)}\\
&\lesssim \|(b_1-(b_1)_B)(b_2-(b_2)_B)f\|_{L^q(2B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(B)}\|f\|_{L^p(2B)}\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.
\end{aligned}$$

Therefore combining the estimates of I_1,I_2,I_3 and I_4, we have

$$I \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.$$

Let us estimate II.

$$\begin{aligned}
&\|\mu^{(b_1,b_2)}_{\Omega,S}f_2\|_{L^q(B)}\\
=\;&\|(b_1-(b_1)_B)(b_2-(b_2)_B)\mu_{\Omega,S}f_2\|_{L^q(B)}+\|(b_1-(b_1)_B)\mu_{\Omega,S}(b_2-(b_2)_B)f_2\|_{L^q(B)}\\
&+\|(b_2-(b_2)_B)\mu_{\Omega,S}(b_1-(b_1)_B)f_2\|_{L^q(B)}+\|\mu_{\Omega,S}(b_1-(b_1)_B)(b_2-(b_2)_B)f_2\|_{L^q(B)}\\
=:\;& II_1+II_2+II_3+II_4.
\end{aligned}$$

Since $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{p}$. Then using Hölder's inequality and (3.6), we have

$$\begin{aligned}
II_1 &\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(B)}\|\mu_{\Omega,S}f_2\|_{L^p(B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b_2-(b_2)_B\|_{L^{q_2}(B)}r^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1}}\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.
\end{aligned}$$

In the following, let us estimate II_2. For $x \in B$, when $\Omega \in L^{\infty}(\mathbb{S}^{n-1})$, from Lemma 2.1 and estimate of (3.3), we have

$$\begin{aligned}
&|\mu_{\Omega,S}(b_2-(b_2)_B)f_2(x)| \\
\lesssim& \sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\int_{2^{j+1}B}|b_2(z)-(b_2)_{B(x_0,r)}||f(z)|dz \\
\lesssim& \sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\|f\|_{L^p(B(x_0,2^{j+1}r))}\|b_2-(b_2)_{B(x_0,r)}\|_{L^{q_2}(B(x_0,2^{j+1}r))}|2^{j+1}B|^{1-\frac{1}{p}-\frac{1}{q_2}} \\
\lesssim& \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-n\lambda_2}}.
\end{aligned} \tag{4.4}$$

For $x \in B$, when $\Omega \in L^{s}(\mathbb{S}^{n-1}), 1<s<\infty$, from Lemma 2.1 and the estimate of (3.5), it follows that

$$\begin{aligned}
&|\mu_{\Omega,S}(b_2-(b_2)_B)f_2(x)| \\
\lesssim& \sum_{j=1}^{\infty}(2^{j+1}r)^{-\frac{n}{s'}}\left(\int_{B(x_0,2^{j+1}r)}(|b_2(z)-(b_2)_{B(x_0,r)}||f(z)|)^{s'}dz\right)^{\frac{1}{s'}} \\
\lesssim& \sum_{j=1}^{\infty}\|f\|_{L^p(B(x_0,2^{j+1}r))}\|b_2-(b_2)_{B(x_0,r)}\|_{L^{q_2}(B(x_0,2^{j+1}r))}|2^{j+1}B|^{-\frac{1}{p}-\frac{1}{q_2}} \\
\lesssim& \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-n\lambda_2}}.
\end{aligned} \tag{4.5}$$

Let $1<\tilde{q}<\infty$ such that $\frac{1}{\tilde{q}}=\frac{1}{p}+\frac{1}{q_2}$, then $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{\tilde{q}}$ and $\max\{2,s'\}<\tilde{q}<\infty$. Thus, from Hölder's inequality, (4.4) and (4.5), we obtain

$$\begin{aligned}
II_2 &\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|\mu_{\Omega,S}(b_2-(b_2)_B)f_2\|_{L^{\tilde{q}}(B)} \\
&\lesssim \|b_1-(b_1)_B\|_{L^{q_1}(B)}\|b\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{\tilde{q}}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-n\lambda_2}} \\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.
\end{aligned} \tag{4.6}$$

Similarly,

$$II_3 \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.$$

Let us estimate II_4. It is analogue to the estimates of (4.4), (4.5) and (4.6), we have the following estimates.

When $x \in B$, $\Omega \in L^\infty(\mathbb{S}^{n-1})$, we have

$$\begin{aligned}
&|\mu_{\Omega,S}(b_1-(b_1)_B)(b_2-(b_2)_B)f_2(x)| \\
\lesssim\ & \sum_{j=1}^{\infty}(2^{j+1}r)^{-n}\int_{2^{j+1}B}|b_1(z)-(b_1)_{B(x_0,r)}||b_2(z)-(b_2)_{B(x_0,r)}||f(z)|dz \\
\lesssim\ & \sum_{j=1}^{\infty}\|f\|_{L^p(B(x_0,2^{j+1}r))}\|b_1-(b_1)_{B(x_0,r)}\|_{L^{q_1}(B(x_0,2^{j+1}r))}\|b_2-(b_2)_{B(x_0,r)}\|_{L^{q_2}(B(x_0,2^{j+1}r))} \\
&|2^{j+1}B|^{-\frac{1}{q}} \\
\lesssim\ & \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.
\end{aligned} \tag{4.7}$$

When $x \in B$, $\Omega \in L^s(\mathbb{S}^{n-1}), 1<s<\infty$, we have

$$\begin{aligned}
&|\mu_{\Omega,S}(b_1-(b_1)_B)(b_2-(b_2)_B)f_2(x)| \\
\lesssim\ & \sum_{j=1}^{\infty}(2^{j+1}r)^{-\frac{n}{s'}}\left(\int_{B(x_0,2^{j+1}r)}(|b_1(z)-(b_1)_{B(x_0,r)}||b_2(z)-(b_2)_{B(x_0,r)}||f(z)|)^{s'}dz\right)^{\frac{1}{s'}} \\
\lesssim\ & \sum_{j=1}^{\infty}\|f\|_{L^p(B(x_0,2^{j+1}r))}\|b_1-(b_1)_{B(x_0,r)}\|_{L^{q_1}(B(x_0,2^{j+1}r))} \\
&\times\|b_2-(b_2)_{B(x_0,r)}\|_{L^{q_2}(B(x_0,2^{j+1}r))}|2^{j+1}B|^{-\frac{1}{p}-\frac{1}{q_1}-\frac{1}{q_2}} \\
\lesssim\ & \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.
\end{aligned} \tag{4.8}$$

Therefore from (4.7) and (4.8), we have

$$II_4 \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.$$

So from the estimates of II_1, II_2, II_3 and II_4, it follows that

$$II \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.$$

Therefore from the estimates of I and II, we deduced that

$$\|\mu^{(b_1,b_2)}_{\Omega,S}f\|_{L^q(B(x_0,r))} \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^2\|f\|_{L^p(B(x_0,l))}\frac{dl}{l^{\frac{n}{p}+1-(\lambda_1+\lambda_2)n}}.$$

Thus the proof of Theorem 4.1 is completed .

Theorem 4.2 Let $\Omega \in L^s(\mathbb{S}^{n-1})$ $(1<s\le\infty)$ satisfy condition (1.1) and $\max\{2,s'\}<q<\infty$. Let $1<p,q_1,q_2,\cdots,p_m<\infty$, such that $\frac{1}{q}=\frac{1}{p}+\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}$, and $b\in LC^{\{x_0\}}_{q_i,\lambda_i}$ for $0\le\lambda_i<\frac{1}{n}, i=1,2,\cdots,m$. Then, if functions φ, $\psi:\mathbb{R}^n\times(0,\infty)\to(0,+\infty)$ satisfy the inequality

$$\int_r^{\infty}\left(1+\ln\frac{l}{r}\right)^m\frac{\operatorname*{ess\,inf}_{l<\tau<\infty}\varphi(x_0,\tau)\tau^{\frac{n}{p}}}{l^{\frac{n}{p}+1-n\lambda}}dl\le C\psi(x_0,r),$$

where $\lambda=\sum\limits_{i=1}^{m}\lambda_i$ and the constant $C>0$ doesn't depend on r. Then $\mu_{\Omega,S}^{\vec{b}}$ is bounded from $LM_{p,\varphi}^{\{x_0\}}$ to $LM_{q,\psi}^{\{x_0\}}$.

Proof Taking $v_1(l)=\varphi(x_0,l)^{-1}l^{-\frac{n}{p}}$, $v_2(l)=\psi(x_0,l)^{-1}$, $g(l)=\|f\|_{L^q(B(x_0,l))}$ and $w(l)=(1+\ln\frac{l}{r})^m l^{n\lambda-\frac{n}{p}-1}$. It is easy to see that

$$\operatorname*{ess\,sup}_{l>0} v_2(l)\int_l^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}\limits_{s<\tau<\infty} v_1(\tau)}<\infty.$$

Thus by Lemma 2.2, we have

$$\operatorname*{ess\,sup}_{l>0} v_2(l)H_w g(l)\le C\operatorname*{ess\,sup}_{l>0} v_1(l)g(l).$$

So

$$\begin{aligned}
&\|\mu_{\Omega,S}^{\vec{b}}f\|_{LM_{q,\psi}^{\{x_0\}}}\\
=\;&\sup_{r>0}\psi(x_0,r)^{-1}|B(x_0,r)|^{-\frac{1}{q}}\|\mu_{\Omega,S}^{\vec{b}}f\|_{L^q(B(x_0,r))}\\
\lesssim\;&\prod_{i=1}^{m}\|b_i\|_{LC_{q_i,\lambda_i}^{\{x_0\}}}\sup_{r>0}\psi(x_0,r)^{-1}\int_{2r}^{\infty}\left(1+\ln\frac{l}{r}\right)^m l^{n\lambda-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,l))}dl\\
\lesssim\;&\prod_{i=1}^{m}\|b_i\|_{LC_{q_i,\lambda_i}^{\{x_0\}}}\sup_{r>0}\varphi(x_0,r)^{-1}r^{-\frac{n}{p}}\|f\|_{L^p(B(x_0,r))}\\
=\;&\prod_{i=1}^{m}\|b_i\|_{LC_{q_i,\lambda_i}^{\{x_0\}}}\|f\|_{LM_{p,\varphi}^{\{x_0\}}}.
\end{aligned}$$

Thus the proof of Theorem 4.2 is finished.

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Lusin面积积分与局部Campanato 函数生成的交换子在广义局部Morrey空间的有界性

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摘要: 本文研究了Lusin面积积分及其交换子在广义局部Morrey 空间的有界性. 利用对Lusin面积积分$\mu_{\Omega,S}$的逐点估计及Hardy不等式, 研究了Lusin面积积分$\mu_{\Omega,S}$ 在广义局部Morrey 空间的有界性. 类似地, 还得到了Lusin面积积分$\mu_{\Omega,S}$与局部Campanato 函数生成的交换子在广义局部Morrey 空间的有界性, 推广了已有的结果.

关键词: Lusin面积积分; 交换子; 局部Campanato 函数; 广义局部Morrey 空间

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