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A NEW PATH THEOREM AND A NEW ITERATIVE SCHEME FOR INFINITE M-ACCRETIVE OPERATORS AND COMPUTATIONAL EXPERIMENT

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Abstract: In this paper, we study the iterative construction of the common zero of infinite m-accretive operators. By using the technique of Banach limit and a new path convergence theorem, the iterative sequence is newly constructed proved to be strongly convergent to the common zero of infinite m-accretive operators, which is also the solution of one kind variational inequalities. The computational experiment is conducted by using codes of Visual Basic six to demonstrate the feasibility of the iterative scheme. The restrictions on the iterative parameters are weaker and some new techniques can be found, which extends and complements the corresponding work.

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1 Introduction

In this paper, we assume that E is a real Banach space with E^* being its dual space. "$\to$" denotes strong convergence and $\langle x, f\rangle$ denotes the value of $f \in E^*$ at $x \in E$.

The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx := \{f \in E^* : \langle x, f\rangle = \|x\|^2,\ \|f\| = \|x\|\}, \qquad x \in E.$$

If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset in E [1].

For a mapping $A : D(A) \sqsubseteq E \to E$, we use $\mathrm{Fix}(A)$ and $A^{-1}0$ to denote its fixed point set and the zero point set. That is, $\mathrm{Fix}(A) := \{x \in D(A) : Ax = x\}$ and $A^{-1}0 := \{x \in D(A) : Ax = 0\}$. The mapping $A : D(A) \sqsubseteq E \to E$ is said to be

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Biography: Wei Li (1967–), female, born at Leting, Hebei, professor, major in nonlinear analysis.

(1) nonexpansive if

$$\|Ax - Ay\| \le \|x - y\|, \ \forall x, y \in D(A);$$

(2) contraction with coefficient k if there exists $0 < k < 1$ such that

$$\|Ax - Ay\| \le k\|x - y\|, \ \forall x, y \in D(A);$$

(3) accretive if for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \ge 0;$$

(4) m-accretive if A is accretive and $R(I + \lambda A) = E$, $\forall \lambda > 0$;

(5) strongly positive operator with coefficient $\overline{\gamma} > 0$ [2] if $D(A) = E$, where E is a real smooth Banach space and

$$\langle Ax, Jx\rangle \ge \overline{\gamma}\|x\|^2, \ \forall x \in E.$$

In this case,

$$\|aI - bA\| = \sup_{\|x\| \le 1} |\langle (aI - bA)x, Jx\rangle|,$$

where I is the identity mapping, $a \in [0,1]$ and $b \in [-1,1]$;

(6) demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(A)$ such that $x_n \rightharpoonup x \in D(A)$ and $Ax_n \to p$ then $Ax = p$, here '$\rightharpoonup$' denotes weak convergence in E.

If A is accretive, then we can define, for each $r > 0$, a single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A := (I + rA)^{-1}$, which is called the resolvent of A [1]. It is well known J_r^A is non-expansive and $A^{-1}0 = \mathrm{Fix}(J_r^A)$.

Let C be a nonempty, closed and convex subset of E and Q be a mapping of E onto C. Then Q is said to be sunny [3] if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in E$ and $t \ge 0$.

A mapping Q of E into E is said to be a retraction [3] if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q.

A subset C of E is said to be a sunny nonexpansive retract of E [3] if there exists a sunny nonexpansive retraction of E onto C and it is called a nonexpansive retract of E if there exists a nonexpansive retraction of E onto C.

Finding the solution of the problem $0 \in A_i x$ $(i \in N^+)$ is one of hot topics in applied mathematics, where A_i is accretive, since the solutions correspond to the equilibrium points of some evolution systems. Based on this reason, we shall first prove a new path convergence theorem and then present a new semi-implicit iterative scheme for approximating the common zero of infinite m-accretive operators. Some new ideas and proof techniques can be found based on weaker restrictions than the recent works in [4–7].

Lemma 1.1 [8] Assume F is a strongly positive bounded operator with coefficient $\overline{\gamma} > 0$ on a real smooth Banach space E and $0 < \rho \le \|F\|^{-1}$. Then $\|I - \rho F\| \le 1 - \rho\overline{\gamma}$.

Lemma 1.2 [1] Let E be a Banach space and C be a nonempty closed and convex subset of E. Let $f : C \to C$ be a contraction. Then f has a unique fixed point $u \in C$.

Lemma 1.3 [9] Let E be a real uniformly convex Banach space, C be a nonempty closed and convex subset of E and $B : C \to E$ be a nonexpansive mapping such that $\mathrm{Fix}(B) \neq \emptyset$, then $I - B$ is demiclosed at zero.

Lemma 1.4 [10] Let E be a real strictly convex Banach space and C be its nonempty closed and convex subset. Let $B_m : C \to C$ be a nonexpansive mapping for each $m \geq 1$. Let $\{a_m\}$ be a real number sequence in (0,1) such that $\sum\limits_{m=1}^{\infty} a_m = 1$. Suppose that $\bigcap\limits_{m=1}^{\infty} \mathrm{Fix}(B_m) \neq \emptyset$. Then $\sum\limits_{m=1}^{\infty} a_m B_m$ is nonexpansive with

$$\mathrm{Fix}(\sum_{m=1}^{\infty} a_m B_m) = \bigcap_{m=1}^{\infty} \mathrm{Fix}(B_m).$$

Lemma 1.5 [11] In a real Banach space E, the following inequality holds

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,\ \ \forall x, y \in E, j(x + y) \in J(x + y).$$

Lemma 1.6 [6] Let $r, t > 0$. If E is uniformly convex, then there exists a continuous strictly increasing and convex function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ so that

$$\|J_r^A x - J_r^A y\|^2 \leq \|x - y\|^2 - \varphi(\|(I - J_r^A)x - (I - J_r^A)y\|)$$

for all $x, y \in R(I + rA)$ with $\max\{\|x\|, \|y\|\} \leq t$, where $A : E \to E$ is m-accretive.

Lemma 1.7 [12] Let $\{s_n\}$ be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_k}\}$ so that $s_{n_k} \leq s_{n_k+1}$ for all $k \geq 0$. For every $n > n_0$, define an integer sequence $\{\tau(n)\}$ as

$$\tau(n) = \max\{n_0 \leq k \leq n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$, $\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}$.

2 A New Path Theorem and a New Strong Convergence Theorem

Theorem 2.1 Suppose E is a real uniformly smooth and uniformly convex Banach space, C is a nonempty, closed and convex sunny nonexpansive retract of E, and Q_C is the sunny nonexpansive retraction of E onto C. Let $f_i : E \to E$ be a contractive mapping with coefficient $k \in (0, 1)$ and $F_i : E \to E$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$, where $i \in N^+$. Let $B : C \to C$ be a nonexpansive mapping. Suppose $\{a_n\}$ and $\{b_n\}$ are real number sequences in (0,1) with $\sum\limits_{i=1}^{\infty} a_i = 1$ and $\sum\limits_{i=1}^{\infty} b_i = 1$, $\sum\limits_{i=1}^{\infty} \|f_i\| < +\infty$, $\sum\limits_{i=1}^{\infty} b_i\|F_i\| < +\infty$, $0 < \eta < \frac{\overline{\gamma}}{2k}$ and $\mathrm{Fix}(B) \neq \emptyset$. If $\forall t \in (0, 1)$, define $W_t : E \to E$ by

$$W_t x := t\eta \sum_{i=1}^{\infty} a_i f_i(x) + (I - t\sum_{i=1}^{\infty} b_i F_i)BQ_C x, \tag{2.1}$$

then W_t has a fixed point x_t, for each $0 < t \le (\sum\limits_{i=1}^{\infty} b_i\|F_i\|)^{-1}$, which is convergent strongly to the fixed point of B, as $t \to 0$. That is, $\lim\limits_{t\to 0} x_t = p_0 \in \mathrm{Fix}(B)$. Moreover, p_0 is the unique solution of the following variational inequality: $\forall z \in \mathrm{Fix}(B)$,

$$\langle(\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i)p_0, J(p_0 - z)\rangle \le 0. \tag{2.2}$$

Proof **Step 1** $\sum\limits_{i=1}^{\infty} b_i F_i : E \to E$ is a strongly positive linear bounded operator with coefficient $\overline{\gamma}$.

It is easy to check $\sum\limits_{i=1}^{\infty} b_i F_i : E \to E$ is linear bounded, so we are left to show that $\sum\limits_{i=1}^{\infty} b_i F_i : E \to E$ is strongly positive with coefficient $\overline{\gamma}$.

It follows from the property of F_i that

$$\langle\sum_{i=1}^{\infty} b_i F_i x, Jx\rangle = \sum_{i=1}^{\infty} b_i \langle F_i x, Jx\rangle \ge \sum_{i=1}^{\infty} b_i \overline{\gamma}\|x\|^2 = \overline{\gamma}\|x\|^2.$$

Thus $\sum\limits_{i=1}^{\infty} b_i F_i : E \to E$ is a strongly positive operator with coefficient $\overline{\gamma}$.

Step 2 W_t is a contraction for $0 < t < (\sum\limits_{i=1}^{\infty} b_i\|F_i\|)^{-1}$.

In fact, noticing Lemma 1.1, $\sum\limits_{i=1}^{\infty} a_i = 1$ and $\sum\limits_{i=1}^{\infty} b_i = 1$, we have

$$\begin{aligned}\|W_t x - W_t y\| &\le t\eta \sum_{i=1}^{\infty} a_i \|f_i(x) - f_i(y)\| + \|(I - t\sum_{i=1}^{\infty} b_i F_i)(BQ_C x - BQ_C y)\| \\ &\le kt\eta\|x - y\| + (1 - t\overline{\gamma})\|x - y\| = [1 - t(\overline{\gamma} - k\eta)]\|x - y\|,\end{aligned}$$

which implies that W_t is a contraction since $0 < \eta < \frac{\overline{\gamma}}{2k}$. Then Lemma 1.2 implies that W_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation $x_t = t\eta \sum\limits_{i=1}^{\infty} a_i f_i(x_t) + (I - t\sum\limits_{i=1}^{\infty} b_i F_i)BQ_C x_t$.

Step 3 $\{x_t\}$ is bounded for $t \in (0, (\sum\limits_{i=1}^{\infty} b_i\|F_i\|)^{-1})$. For $p \in \mathrm{Fix}(B) \subset C$, we have $p = BQ_C p$, then

$$\begin{aligned}\|x_t - p\| &= \|(I - t\sum_{i=1}^{\infty} b_i F_i)(BQ_C x_t - p) + t(\eta \sum_{i=1}^{\infty} a_i f_i(x_t) - \sum_{i=1}^{\infty} b_i F_i p)\| \\ &\le (1 - t\overline{\gamma})\|x_t - p\| + t\|\eta \sum_{i=1}^{\infty} a_i (f_i(x_t) - f_i(p))\| + t\|\eta \sum_{i=1}^{\infty} a_i f_i(p) - \sum_{i=1}^{\infty} b_i F_i p\| \\ &\le [1 - t(\overline{\gamma} - k\eta)]\|x_t - p\| + t\|\eta \sum_{i=1}^{\infty} a_i f_i(p) - \sum_{i=1}^{\infty} b_i F_i p\|.\end{aligned}$$

This ensures that

$$\|x_t - p\| \le \frac{(\eta \sum_{i=1}^{\infty} \|f_i\| + \sum_{i=1}^{\infty} \|F_i\|)\|p\|}{\overline{\gamma} - k\eta}.$$

Thus $\{x_t\}$ is bounded, and then both $\{\sum_{i=1}^{\infty} a_i f_i(x_t)\}$ and $\{\sum_{i=1}^{\infty} b_i F_i BQ_C x_t\}$ are bounded.

Step 4 $x_t - BQ_C x_t \to 0$, as $t \to 0$. From Step 3, we have

$$\|x_t - BQ_C x_t\| = t\|\eta \sum_{i=1}^{\infty} a_i f_i(x_t) - \sum_{i=1}^{\infty} b_i F_i BQ_C x_t\| \to 0,$$

as $t \to 0$.

Step 5 If variational inequality (2.2) has solutions, the solution must be unique.

Suppose both $u_0 \in \mathrm{Fix}(B)$ and $v_0 \in \mathrm{Fix}(B)$ are the solutions of the variational inequality(2.2). Then we have

$$\langle (\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i) v_0, J(v_0 - u_0)\rangle \le 0 \tag{2.3}$$

and

$$\langle (\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i) u_0, J(u_0 - v_0)\rangle \le 0. \tag{2.4}$$

Adding up (2.3) and (2.4), we obtain that

$$\langle (\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i) u_0 - (\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i) v_0, J(u_0 - v_0)\rangle \le 0. \tag{2.5}$$

Since

$$\begin{aligned}
&\langle (\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i) u_0 - (\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i) v_0, J(u_0 - v_0)\rangle \\
=& \sum_{i=1}^{\infty} b_i \langle F_i u_0 - F_i v_0, J(u_0 - v_0)\rangle - \eta \sum_{i=1}^{\infty} a_i \langle f_i(u_0) - f_i(v_0), J(u_0 - v_0)\rangle \\
\ge& \overline{\gamma}\|u_0 - v_0\|^2 - k\eta\|u_0 - v_0\|^2 = (\overline{\gamma} - k\eta)\|u_0 - v_0\|^2,
\end{aligned}$$

then (2.5) implies that $u_0 = v_0$.

Step 6 $x_t \to p_0 \in \mathrm{Fix}(B)$, as $t \to 0$, which satisfies the variational inequality (2.2).

Assume $t_n \to 0$. Set $x_n := x_{t_n}$ and defined $\mu : E \to \mathbb{R}$ by $\mu(x) = LIM\|x_n - x\|^2,\ x \in E$, where LIM is the Banach limit on l^∞. Let

$$K = \{x \in E : \mu(x) = \min_{x \in E} LIM\|x_n - x\|^2\}.$$

It is easily seen that K is a nonempty closed convex bounded subset of E. Since $x_n - BQ_C x_n \to 0$, then for $x \in K$,

$$\mu(BQ_C x) = LIM\|x_n - BQ_C x\|^2 \le LIM\|x_n - x\|^2 = \mu(x),$$

it follows that $BQ_C(K)\subset K$; that is, K is invariant under BQ_C. Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, BQ_C has a fixed point, say p_0, in K. That is, $BQ_Cp_0=p_0\in C$ which ensures that $p_0=Bp_0$ from the definition of B and then $p_0\in \mathrm{Fix}(B)$. Since p_0 is also a minimizer of μ over E, it follows that, for $t\in(0,1)$,

$$\begin{aligned}
0\leq&\frac{\mu(p_0+\eta t\sum\limits_{i=1}^{\infty}a_if_i(p_0)-t\sum\limits_{i=1}^{\infty}b_iF_ip_0)-\mu(p_0)}{t}\\
=&LIM\frac{\|x_n-p_0-\eta t\sum\limits_{i=1}^{\infty}a_if_i(p_0)+t\sum\limits_{i=1}^{\infty}b_iF_ip_0\|^2-\|x_n-p_0\|^2}{t}\\
=&LIM\frac{\langle x_n-p_0-\eta t\sum\limits_{i=1}^{\infty}a_if_i(p_0)+t\sum\limits_{i=1}^{\infty}b_iF_ip_0, J(x_n-p_0-\eta t\sum\limits_{i=1}^{\infty}a_if_i(p_0)+t\sum\limits_{i=1}^{\infty}b_iF_ip_0)\rangle-\|x_n-p_0\|^2}{t}\\
=&LIM\{\frac{\langle x_n-p_0, J(x_n-p_0-\eta t\sum\limits_{i=1}^{\infty}a_if_i(p_0)+t\sum\limits_{i=1}^{\infty}b_iF_ip_0)\rangle-\|x_n-p_0\|^2}{t}\\
&+\frac{t\langle\sum\limits_{i=1}^{\infty}b_iF_ip_0-\eta\sum\limits_{i=1}^{\infty}a_if_i(p_0), J(x_n-p_0-\eta t\sum\limits_{i=1}^{\infty}a_if_i(p_0)+t\sum\limits_{i=1}^{\infty}b_iF_ip_0)\rangle}{t}\}.
\end{aligned}$$

Since E is uniformly smooth, then by letting $t\to 0$, we find the two limits above can be interchanged and obtain

$$LIM\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0, J(x_n-p_0)\rangle\leq 0. \tag{2.6}$$

Since $x_t-p_0=t(\eta\sum\limits_{i=1}^{\infty}a_if_i(x_t)-\sum\limits_{i=1}^{\infty}b_iF_ip_0)+(I-t\sum\limits_{i=1}^{\infty}b_iF_i)(BQ_Cx_t-p_0)$, then

$$\begin{aligned}
&\|x_t-p_0\|^2\\
=&t\langle\eta\sum_{i=1}^{\infty}a_if_i(x_t)-\sum_{i=1}^{\infty}b_iF_ip_0, J(x_t-p_0)\rangle+\langle(I-t\sum_{i=1}^{\infty}b_iF_i)(BQ_Cx_t-p_0), J(x_t-p_0)\rangle\\
\leq&t\eta\langle\sum_{i=1}^{\infty}a_if_i(x_t)-\sum_{i=1}^{\infty}a_if_i(p_0), J(x_t-p_0)\rangle+t\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0, J(x_t-p_0)\rangle\\
&+\|I-t\sum_{i=1}^{\infty}b_iF_i\|\|x_t-p_0\|^2\\
\leq&[1-t(\overline{\gamma}-\eta k)]\|x_t-p_0\|^2+t\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0, J(x_t-p_0)\rangle.
\end{aligned}$$

Therefore

$$\|x_t-p_0\|^2\leq\frac{1}{\overline{\gamma}-\eta k}\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0, J(x_t-p_0)\rangle.$$

Hence by (2.6),

$$LIM\|x_n-p_0\|^2\leq\frac{1}{\overline{\gamma}-\eta k}LIM\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0, J(x_n-p_0)\rangle\leq 0,$$

which implies that $LIM\|x_n - p_0\|^2 = 0$, and then there exists a subsequence which is still denoted by $\{x_n\}$ such that $x_n \to p_0$.

Next, we shall show that p_0 solves the variational inequality (2.2).

Since $x_t = t\eta \sum\limits_{i=1}^{\infty} a_i f_i(x_t) + (I - t\sum\limits_{i=1}^{\infty} b_i F_i)BQ_C x_t$, then

$$(\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i)x_t = -\frac{1}{t}(I - t\sum_{i=1}^{\infty} b_i F_i)(I - BQ_C)x_t.$$

$\forall z \in \mathrm{Fix}(B)$, we have

$$\begin{aligned}
&\langle(\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i)x_t, J(x_t - z)\rangle = -\frac{1}{t}\langle(I - t\sum_{i=1}^{\infty} b_i F_i)(I - BQ_C)x_t, J(x_t - z)\rangle \\
=& -\frac{1}{t}\langle(I - BQ_C)x_t - (I - BQ_C)z, J(x_t - z)\rangle + \langle\sum_{i=1}^{\infty} b_i F_i(I - BQ_C)x_t, J(x_t - z)\rangle \\
=& -\frac{1}{t}[\|x_t - z\|^2 - \langle BQ_C x_t - BQ_C z, J(x_t - z)\rangle] + \langle\sum_{i=1}^{\infty} b_i F_i(I - BQ_C)x_t, J(x_t - z)\rangle \\
\le& \langle\sum_{i=1}^{\infty} b_i F_i(I - BQ_C)x_t, J(x_t - z)\rangle.
\end{aligned}$$

Taking the limits on both sides of the above inequality,

$$\langle(\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i)p_0, J(p_0 - z)\rangle \le 0$$

since $x_n \to p_0$ and J is uniformly continuous on each bounded subsets of E.

Thus p_0 satisfies the variational inequality (2.2).

Now assume there exists another subsequence $\{x_m\}$ of $\{x_t\}$ satisfying $x_m \to q_0$. Then Step 4 implies that $BQ_C x_m \to q_0$. From Lemma 1.3, we know that $I - BQ_C$ is demiclosed at zero, then $q_0 = BQ_C q_0$ which ensures that $q_0 \in \mathrm{Fix}(B)$. Repeating the above proof, we can also know that q_0 solves variational inequality (2.2). Thus $p_0 = q_0$ in view of Step 5.

Hence $x_t \to p_0$, as $t \to 0$, which is the unique solution of the variational inequality (2.2).

This completes the proof.

Theorem 2.2 Let E, C, Q_C, f_i, F_i, k and $\overline{\gamma}$ be the same as those in Theorem 2.1, and let $A_i : C \to C$ be m-accretive operator for $i \in N^+$. Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\beta_n\}$, $\{\zeta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real number sequences in (0,1) with $\sum\limits_{i=1}^{\infty} a_i = \sum\limits_{i=1}^{\infty} b_i = \sum\limits_{i=1}^{\infty} c_i = 1$. Let $\{r_{n,i}\} \subset (0, +\infty)$ for $i \in N^+$, $\{\varepsilon'_n\} \subset E$ and $\{\varepsilon''_n\} \subset C$. Suppose

$$D := \bigcap_{i=1}^{\infty} A_i^{-1}0 \neq \emptyset, \ \ \sum_{i=1}^{\infty} \|f_i\| < +\infty, \ \ \sum_{i=1}^{\infty} b_i\|F_i\| < +\infty$$

and $0 < \eta < \frac{\overline{\gamma}}{2k}$.

Let $\{x_n\}$ be generated by the following iterative scheme

$$\begin{cases} x_0 \in E, \\ y_n = Q_C[(1-\alpha_n)(x_n + \varepsilon'_n)], \\ z_n = \beta_n y_n + \gamma_n \sum\limits_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i}(\dfrac{y_n + z_n}{2}) + \delta_n \varepsilon''_n, \\ x_{n+1} = \zeta_n \eta \sum\limits_{i=1}^{\infty} a_i f_i(x_n) + (I - \zeta_n \sum\limits_{i=1}^{\infty} b_i F_i) \sum\limits_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} z_n, \quad n \geq 0. \end{cases} \tag{A}$$

Then $\{x_n\}$ converges strongly to the unique element $p_0 \in D$, which satisfies the following variational inequality: $\forall z \in D$,

$$\langle (\sum_{i=1}^{\infty} b_i F_i - \eta \sum_{i=1}^{\infty} a_i f_i) p_0, J(p_0 - z) \rangle \leq 0 \tag{2.7}$$

under the assumptions that

(i) $\sum\limits_{n=0}^{\infty} \alpha_n < +\infty$, $\sum\limits_{n=0}^{\infty} \delta_n < +\infty$, $\sum\limits_{n=0}^{\infty} \|\varepsilon'_n\| < +\infty$, $\sum\limits_{n=0}^{\infty} \|\varepsilon''_n\| < +\infty$;

(ii) $\sum\limits_{n=0}^{\infty} \zeta_n = +\infty$ and $\zeta_n \to 0$ as $n \to \infty$;

(iii) $\frac{\alpha_n}{\zeta_n} \to 0$, $\frac{\delta_n}{\zeta_n} \to 0, \frac{\|\varepsilon'_n\|}{\zeta_n} \to 0$;

(iv) $\delta_n + \beta_n + \gamma_n \equiv 1$, for $n \geq 0$.

Proof We shall split the proof into five steps.

Step 1 $\{x_n\}$ is well-defined.

In fact, it suffices to show that $\{z_n\}$ is well-defined.

For $t, s \in (0,1)$, define $U_{t,s} : C \to C$ by $U_{t,s} x := tu + sB(\frac{u+x}{2}) + (1-t-s)v$, where $B : C \to C$ is nonexpansive and $x, u, v \in C$. Then

$$\|U_{t,s}x - U_{t,s}y\| \leq s\|\frac{u+x}{2} - \frac{u+y}{2}\| \leq \frac{s}{2}\|x - y\|.$$

Thus $U_{t,s}$ is a contraction, which ensures from Lemma 1.2 that there exists $x_{t,s} \in C$ such that $U_{t,s}x_{t,s} = x_{t,s}$. That is, $x_{t,s} = tu + sB(\frac{u+x_{t,s}}{2}) + (1-t-s)v$.

Since $J_{r_{n,i}}^{A_i}$ is nonexpansive and $\sum\limits_{i=1}^{\infty} c_i = 1$, then $\sum\limits_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i}$ is nonexpansive, which implies that $\{z_n\}$ is well-defined, and then $\{x_n\}$ is well-defined.

Step 2 $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all bounded.

$\forall p \in D$, we see that for $n \geq 0$,

$$\|y_n - p\| \leq (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\|\varepsilon'_n\| + \alpha_n\|p\|. \tag{2.8}$$

Therefore for $p \in D$ and $n \geq 0$, we have

$$\begin{aligned} \|z_n - p\| &\leq \beta_n\|y_n - p\| + \gamma_n\|\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i}(\frac{y_n + z_n}{2}) - p\| + \delta_n\|\varepsilon''_n - p\| \\ &\leq (\beta_n + \frac{\gamma_n}{2})\|y_n - p\| + \frac{\gamma_n}{2}\|z_n - p\| + \delta_n\|\varepsilon''_n - p\| \\ &\leq (1 - \frac{\gamma_n}{2})\|y_n - p\| + \frac{\gamma_n}{2}\|z_n - p\| + \delta_n\|\varepsilon''_n - p\|, \end{aligned}$$

which implies that

$$\|z_n - p\| \leq \|y_n - p\| + \frac{2\delta_n}{2-\gamma_n}\|\varepsilon''_n - p\| \leq \|y_n - p\| + 2\|\varepsilon''_n\| + \frac{2\delta_n}{2-\gamma_n}\|p\|. \tag{2.9}$$

Using Lemma 1.1, (2.8) and (2.9), we have for $n \geq 0$,

$$\begin{aligned}
&\|x_{n+1} - p\| \\
\leq&\zeta_n\eta\sum_{i=1}^{\infty} a_i\|f_i(x_n) - f_i(p)\| + \zeta_n\|\eta\sum_{i=1}^{\infty} a_i f_i(p) - \sum_{i=1}^{\infty} b_i F_i(p)\| \\
&+ \|I - \zeta_n\sum_{i=1}^{\infty} b_i F_i\|\|z_n - p\| \\
\leq&\zeta_n\eta k\|x_n - p\| + \zeta_n\|\eta\sum_{i=1}^{\infty} a_i f_i(p) - \sum_{i=1}^{\infty} b_i F_i(p)\| + (1-\zeta_n\overline{\gamma})\|z_n - p\| \\
\leq&[1-\zeta_n(\overline{\gamma} - k\eta)]\|x_n - p\| + \zeta_n\|\eta\sum_{i=1}^{\infty} a_i f_i(p) - \sum_{i=1}^{\infty} b_i F_i(p)\| \\
&+ \|\varepsilon'_n\| + 2\|\varepsilon''_n\| + \|p\|(\alpha_n + \frac{2\delta_n}{2-\gamma_n}).
\end{aligned} \tag{2.10}$$

By using the inductive method, we can easily get the following result from (2.10)

$$\begin{aligned}
\|x_{n+1} - p\| \leq& \max\{\|x_0 - p\|, \frac{\|\eta\sum\limits_{i=1}^{\infty} a_i f_i(p) - \sum\limits_{i=1}^{\infty} b_i F_i(p)\|}{\overline{\gamma} - k\eta}\} \\
&+ \sum_{k=0}^{n}\|\varepsilon'_k\| + 2\sum_{k=0}^{n}\|\varepsilon''_k\| + \|p\|(\sum_{k=0}^{n}\alpha_k + \sum_{k=0}^{n}\frac{2\delta_k}{2-\gamma_k}).
\end{aligned}$$

Therefore from assumption (i), we know that $\{x_n\}$ is bounded. Set $M = \sup\{\|x_n\|, \|\varepsilon'_n\| : n \geq 0\}$. Then M is a positive constant.

Step 3 $\mathrm{Fix}(\sum\limits_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i}) = D$.

It is obvious that $A_i^{-1}0 = \mathrm{Fix}(J_{r_{n,i}}^{A_i})$ for $i \in N^+$. Lemma 1.4 implies that $\bigcap\limits_{i=1}^{\infty}\mathrm{Fix}(J_{r_{n,i}}^{A_i}) = \mathrm{Fix}(\sum\limits_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i})$. Thus $\mathrm{Fix}(\sum\limits_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i}) = D$.

Step 4

$$\|x_{n+1} - p_0\|^2 \leq (1-\omega_n^{(1)})\|x_n - p_0\|^2 + \omega_n^{(1)}\omega_n^{(2)} - \omega_n^{(3)}, \tag{2.11}$$

where p_0 is the unique solution of variational inequality (2.7), $M' = 4M + 2\|p_0\|$,

$$\begin{aligned}
\omega_n^{(1)} &= \frac{\zeta_n(\overline{\gamma} - 2\eta k)}{1 - \zeta_n \eta k},\\
\omega_n^{(2)} &= \frac{1}{\zeta_n(\overline{\gamma} - 2\eta k)}(\|\varepsilon_n'\|M' + \alpha_n M'\|p_0\| + \frac{2\delta_n}{2-\gamma_n}\|\varepsilon_n'' - p_0\|^2)\\
&\quad + \frac{2\langle \eta \sum\limits_{i=1}^{\infty} a_i f_i(p_0) - \sum\limits_{i=1}^{\infty} b_i F_i(p_0), J(x_{n+1} - p_0)\rangle}{\overline{\gamma} - 2\eta k},\\
\omega_n^{(3)} &= \frac{(1 - \zeta_n\overline{\gamma})^2}{1 - \zeta_n \eta k}\frac{2\gamma_n}{2-\gamma_n}\sum_{i=1}^{\infty} c_i \varphi(\|\frac{y_n + z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n + z_n}{2})\|).
\end{aligned}$$

Since $\sum\limits_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} : C \to C$ is nonexpansive, then using Theorem 2.1, we know that there exists z_t such that

$$z_t = t\eta \sum_{i=1}^{\infty} a_i f_i(z_t) + (I - \sum_{i=1}^{\infty} b_i F_i)\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t \tag{2.12}$$

for $t \in (0, (\sum\limits_{i=1}^{\infty} b_i \|F_i\|)^{-1})$. Moreover, $z_t \to p_0 \in D$, as $t \to 0$, which is the unique solution of the variational inequality (2.7).

For this $p_0 \in D$, using Lemma 1.5, we have

$$\begin{aligned}
&\|y_n - p_0\|^2 \le \|(1-\alpha_n)(x_n + \varepsilon_n') - p_0\|^2\\
\le&(1-\alpha_n)^2\|x_n - p_0\|^2 + 2\langle (1-\alpha_n)\varepsilon_n' - \alpha_n p_0, J[(1-\alpha_n)(x_n + \varepsilon_n') - p_0]\rangle\\
\le&(1-\alpha_n)^2\|x_n - p_0\|^2 + \|\varepsilon_n'\|M' + \alpha_n\|p_0\|M'.
\end{aligned} \tag{2.13}$$

And using Lemma 1.6,

$$\begin{aligned}
&\|z_n - p_0\|^2 \le \beta_n\|y_n - p_0\|^2 + \gamma_n \sum_{i=1}^{\infty} c_i \|J_{r_{n,i}}^{A_i}(\frac{y_n + z_n}{2}) - p_0\|^2 + \delta_n\|\varepsilon_n'' - p_0\|^2\\
\le&\beta_n\|y_n - p_0\|^2 + \gamma_n \sum_{i=1}^{\infty} c_i[\|\frac{y_n + z_n}{2} - p_0\|^2 - \varphi(\|(I - J_{r_{n,i}}^{A_i})(\frac{y_n + z_n}{2})\|)] + \delta_n\|\varepsilon_n'' - p_0\|^2\\
\le&(\beta_n + \frac{\gamma_n}{2})\|y_n - p_0\|^2 + \delta_n\|\varepsilon_n'' - p_0\|^2 - \gamma_n \sum_{i=1}^{\infty} c_i \varphi(\|(I - J_{r_{n,i}}^{A_i})(\frac{y_n + z_n}{2})\|) + \frac{\gamma_n}{2}\|z_n - p_0\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z_n - p_0\|^2 \le&\frac{2\beta_n + \gamma_n}{2 - \gamma_n}\|y_n - p_0\|^2\\
&+ \frac{2\delta_n}{2-\gamma_n}\|\varepsilon_n'' - p_0\|^2 - \frac{2\gamma_n}{2-\gamma_n}\sum_{i=1}^{\infty} c_i \varphi(\|\frac{y_n + z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n + z_n}{2})\|)\\
\le&(1-\alpha_n)^2\|x_n - p_0\|^2 + \|\varepsilon_n'\|M' + \alpha_n\|p_0\|M' + \frac{2\delta_n}{2-\gamma_n}\|\varepsilon_n'' - p_0\|^2\\
&- \frac{2\gamma_n}{2-\gamma_n}\sum_{i=1}^{\infty} c_i \varphi(\|\frac{y_n + z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n + z_n}{2})\|).
\end{aligned} \tag{2.14}$$

Now, noticing Step 1 in Theorem 2.1 and using Lemma 1.5 again,

$$\begin{aligned}
&\|x_{n+1}-p_0\|^2\\
=&\|(I-\zeta_n\sum_{i=1}^{\infty}b_iF_i)\sum_{i=1}^{\infty}c_iJ_{r_{n,i}}^{A_i}(z_n-p_0)+\zeta_n(\eta\sum_{i=1}^{\infty}a_if_i(x_n)-\sum_{i=1}^{\infty}b_iF_ip_0)\|^2\\
\le&(1-\zeta_n\overline{\gamma})^2\|z_n-p_0\|^2+2\zeta_n\langle\eta\sum_{i=1}^{\infty}a_if_i(x_n)-\sum_{i=1}^{\infty}b_iF_ip_0,J(x_{n+1}-p_0)\rangle\\
\le&(1-\zeta_n\overline{\gamma})\|x_n-p_0\|^2+\|\varepsilon_n'\|M'+\alpha_n\|p_0\|M'\\
&-(1-\zeta_n\overline{\gamma})^2\frac{2\gamma_n}{2-\gamma_n}\sum_{i=1}^{\infty}c_i\varphi(\|\frac{y_n+z_n}{2}-J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|)\\
&+\frac{2\delta_n}{2-\gamma_n}\|\varepsilon_n''-p_0\|^2+2\zeta_n\langle\eta\sum_{i=1}^{\infty}a_i(f_i(x_n)-f_i(p_0)),J(x_{n+1}-p_0)\rangle\\
&+2\zeta_n\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0,J(x_{n+1}-p_0)\rangle\\
\le&(1-\zeta_n\overline{\gamma})\|x_n-p_0\|^2+\|\varepsilon_n'\|M'+\alpha_n\|p_0\|M'\\
&-(1-\zeta_n\overline{\gamma})^2\frac{2\gamma_n}{2-\gamma_n}\sum_{i=1}^{\infty}c_i\varphi(\|\frac{y_n+z_n}{2}-J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|)\\
&+\frac{2\delta_n}{2-\gamma_n}\|\varepsilon_n''-p_0\|^2+2\zeta_n\eta k\|x_n-p_0\|\|x_{n+1}-p_0\|\\
&+2\zeta_n\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0,J(x_{n+1}-p_0)\rangle\\
\le&(1-\zeta_n\overline{\gamma})\|x_n-p_0\|^2+\|\varepsilon_n'\|M'+\alpha_n\|p_0\|M'\\
&-(1-\zeta_n\overline{\gamma})^2\frac{2\gamma_n}{2-\gamma_n}\sum_{i=1}^{\infty}c_i\varphi(\|\frac{y_n+z_n}{2}-J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|)\\
&+\frac{2\delta_n}{2-\gamma_n}\|\varepsilon_n''-p_0\|^2+\zeta_n\eta k(\|x_n-p_0\|^2+\|x_{n+1}-p_0\|^2)\\
&+2\zeta_n\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0,J(x_{n+1}-p_0)\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|x_{n+1}-p_0\|^2\\
\le&\frac{1-\zeta_n\overline{\gamma}+\zeta_n\eta k}{1-\zeta_n\eta k}\|x_n-p_0\|^2+\frac{1}{1-\zeta_n\eta k}(\|\varepsilon_n'\|M'+\alpha_n\|p_0\|M'+\frac{2\delta_n}{2-\gamma_n}\|\varepsilon_n''-p_0\|^2)\\
&+\frac{2\zeta_n}{1-\zeta_n\eta k}\langle\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_ip_0,J(x_{n+1}-p_0)\rangle\\
&-\frac{(1-\zeta_n\overline{\gamma})^2}{1-\zeta_n\eta k}\frac{2\gamma_n}{2-\gamma_n}\sum_{i=1}^{\infty}c_i\varphi(\|\frac{y_n+z_n}{2}-J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|)
\end{aligned}$$

$$=(1-\omega_n^{(1)})\|x_n-p_0\|^2+\omega_n^{(1)}\omega_n^{(2)}-\omega_n^{(3)}.$$

It follows from assumption(ii) that $\omega_n^{(1)}\to 0$, as $n\to+\infty$.

Step 5 $x_n\to p_0$, as $n\to+\infty$, where p_0 is the same as that in Step 4.

Our next discussion will be divided into two cases:

Case 1 $\{\|x_n-p_0\|\}$ is decreasing.

If $\{\|x_n-p_0\|\}$ is decreasing, we know from the result of Step 4 that

$$0\le\omega_n^{(3)}\le\omega_n^{(1)}(\omega_n^{(2)}-\|x_n-p_0\|^2)+(\|x_n-p_0\|^2-\|x_{n+1}-p_0\|^2)\to 0,$$

which ensures that

$$\sum_{i=1}^{\infty}c_i\varphi(\|\frac{y_n+z_n}{2}-J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|)\to 0$$

as $n\to+\infty$. Then from the property of φ, we know that

$$\sum_{i=1}^{\infty}c_i\|\frac{y_n+z_n}{2}-J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|\to 0$$

as $n\to+\infty$. Since

$$\|y_n-z_n\|\le\gamma_n\sum_{i=1}^{\infty}c_i\|J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})-y_n\|+\delta_n\|\varepsilon_n''-y_n\|$$

$$\le\gamma_n\sum_{i=1}^{\infty}c_i\|J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})-\frac{y_n+z_n}{2}\|+\gamma_n\|\frac{y_n+z_n}{2}-y_n\|+\delta_n\|\varepsilon_n''-y_n\|,$$

then

$$\|y_n-z_n\|\le\frac{2}{2-\gamma_n}[\gamma_n\sum_{i=1}^{\infty}c_i\|J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})-\frac{y_n+z_n}{2}\|+\delta_n\|\varepsilon_n''-y_n\|]\to 0$$

as $n\to+\infty$. Therefore

$$\|y_n-\sum_{i=1}^{\infty}c_iJ_{r_{n,i}}^{A_i}y_n\|$$

$$\le\|y_n-z_n\|+\|z_n-\sum_{i=1}^{\infty}c_iJ_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|+\|\sum_{i=1}^{\infty}c_iJ_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})-\sum_{i=1}^{\infty}c_iJ_{r_{n,i}}^{A_i}y_n\|$$

$$\le\frac{3}{2}\|y_n-z_n\|+\beta_n\|y_n-\sum_{i=1}^{\infty}c_iJ_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|+\delta_n\|\varepsilon_n''-\sum_{i=1}^{\infty}c_iJ_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|\to 0.$$

Next, we shall show that

$$\limsup_{n\to+\infty}\langle\eta\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}b_iF_i(p_0),J(x_{n+1}-p_0)\rangle\le 0. \tag{2.15}$$

Let z_t be the same as that in (2.12). Since $\|z_t\| \leq \|z_t - p_0\| + \|p_0\|$, then $\{z_t\}$ is bounded, as $t \to 0$. Using Lemma 1.5, we have

$$
\begin{aligned}
&\|z_t - y_n\|^2 = \|z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n + \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n - y_n\|^2 \\
\leq &\|z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n\|^2 + 2\langle \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n - y_n, J(z_t - y_n)\rangle \\
= &\|t\eta \sum_{i=1}^{\infty} a_i f_i(z_t) + (I - t\sum_{i=1}^{\infty} b_i F_i) \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n\|^2 \\
&+ 2\langle \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n - y_n, J(z_t - y_n)\rangle \\
\leq &\| \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n\|^2 \\
&+ 2t\langle \eta \sum_{i=1}^{\infty} a_i f_i(z_t) - \sum_{i=1}^{\infty} b_i F_i(\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t), J(z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle \\
&+ 2\langle \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n - y_n, J(z_t - y_n)\rangle \\
\leq &\|z_t - y_n\|^2 + 2t\langle \eta \sum_{i=1}^{\infty} a_i f_i(z_t) - \sum_{i=1}^{\infty} b_i F_i(\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t), J(z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle \\
&+ 2\| \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n - y_n\| \|z_t - y_n\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&t\langle \sum_{i=1}^{\infty} b_i F_i(\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t) - \eta \sum_{i=1}^{\infty} a_i f_i(z_t), J(z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle \\
\leq &\| \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n - y_n\| \|z_t - y_n\|.
\end{aligned}
$$

So

$$
\lim_{t\to 0} \limsup_{n\to+\infty} \langle \sum_{i=1}^{\infty} b_i F_i(\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t) - \eta \sum_{i=1}^{\infty} a_i f_i(z_t), J(z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle \leq 0.
$$

Since $z_t \to p_0$, then

$$
\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t \to \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C p_0 = p_0
$$

as $t \to 0$. Noticing that

$$
\begin{aligned}
&\langle\sum_{i=1}^{\infty} b_i F_i(p_0) - \eta\sum_{i=1}^{\infty} a_i f_i(p_0), J(p_0 - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle \\
=&\langle\sum_{i=1}^{\infty} b_i F_i(\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C p_0) - \eta\sum_{i=1}^{\infty} a_i f_i(p_0), J(p_0 - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n) - J(z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle \\
&+ \langle\sum_{i=1}^{\infty} b_i F_i(\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C p_0) - \eta\sum_{i=1}^{\infty} a_i f_i(p_0) - \sum_{i=1}^{\infty} b_i F_i(\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t) \\
&+ \eta\sum_{i=1}^{\infty} a_i f_i(z_t), J(z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle \\
&+ \langle\sum_{i=1}^{\infty} b_i F_i(\sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} Q_C z_t) - \eta\sum_{i=1}^{\infty} a_i f_i(z_t), J(z_t - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle,
\end{aligned}
$$

then we have

$$\limsup_{n\to+\infty}\langle\sum_{i=1}^{\infty} b_i F_i(p_0) - \eta\sum_{i=1}^{\infty} a_i f_i(p_0), J(p_0 - \sum_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n)\rangle \le 0.$$

Since $y_n - z_n \to 0$, then $x_{n+1} - \sum\limits_{i=1}^{\infty} c_i J_{r_{n,i}}^{A_i} y_n \to 0$, which implies that

$$\limsup_{n\to+\infty}\langle\sum_{i=1}^{\infty} b_i F_i(p_0) - \eta\sum_{i=1}^{\infty} a_i f_i(p_0), J(p_0 - x_{n+1})\rangle \le 0.$$

Assumption (ii) and (2.15) ensure that $\limsup\limits_{n\to\infty}\omega_n^{(2)} \le 0$.

Employing (2.11) again, we have

$$\|x_n - p_0\|^2 \le \frac{\|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2}{\omega_n^{(1)}} + \omega_n^{(2)}.$$

Assumption (ii) implies that $\liminf\limits_{n\to\infty}\frac{\|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2}{\omega_n^{(1)}} = 0$. Then

$$\lim_{n\to\infty}\|x_n - p_0\|^2 \le \liminf_{n\to\infty}\frac{\|x_n - p_0\|^2 - \|x_{n+1} - p_0\|^2}{\omega_n^{(1)}} + \limsup_{n\to\infty}\omega_n^{(2)} \le 0.$$

Then the result that $x_n \to p_0$ follows.

Case 2 If $\{\|x_n - p_0\|\}$ is not eventually decreasing, then we can find a subsequence $\{\|x_{n_m} - p_0\|\}$ so that $\|x_{n_m} - p_0\| \le \|x_{n_{m+1}} - p_0\|$ for all $m \ge 1$. From Lemma 1.7, we can define a subsequence $\{\|x_{\tau(n)} - p_0\|\}$ so that $\max\{\|x_{\tau(n)} - p_0\|, \|x_n - p_0\|\} \le \|x_{\tau(n)+1} - p_0\|$ for all $n > n_1$. This enable us to deduce that (similar to Case 1)

$$0 \le \omega_{\tau(n)}^{(3)} \le \omega_{\tau(n)}^{(1)}(\omega_{\tau(n)}^{(2)} - \|x_{\tau(n)} - p_0\|^2) + (\|x_{\tau(n)} - p_0\|^2 - \|x_{\tau(n)+1} - p_0\|^2) \to 0,$$

and then copy Case 1, we have $\lim_{n\to\infty}\|x_{\tau(n)}-p_0\|=0$. Thus $0\le\|x_n-p_0\|\le\|x_{\tau(n)+1}-p_0\|\to 0$, as $n\to\infty$. This completes the proof.

Remark 2.1 In similar studies, e.g. [5], they usually have the following strong restrictions on the parameters: $\sum_{i=1}^{\infty}|r_{n+1,i}-r_{n,i}|<+\infty$, $r_{n,i}\ge\varepsilon>0$, $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$ and $\gamma_{n+1}-\gamma_n\to 0$. By using new tools of Lemmas 1.6 and 1.7, these strong restrictions are deleted in our paper.

Remark 2.2 In scheme (A), let $E=C=(-\infty,+\infty), a_i=c_i=\frac{1}{2^i}, b_i=\frac{7}{8^i}$, $f_i(x)=\frac{x}{2^i}, k=\frac{1}{2}$, $F_ix=\frac{4^i}{7}x, \overline{\gamma}=\frac{4}{7}, \eta=\frac{1}{7}, \alpha_n=\beta_n=\delta_n=\varepsilon_n'=\varepsilon_n''=\frac{1}{(n+1)^2}, \zeta_n=\frac{1}{n+1}, r_{n,i}=(n+1)2^i$ and $A_ix=\frac{x}{2^i}$ for $i\in N^+$ and $n\ge 0$. Then all of the assumptions in Theorem 2.2 are satisfied and $D=\{0\}$. By using Visual Basic six, we get Table 2.1 and Figure 2.1 below, from which we can see the convergence of $\{x_n\}$.

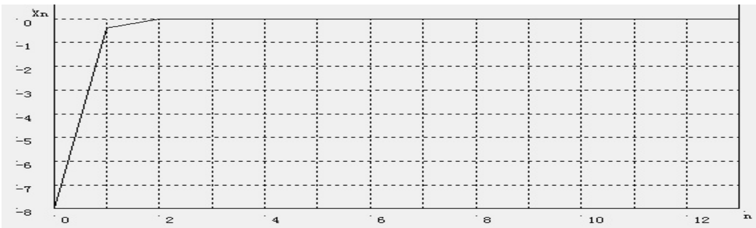

Figure 2.1: Convergence of $\{x_n\}$

Table 2.1: Numerical Results of $\{x_n\}$ with Initial $x_0=-8.0$

n	y_n	z_n	x_n
0	0.000000	0.8000000	−8.000000
1	−0.0982143	0.0324675	−0.3809524
2	0.0955130	0.0357167	−0.00365904
3	0.0641200	0.0148211	0.00589470
4	0.0406016	0.0068630	0.00229334
5	0.0279171	0.0036785	0.00093691
6	0.0204279	0.0021893	0.00044536
7	0.0156147	0.0014041	0.00023759
8	0.0123295	0.0009523	0.00013792
9	0.0099845	0.0006747	0.00008538
10	0.0082513	0.0004949	0.00005561
11	0.0069337	0.0003735	0.00003773
12	0.0059085	0.0002887	0.00002649

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新路径定理和无穷个m增生算子的新迭代设计及计算试验

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摘 要: 本文研究了无穷个m增生算子公共零点的迭代构造问题. 利用Banach极限的技巧和新路径收敛定理, 证明了新构造的迭代序列强收敛到无穷个m增生算子的公共零点的结论, 同时证明了这个公共零点还是一类变分不等式的解. 利用Visual Basic 6 编程, 进行了计算试验用以验证迭代构造的合理性. 迭代参数的限定条件更弱且采用了新的证明技巧, 推广和补充了以往的相关研究成果.

关键词: Banach极限; 增生算子; 保核收缩映射; 强正算子; Visual Basic 6

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