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THE EXISTENCE OF MILD SOLUTION FOR IMPULSIVE FRACTIONAL NEUTRAL FUNCTION INTEGRO-DIFFERENTIAL EVOLUTION EQUATION WITH INFINITE DELAY OF ORDER $0 < \alpha < 1$

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Abstract: In this paper, we investigate the existence of mild solution for impulsive fractional neutral function integro-differential evolution equations with infinite delay of order $0 < \alpha < 1$ in a Banach space. The main mathematical techniques used here include the fractional calculus, properties of solution operators, and Mönch's fixed point theorem via measures of noncompactness. Without assuming that the solution operators are compact, we prove the existence of mild solution to such equations.

Keywords: impulsive fractional neutral function integro-differential evolution equations; mild solution; fixed point theorem; Hausdorff measure of noncompactness

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1 Introduction

Fractional differential equation, as an excellent tool for describing memory and hereditary properties of various materials and processes in natural sciences and engineering, received a great deal of attention in the literature [1–5]. In particular, the existence of solutions to fractional order differential equations attracted researches attention. For example, the existence of solution for fractional semilinear differential or integro differential equations was extensively investigated [6–10]. Recently, mixed type integro-differential systems with and without delay conditions were studied [11–13]. Ravichandran and Baleanu [13] considered the existence of solution for the following fractional neutral functional integro-differential evolution equations with infinite delay in Banach spaces

$$\begin{cases} {}^cD_t^\alpha[x(t)-g(t,x_t)] = Ax(t) + f\big(t, x_t, \displaystyle\int_0^t h(t,s,x_s)ds\big), \quad t \in J = [0,b], \\ x_0 = \phi \in B_h, \quad t \in (-\infty, 0], \end{cases}$$

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where ${}^cD_t^\alpha$ is the Caputo fraction derivative of order $0 < \alpha < 1$. By using properties of solution operators and Mönch's fixed point theorem via measures of noncompactness, the authors developed the existence theorem for such equations.

Motivated by this work, we use Mönch's fixed point theorem via measure of noncompactness to investigate the existence of mild solution for the following impulsive fractional neutral function integro-differential evolution equations with infinite delay in a Banach space X.

$$\begin{cases} {}^cD_t^\alpha[x(t) - g(t,x_t)] = Ax(t) + f\big(t, x_t, \int_0^t h(t,s,x_s)ds\big), \quad t \in J = [0,T], \quad t \neq t_k, \\ \Delta x\mid_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,2,\cdots,m, \\ x_0 = \phi \in B_v, \ \ t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where $T > 0$, ${}^cD_t^\alpha$ is the Caputo fraction derivative of order $0 < \alpha < 1$, A is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$ in a Banach space X, $f : J \times B_v \times X$, $g : J \times B_v$ and $h : J \times J \times B_v$ are given functions, where B_v is the phase space defined in Section 2. The impulsive functions $I_k : X \to X$ $(k = 1, 2, \cdots, m)$ is an appropriate functions. $\Delta x\mid_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The histories $x_t : (-\infty, 0] \to X$, defined by $x_t(s) = x(t + s), s \leq 0$, belong to some abstract phase space B_v.

The rest of the paper is organized as follows: in Section 2, some basic definitions, notations and preliminary facts that are used throughout the paper are presented. The definition of mild solution is given in Section 3. The main results are drown in Section 4, in which we present the existence results for impulsive fractional neutral function integro differential evolution equation of order $0 < \alpha < 1$ with infinite delay.

2 Preliminaries

2.1 Definitions and Theorems

Let X be a complex Banach space, whose norm is denoted by $\|\cdot\|$. Suppose $L(X)$ is the Banach space of all bounded linear operators from X into X, whose corresponding norm is denoted by $\|\cdot\|_{L(X)}$. Let $C(J,X)$ denote the space of all continuous functions from J into X, whose supremum norm is given by $\|\cdot\|_{C(J,X)}$. We use $B_r(x,X)$ to denote the closed ball in X with center at x and radius r.

In the paper, we assume that $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $T(\cdot)$. Thus there exists a constant $M \leq 1$. Without loss of generality, we assume that $0 \in \rho(A)$. Then we can define the fractional power A^α for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$ (see [14]). A^α admits the following properties.

(1) $D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in D(A^\alpha)$.

(2) $T(t) : X \to X_\alpha$ for $t \geq 0$.

(3) $A^\alpha T(t)x = T(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \geq 0$.

(4) For every $t > 0$, $A^\alpha T(t)$ is bounded on X and there exists $M_\alpha > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha t^{-\alpha} e^{-\delta t}.$$

(5) $A^{-\alpha}$ is a bounded linear operator for $0 \leq \alpha \leq 1$ in X.

Before introducing a fractional order functional differential equation with infinite delay, we define the abstract phase space B_v. Let $v : (\infty, 0] \to (0, \infty)$ be a continuous function that satisfies

$$l = \int_{-\infty}^{0} v(t)dt < +\infty.$$

The Banach space $(B_v, \|\cdot\|_{B_v})$ induced by v is then given by

$$B_v := \{\varphi : (-\infty, 0) \to X : \text{for any } c > 0,\ \varphi(\theta) \text{ is a bounded and measurable function on [-c, 0], and } \int_{-\infty}^{0} v(s) \sup_{s\leq\theta\leq 0} \|\varphi(\theta)\| ds < +\infty\}$$

endowed with the norm $\|\varphi\|_{B_v} := \int_{-\infty}^{0} v(s) \sup_{s\leq\theta\leq 0} \|\varphi(\theta)\| ds.$

Define the following space

$$B'_v := \{\varphi : (-\infty, T] \to X : \varphi_k \in C^1(J_k, X), k = 0, 1, 2, \cdots, m, \text{ and there exist } \varphi(t_k^-) \text{ and } \varphi(t_k^+) \text{ with } \varphi(t_k) = \varphi(t_k^-), \varphi_0 = \phi \in B_v\},$$

where φ_k is the restriction of φ to J_k, $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$.

We use $\|\cdot\|_{B'_v}$ to denote a seminorm in the space B'_v defined by

$$\|\varphi\|_{B'_v} := \|\phi\|_{B_v} + \max\{\|\varphi_k\|_{J_k},\ \ k = 0, 1, \cdots, m\},$$

where

$$\phi = \varphi_0, \|\varphi_k\|_{J_k} = \sup_{s\in J_k}\{\|\varphi_k(s)\|\}.$$

Generally, the Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i}\int_{H_\alpha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu,\ \alpha, \beta > 0,\ z \in C,$$

where H_α is a Hankel path, a contour that starts and ends at $-\infty$, and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ counterclockwise.

Now we consider some definitions about fractional differential equations.

Definition 2.1 The fractional integral of order α with the the lower limit zero of a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0,$$

provided that the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is a gamma function defined by

$$\Gamma(\alpha)=\int_0^{\infty} t^{\alpha-1}e^{-t}dt.$$

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha>0, n-1<\alpha<n, n\in N$, is defined as

$${}^{(R-L)}D_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 The Caputo derivative of order α for a function $f:[0,\infty)\to R$ can be written as

$${}^cD^{\alpha}f(t)=D^{\alpha}\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right),\quad t>0,\quad n-1<\alpha<n.$$

Lemma 2.1 (see [15]) Assume $x\in B_v^{'}$, then for $t\in J,\ x_t\in B_v$. Moreover,

$$l\|x(t)\|\le\|x_t\|_{B_v}\le\|\phi\|_{B_v}+l\sup_{s\in[0,t]}\|x(s)\|,$$

where

$$l=\int_{-\infty}^0 v(t)dt<+\infty.$$

Next, we consider some definitions and properties of the measures of noncompactness.

The Hausdorff measure of noncompactness $\beta(\cdot)$ defined on each bounded subset $\mathscr{B}$ of Banach space X is given by

$$\beta(\mathscr{B})=\inf\{\varepsilon>0;\mathscr{B}\quad \text{has a finite }\ \varepsilon\text{ - net in }\ X\}.$$

Some basic properties of $\beta(\cdot)$ are given in the following lemma.

Lemma 2.2 (see [16–18]) If X is a real Banach space and $\mathscr{B},\mathscr{D}\subset X$ are bounded, then the following properties are satisfied:

(1) monotone: if for all bounded subsets $\mathscr{B},\mathscr{D}$ of X, $\mathscr{B}\subseteq\mathscr{D}$ implies $\beta(\mathscr{B})\le\beta(\mathscr{D})$;

(2) nonsingular: $\beta(\{x\}\cup\mathscr{B})=\beta(\mathscr{B})$ for every $x\in X$ and every nonempty subset $\mathscr{B}\subset X$;

(3) regular: $\mathscr{B}$ is precompact if and only if $\beta(\mathscr{B})=0$;

(4) $\beta(\mathscr{B}+\mathscr{D})\le\beta(\mathscr{B})+\beta(\mathscr{D})$, where $\mathscr{B}+\mathscr{D}=\{x+y;x\in\mathscr{B},y\in\mathscr{D}\}$;

(5) $\beta(\mathscr{B}\cup\mathscr{D})\le\max\{\beta(\mathscr{B}),\beta(\mathscr{D})\}$;

(6) $\beta(\lambda\mathscr{B})\le|\lambda|\beta(\mathscr{B})$;

(7) if $W\subset C(J,X)$ is bounded and equicontinuous, then $t\to\beta(W(t))$ is continuous on J, and

$$\beta(W)\le\max_{t\in J}\beta(W(t)),\tag{2.1}$$

$$\beta\left(\int_0^t W(s)ds\right)\le\int_0^t\beta(W(s))ds\ \text{ for all }\ t\in J,\tag{2.2}$$

where

$$\int_0^t W(s)ds = \left\{ \int_0^t u(s)ds : \text{ for all } u \in W, t \in J \right\};$$

(8) if $\{u_n\}_1^\infty$ is a sequence of Bochner integrable functions from J into X with $\|u_n(t)\| \leq \widehat{m}(t)$ for almost all $t \in J$ and every $n \geq 1$, where $\widehat{m}(t) \in L(J, R^+)$, then the function $\psi(t) = \beta(\{u_n\}_{n=1}^\infty))$ belongs to $L(J, R^+)$ and satisfies

$$\beta\left(\left\{ \int_0^t u_n(s)ds : n \geq 1 \right\} \right) \leq 2 \int_0^t \psi(s)ds; \tag{2.3}$$

(9) if W is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^\infty \subset W$, such that

$$\beta(W) \leq 2\beta(\{u_n\}_{n=1}^\infty) + \varepsilon. \tag{2.4}$$

The following lemmas about the Hausdorff measure of noncompactness will be used in proving our main results.

Lemma 2.3 (see [19]) Let D be a closed convex subset of a Banach space X and $0 \in D$. Assume that $F : D \to X$ is a continuous map which satisfies the Mönch's condition, that is, $M \subseteq D$ is countable, $M \subseteq \overline{\text{co}}\,(0 \cup F(M)) \Rightarrow \overline{M}$ is compact. Then F has a fixed point in D.

2.2 Properties of Solution Operators

Lemma 2.4 (see [20]) If A is a sectorial operator of type (M, θ, α, μ), then

$$S_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda = E_{\alpha,1}(At^\alpha) = \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(1+\alpha k)}, \tag{2.5}$$

$$T_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = t^{\alpha-1} \sum_{k=0}^\infty \frac{(At^\alpha)^k}{\Gamma(\alpha+\alpha k)}. \tag{2.6}$$

Lemma 2.5 (see [13]) Assuming A is the infinitesimal generator of an analytic semigroup, given by $T(t)_{t\geq 0}$ and $0 \in \rho(A)$, then we now have

$$S_\alpha(t) = \int_0^\infty \phi_\alpha(r) T(t^\alpha r) dr$$

and

$$T_\alpha(t) = \alpha \int_0^\infty r \phi_\alpha(r) t^{\alpha-1} T(t^\alpha r) dr,$$

where

$$\phi_\alpha(r) = \frac{1}{\alpha} r^{-1-\frac{1}{\alpha}} \psi_\alpha(r^{-\frac{1}{\alpha}}) \geq 0,$$

$$\psi_\alpha(r) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} r^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha).$$

Lemma 2.6 (see [21])

$$\|A^{1-\beta}T_\alpha(t)\| \leq \frac{\alpha M_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)t^{1-\alpha\beta}}, \tag{2.7}$$

where $\alpha, \beta \in (0,1)$.

Lemma 2.7 Suppose A is a sectorial operator of type (M, θ, α, μ). If $0 < \alpha < 1$, then

$$^cD_t^\alpha[S_\alpha(t)x_0] = AS_\alpha(t)x_0$$

and

$$^cD_t^\alpha\left(\int_0^t T_\alpha(t-\theta)f(\theta)d\theta\right) = A\int_0^t T_\alpha(t-\theta)f(\theta)d\theta + f(t).$$

3 Definition of Mild Solution

Theorem 3.1 If f satisfies a uniform Hölder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator of type (M, θ, α, μ), then the Cauchy problem (1.1) has a solution, given by

$$x(t) = \begin{cases} x_0 = \phi \in B_v, \qquad t \in (-\infty, 0], \\ S_\alpha(t)[\phi(0) - g(0,\phi(0))] + g(t, x_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s,x_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds, \quad t \in (0, t_1], \\ S_\alpha(t)[\phi(0) - g(0,\phi(0))] + g(t, x_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s,x_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(x(t_k^-)), \;\; t \in (t_k, t_{k+1}]. \end{cases} \tag{3.1}$$

Proof For all $t \in (t_k, t_{k+1}]$ where $k = 0, 1, \cdots, m$ by Lemma 2.7, we obtain

$$\begin{aligned} & {}^cD_t^\alpha[x(t) - g(t,x_t)] \\ = & {}^cD_t^\alpha\Bigg[S_\alpha(t)[\phi(0) - g(0,\phi(0))] + \int_0^t T_\alpha(t-s)Ag(s,x_s)ds \\ & + \int_0^t T_\alpha(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(x(t_k^-))\Bigg] \\ = & AS_\alpha(t)[\phi(0) - g(0,\phi(0))] + A\int_0^t T_\alpha(t-s)Ag(s,x_s)ds + Ag(t,x_t) \\ & + A\int_0^t T_\alpha(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds \\ & + f\left(t, x_t, \int_0^t h(t,s,x_s)ds\right) + A\sum_{i=1}^k S_\alpha(t-t_i)I_i(x(t_k^-)) \end{aligned}$$

$$
\begin{aligned}
= \;& A\left(S_\alpha(t)[\phi(0)-g(0,\phi(0))]+g(t,x_t)+\int_0^t AT_\alpha(t-s)g(s,x_s)ds\right.\\
&\left.+\int_0^t T_\alpha(t-s)f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\right)ds+\sum_{i=1}^{k}S_\alpha(t-t_i)I_i(x(t_k^-))\right)\\
&+f\left(t,x_t,\int_0^t h(t,s,x_s)ds\right).
\end{aligned}
$$

Thus expression (3.1) satisfies the first equation of problem (1.1).

For $k=1,2,\cdots,m$, it follows from (3.1) that

$$
\begin{aligned}
x(t_k^-) &= S_\alpha(t_k)[\phi(0)-g(0,\phi(0))]+g(t_k,x_{t_k})+\int_0^{t_k} AT_\alpha(t_k-s)g(s,x_s)ds\\
&\quad+\int_0^{t_k} T_\alpha(t_k-s)f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\right)ds+\sum_{i=1}^{k-1}S_\alpha(t_k-t_i)I_i(x(t_k^-)),\\
x(t_k^+) &= S_\alpha(t_k)[\phi(0)-g(0,\phi(0))]+g(t_k,x_{t_k})+\int_0^{t_k} AT_\alpha(t_k-s)g(s,x_s)ds\\
&\quad+\int_0^{t_k} T_\alpha(t_k-s)f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\right)ds+\sum_{i=1}^{k}S_\alpha(t_k-t_i)I_i(x(t_k^-)).
\end{aligned}
$$

Therefore we have

$$
\Delta x(t_k)=x(t_k^+)-x(t_k^-)=S_\alpha(t_k-t_k)I_k(x(t_k^-))=I_k(x(t_k^-)).
$$

Consequently, all the conditions of problem (1.1) are satisfied. Hence (3.1) is a solution of problem (1.1).

Thus the mild solution of equation (1.1) can be defined as follows.

Definition 3.1 (see [22]) A continuous function $x:(-\infty,T]\to X$ is said to be mild solution of system (1.1) if $x=\phi\in B_v$ on $(-\infty,0]$, the impulsive condition $\Delta x(t_k)=I_k(x(t_k^-)), k=1,2,\cdots,m$ is satisfied, the restriction of $x(\cdot)$ to the interval J_k is continuous, and $x(t)$ satisfies the following integral equation

$$
x(t)=\begin{cases}
x_0=\phi\in B_v, \qquad t\in(-\infty,0],\\
S_\alpha(t)[\phi(0)-g(0,\phi(0))]+g(t,x_t)+\int_0^t AT_\alpha(t-s)g(s,x_s)ds\\
+\int_0^t T_\alpha(t-s)f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\right)ds, \quad t\in(0,t_1],\\
S_\alpha(t)[\phi(0)-g(0,\phi(0))]+g(t,x_t)+\int_0^t AT_\alpha(t-s)g(s,x_s)ds\\
+\int_0^t T_\alpha(t-s)f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\right)ds+\sum_{i=1}^{k}S_\alpha(t-t_i)I_i(x(t_k^-)), \quad t\in(t_k,t_{k+1}].
\end{cases}
$$

4 Existence Results

In this section, we present the main results of this article. We first consider the following hypotheses.

If A is a sectorial operator of type (M, θ, α, μ), then $\|S_\alpha(t)\| \le Me^{\omega t}$ and $\|T_\alpha(t)\| \le Ce^{\omega t}(1 + t^{\alpha-1})$. Let

$$M_S := \sup_{t \in J} \|S_\alpha(t)\|_{L(X)} \quad \text{and} \quad M_T := \sup_{t \in J} Ce^{\omega t}(1 + t^{\alpha-1}),$$

where $L(X)$ is the Banach space of bounded linear operator from X into X equipped with its natural topology. Thus we have

$$\|S_\alpha(t)\| \le M_S, \quad \|T_\alpha(t)\| \le t^{\alpha-1} M_T.$$

We assume the following conditions hold

(H1) The function $f : J \times B_v \times X \to X$ satisfies the following conditions:

(i) $f(\cdot, \phi, x)$ is measurable for all $(\phi, x) \in B_v \times X$ and $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in J$.

(ii) There exist a constant $\alpha_1 \in (0, \alpha)$, $m \in L^{\frac{1}{\alpha_1}}(J, R^+)$ and a positive integrable function $\Omega : R^+ \to R^+$ such that

$$\|f(t, \phi, x)\| \le m(t)\Omega(\|\phi\|_{B_v} + \|x\|)$$

for all $(t, \phi, x) \in J \times B_v \times X$, where Ω satisfies

$$\liminf_{n \to \infty} \frac{\Omega(n)}{n} = 0.$$

(iii) There exist a constant $\alpha_2 \in (0, \alpha)$ and a function $\eta \in L^{\frac{1}{\alpha_2}}(J, R^+)$ such that, for any bounded subset $D_1 \subset X, F_1 \subset B_v$,

$$\beta(f(t, F_1, D_1)) \le \eta(t)[\sup_{\theta \in (-\infty, 0]} \beta(F_1(\theta)) + \beta(D_1)]$$

for a.e. $t \in J$, where $F_1(\theta) = \{v(\theta) : v \in F_1\}$ and β is the Hausdorff MNC.

(H2) The function $h : J \times J \times B_v \to X$ satisfies the following conditions:

(i) $h(\cdot, t, x)$ is measurable for all $(t, x) \in J \times B_v$ and $h(t, \cdot, \cdot)$ is continuous for a.e. $t \in J$.

(ii) There exist a constant $H_0 > 0$ such that

$$\|h(t, s, \phi)\| \le H_0(1 + \|\phi\|_{B_v})$$

for all $t, s \in J, \phi \in B_v$.

(iii) There exists $\xi \in L^1(J^2, R^+)$ such that for any bounded subset $F_2 \subset B_v$,

$$\beta(h(t, s, F_2)) \le \xi(t, s)[\sup_{\theta \in (-\infty, 0]} \beta(F_2(\theta))]$$

for a.e. $t \in J$, with $\xi^* = \sup_{t \in J} \int_0^s \xi(t, \tau)d\tau < +\infty$.

(H3) The function $g : J \times B_v \to X$ satisfies the following conditions:

(i) g is continuous and there exist a constant $H_1 > 0$ and $0 < \beta < 1$ such that g is X_β valued and

$$\|A^\beta g(t,x) - A^\beta g(t,y)\| \le H_1 \|x-y\|_{B_v}, x, y \in B_v,\ t \in J = [0,t],$$
$$\|A^\beta g(t,x)\| \le H_1(1 + \|x\|_{B_v}).$$

(ii) There exist a constant $\alpha_3 \in (0,\alpha)$ and $g^* \in L^{\frac{1}{\alpha_3}}(J, R^+)$ such that, for any bounded subset $F_3 \subset B_v$,

$$\beta(A^\beta g(t,F_3)) \le g^*(t) \sup_{\theta\in(-\infty,0]} \beta(F_3(\theta)), \quad G = \sup_{t\in J} g^*(t).$$

(H4) $I_i : X \to X$ be continuous operators and there exist positive numbers c_i, K_i such that

$$\|I_i(x)\|_X \le c_i \|x\|_{B_v'} \text{ for all } x \in B_v',\ i = 1,2,\cdots,m,$$
$$\beta(I_i(F_4)) \le K_i \sup_{\theta\in(-\infty,T]} \beta(F_4(\theta)),\ i = 1,2,\cdots,m$$

for every bounded subset F_4 of B_v'.

(H5)

$$\left[H_1(\|A^{-\beta}\| + \frac{\alpha M_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta}) + M_S \sum_{i=1}^m c_i\right] l < 1,$$
$$M^* = M_S \sum_{i=1}^m K_i + G + 2\frac{\alpha M_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta}\|g^*\| + 2M_T(1+2\xi^*)\frac{T^\alpha}{\alpha}\|\eta\| < 1.$$

Theorem 4.1 Suppose conditions (H1)–(H5) are satisfied. Then system(1.1) has at least one solution on J.

Proof We define the operator $\Gamma : B_v' \to B_v'$ by

$$\Gamma x(t) = \begin{cases} x_0 = \phi \in B_v, \qquad t \in (-\infty, 0], \\ S_\alpha(t)[\phi(0) - g(0,\phi(0))] + g(t,x_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s,x_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right) ds, \quad t \in (0,t_1], \\ S_\alpha(t)[\phi(0) - g(0,\phi(0))] + g(t,x_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s,x_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right) ds + \sum_{i=1}^k S_\alpha(t-t_i)I_i(x(t_k^-)),\ t \in (t_k, t_{k+1}]. \end{cases}$$

The operator Γ has a fixed point if and only if system (1.1) has a solution. For $\phi \in B_v$, denote

$$\hat{\phi}(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ S_\alpha(t)\phi(0), \quad t \in J. \end{cases}$$

Then $\hat{\phi}(t) \in B_v^{'}$.

Let $x(t) = y(t) + \hat{\phi}(t),\ -\infty < t \leq T$. It is easy to see that y satisfies $y_0 = 0, t \in (-\infty, 0]$ and

$$y(t) = \begin{cases} S_\alpha(t)[-g(0,\phi(0))] + g(t, y_t + \hat{\phi}_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s, y_s + \hat{\phi}_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s)f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right)ds, \quad t \in (0, t_1], \\ S_\alpha(t)[-g(0,\phi(0))] + g(t, y_t + \hat{\phi}_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s, y_s + \hat{\phi}_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s)f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right)ds \\ + \displaystyle\sum_{i=1}^k S_\alpha(t-t_i)I_i(y(t_i^-) + \hat{\phi}(t_i^-)),\ t \in (t_k, t_{k+1}] \end{cases}$$

if and only if $x(t)$ satisfies $x(t) = \phi(t), t \in (-\infty, 0]$ and

$$x(t) = \begin{cases} S_\alpha(t)[\phi(0) - g(0,\phi(0))] + g(t, x_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s, x_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds, \quad t \in (0, t_1], \\ S_\alpha(t)[\phi(0) - g(0,\phi(0))] + g(t, x_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s, x_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds + \displaystyle\sum_{i=1}^k S_\alpha(t-t_i)I_i(x(t_k^-)),\ t \in (t_k, t_{k+1}]. \end{cases}$$

Define the Banach space $(B_v^{''}, \|\cdot\|_{B_v^{''}})$ induced by $B_v^{'}$, $B_v^{''} = \{y : y \in B_v^{'}, y_0 = 0\}$ with the norm

$$\|y(t)\|_{B_v^{''}} = \sup\{\|y(s)\|_X, s \in [0,T]\}.$$

Let $B_r = \{y \in B_v^{''} : \|y\|_{B_v^{'}} \leq r\}$. Then for each r, B_r is a bounded, close and convex subset. For any $y \in B_r$, it follows from Lemma 2.1 that

$$\|y_t + \hat{\phi}_t\|_{B_v} \leq \|y_t\|_{B_v} + \|\hat{\phi}_t\|_{B_v} \leq l(r + M_S\|\phi(0)\|) + \|\phi\|_{B_v} = r',$$

we define the operator $N : B_v^{''} \to B_v^{''}$ by

$$Ny(t) = \begin{cases} 0, \qquad t \in (-\infty, 0], \\ S_\alpha(t)[-g(0,\phi(0))] + g(t, y_t + \hat{\phi}_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s, y_s + \hat{\phi}_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s)f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right)ds, \quad t \in (0, t_1], \\ S_\alpha(t)[-g(0,\phi(0))] + g(t, y_t + \hat{\phi}_t) + \displaystyle\int_0^t AT_\alpha(t-s)g(s, y_s + \hat{\phi}_s)ds \\ + \displaystyle\int_0^t T_\alpha(t-s)f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right)ds \\ + \displaystyle\sum_{i=1}^k S_\alpha(t-t_i)I_i(y(t_i^-) + \hat{\phi}(t_i^-)), \qquad t \in (t_k, t_{k+1}]. \end{cases}$$

It is easy to see that operator Γ has a fixed point if and only if operator N has a fixed point. In the following, we prove that N has a fixed point.

Step 1 We prove that there exists some $r > 0$ such that $N(B_r) \subset B_r$. If this is not true, then, for each positive integer r, there exist $y^r \in B_r$ and $t^r \in (-\infty, T]$ such that $\|(Ny^r)(t^r)\| > r$.

On the other hand, it follows from the assumption that

$$\begin{aligned} r &< \|(Ny^r)(t^r)\| \\ &\leq \|S_\alpha(t^r)[-g(0,\phi(0))]\| + \|g(t^r, y_{t^r} + \hat{\phi_{t^r}})\| + \|\int_0^{t^r} AT_\alpha(t^r - s)g(s, y_s + \hat{\phi}_s)ds\| \\ &\quad + \|\int_0^{t^r} T_\alpha(t^r - s) f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau + \hat{\phi}_\tau)d\tau\right) ds\| \\ &\quad + \sum_{i=1}^k \|S_\alpha(t^r - t_i)I_i(y^r(t_i^-) + \hat{\phi}(t_i^-))\| \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Now we estimate I_i, $i = 1, 2, 3, 4, 5$. Assumption (H3) (i) implies

$$\begin{aligned} I_1 &= \|S_\alpha(t^r)[-g(0,\phi(0))]\| \leq M_S\|A^{-\beta}A^\beta g(0,\phi(0)\| \\ &\leq M_S\|A^{-\beta}\|\|A^\beta g(0,\phi(0)\| \leq M_S\|A^{-\beta}\|H_1(1 + \|\phi\|_{B_v}), \\ I_2 &= \|g(t^r, y_{t^r} + \hat{\phi}_{t^r})\| \leq \|A^{-\beta}\|H_1(1 + \|y_{t^r} + \hat{\phi}_{t^r}\|_{B_v}) \\ &\leq \|A^{-\beta}\|H_1(1 + r'). \end{aligned}$$

It follows from assumption (H3) (i) and Hölder's inequality that

$$\begin{aligned} I_3 &= \|\int_0^{t^r} AT_\alpha(t^r - s)g(s, y_s + \hat{\phi}_s)ds\| \\ &= \|\int_0^{t^r} A^{1-\beta}T_\alpha(t^r - s)A^\beta g(s, y_s + \hat{\phi}_s)ds\| \\ &\leq \frac{\alpha M_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} H_1(1 + \|y_s + \hat{\phi}_s\|_{B_v}) \int_0^{t^r} (t^r - s)^{\alpha\beta - 1} ds \\ &\leq \frac{\alpha M_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} H_1(1 + r^{'}) \frac{T^{\alpha\beta}}{\alpha\beta}. \end{aligned}$$

By using assumptions (H1) and (H2), we obtain

$$\begin{aligned} I_4 &= \|\int_0^{t^r} T_\alpha(t^r - s) f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau + \hat{\phi}_\tau)d\tau\right) ds\| \\ &\leq M_T \frac{T^\alpha}{\alpha} m(t)\Omega(r' + TH_0(1 + r')) \\ &\leq M_T \frac{T^\alpha}{\alpha} \Omega(r' + TH_0(1 + r')) \sup_{t\in J} m(t). \end{aligned}$$

Using assumptions (H1) (i) and (H4) yields

$$
\begin{aligned}
I_5 &= \sum_{i=1}^{k} \|S_\alpha(t^r - t_i)\| \|I_i(y^r(t_i^-) + \hat{\phi}(t_i^-))\| \le M_S \sum_{i=1}^{k} \|I_i(y^r(t_i^-) + \hat{\phi}(t_i^-))\| \\
&\le M_S \sum_{i=1}^{m} c_i r'.
\end{aligned}
$$

Combining estimates $I_1 - I_5$, we obtain

$$
\begin{aligned}
r \;&<\; I_1 + I_2 + I_3 + I_4 + I_5 \le M_S \|A^{-\beta}\| H_1(1 + \|\phi\|_{B_v}) + \|A^{-\beta}\| H_1(1 + r') \\
&\quad + \frac{\alpha M_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} H_1(1 + r^{'}) \frac{T^{\alpha\beta}}{\alpha\beta} + M_T \frac{T^\alpha}{\alpha} \Omega(r' + TH_0(1 + r')) \sup_{t\in J} m(t) + M_S \sum_{i=1}^{m} c_i r' \\
&\le \left[H_1(\|A^{-\beta}\| + \frac{\alpha M_{1-\beta}\Gamma(1+\beta) T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta}) + M_S \sum_{i=1}^{m} c_i \right] r^{'} \\
&\quad + H_1(\|A^{-\beta}\| + \frac{\alpha M_{1-\beta}\Gamma(1+\beta) T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta}) + M_S \sum_{i=1}^{m} c_i + M_S \|A^{-\beta}\| H_1(1 + \|\phi\|_{B_v}) \\
&\quad + M_T \frac{T^\alpha}{\alpha} \Omega(r' + TH_0(1 + r')) \sup_{t\in J} m(t)
\end{aligned}
$$

Dividing both sides by r and taking $r \to +\infty$ from

$$
\lim_{r\to\infty} \frac{r^{'}}{r} = \lim_{r\to\infty} \frac{l(r + M_S\|\phi(0)\|) + \|\phi\|_{B_v}}{r} = l \quad \text{and} \quad \lim_{n\to\infty} \inf \frac{\Omega(n)}{n} = 0
$$

yields

$$
\left[H_1(\|A^{-\beta}\| + \frac{\alpha M_{1-\beta}\Gamma(1+\beta) T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta}) + M_S \sum_{i=1}^{m} c_i \right] l \ge 1.
$$

This contradicts (H5). Thus, for some number $r, N(B_r) \subset B_r$.

Step 2 N is continuous on B_r.

Let $\{y^n\}_{n=1}^{+\infty} \subset B_r$, with $y^n \to y$ in B_r as $n \to +\infty$. Then by using hypotheses (H2), (H4) and (H5), we have

(i) $f\left(s, y_s^n + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^n + \hat{\phi}_\tau)d\tau\right) \to f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right), \ n \to \infty.$

(ii) $g\left(t, y_t^n + \hat{\phi}_t\right) \to g\left(t, y_t + \hat{\phi}_t\right), \quad n \to \infty.$

(iii) $\|I_i(y^n(t_i^-) + \hat{\phi}(t_i^-)) - I_i(y(t_i^-) + \hat{\phi}(t_i^-))\| \to 0, \quad n \to \infty, i = 1, 2, \cdots, m.$

Now for every $t \in [0, t_1]$, we have

$$
\begin{aligned}
\|Ny^n(t) - Ny(t)\| \;&\le\; \|g(t, y_t^n + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t)\| \\
&\quad + \frac{\alpha M_{1-\beta}\Gamma(1+\beta) T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta} H_1 \|y_t^n - y_t\| \\
&\quad + M_T \frac{T^\alpha}{\alpha} \| f\left(s, y_s^n + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^n + \hat{\phi}_\tau)d\tau\right) \\
&\quad - f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right) \| \to 0 \ \ (n \to \infty).
\end{aligned}
$$

Moreover, for all $t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m$, we have

$$
\begin{aligned}
\|Ny^n(t) - Ny(t)\| \le\ & \|g(t, y_t^n + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t)\| \\
& + \frac{\alpha M_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta} H_1 \|y_t^n - y_t\| \\
& + M_T \frac{T^\alpha}{\alpha} \|f\left(s, y_s^n + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau^n + \hat{\phi}_\tau) d\tau\right) \\
& - f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau + \hat{\phi}_\tau) d\tau\right)\| \\
& + M_S \sum_{i=1}^{k} \|I_i(y^n(t_i^-) + \hat{\phi}(t_i^-)) - I_i(y(t_i^-) + \hat{\phi}(t_i^-))\| \to 0\ \ (n \to \infty).
\end{aligned}
$$

We thus obtain $\|Ny^n - Ny\| \to 0$ as $n \to \infty$ implying that N is continuous on B_r.

Step 3 The map $N(B_r)$ is equicontinuous on J.

The function $\{Ny : y \in B_r\}$ is equicontinuous at $t = 0$. For $0 < t_1 < t_2 \le T,\ t_1, t_2 \in (t_k, t_{k+1}], k = 1, 2, \cdots, m$ and $y \in B_r$, by assumptions (H1) and (H4), we deduce that

$$
\begin{aligned}
& \|(Ny)(t_2) - (Ny)(t_1)\| \\
\le\ & \|S_\alpha(t_2) - S_\alpha(t_1)\| \|g(0, \phi(0))\| + \|g(t_2, y_{t_2} + \hat{\phi}_{t_2}) - g(t_1, y_{t_1} + \hat{\phi}_{t_1})\| \\
& + \|\int_0^{t_2} AT_\alpha(t_2 - s) g(s, y_s + \hat{\phi}_s) ds - \int_0^{t_1} AT_\alpha(t_1 - s) g(s, y_s + \hat{\phi}_s) ds\| \\
& + \int_0^{t_2} T_\alpha(t_2 - s) f\left(s, y_s^n + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau^n + \hat{\phi}_\tau) d\tau\right) ds \\
& - \int_0^{t_1} T_\alpha(t_1 - s) f\left(s, y_s^n + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau^n + \hat{\phi}_\tau) d\tau\right) ds\| \\
\le\ & G\|S_\alpha(t_2) - S_\alpha(t_1)\| + \|g(t_2, y_{t_2} + \hat{\phi}_{t_2}) - g(t_1, y_{t_1} + \hat{\phi}_{t_1})\| \\
& + \|\int_0^{t_1} A^{1-\beta}[T_\alpha(t_2 - s) - T_\alpha(t_1 - s)] A^\beta g(s, y_s + \hat{\phi}_s) ds\| \\
& + \|\int_{t_1}^{t_2} AT_\alpha(t_2 - s) g(s, y_s + \hat{\phi}_s) ds\| \\
& + \|\int_0^{t_1} [T_\alpha(t_2 - s) - T_\alpha(t_1 - s)] f\left(s, y_s^n + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau^n + \hat{\phi}_\tau) d\tau\right) ds\| \\
& + \|\int_{t_1}^{t_2} T_\alpha(t_2 - s) f\left(s, y_s^n + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau^n + \hat{\phi}_\tau) d\tau\right) ds\| \to 0\ \ as\ \ t_2 \to t_1.
\end{aligned}
$$

Hence $N(B_r)$ is equicontinuous on J.

Step 4 Mönch's condition holds.

Let $N = N_1 + N_2 + N_3$, where

$$
N_1 y(t) = \sum_{i=1}^{k} S_\alpha(t - t_i) I_i(x(t_k^-)), \quad t \in (t_k, t_{k+1}],
$$

$$N_2y(t) = S_\alpha(t)[-g(0,\phi(0))] + g(t, y_t+\hat{\phi}_t) + \int_0^t AT_\alpha(t-s)g(s,y_s+\hat{\phi}_s)ds,$$

$$N_3y(t) = \int_0^t T_\alpha(t-s)f\left(s, y_s+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau+\hat{\phi_\tau})d\tau\right)ds.$$

Assume $W \subseteq B_r$ is countable and $W \subseteq \overline{\text{co}}\,(\{0\}\cup N(W))$. We show that $\beta(W)=0$, where β is the Hausdorff MNC. Without loss of generality, we may suppose that $W = \{y^n\}_{n=1}^\infty$. Since $N(W)$ is equicontinuous on J_k, $W \subseteq \overline{\text{co}}\,(\{0\}\cup N(W))$ is equicontinuous on J_k as well.

By (H4) (ii), we have

$$\begin{aligned}&\beta\left(\{N_1y^n(t)\}_{n=1}^\infty\right) \le M_S\sum_{i=1}^k \beta\left(\{I_i(y_{t_i}^n+\hat{\phi}_{t_i})\}_{n=1}^\infty\right)\\ \le& M_S\sum_{i=1}^m K_i \sup_{-\infty<\theta\le 0}\beta\left(y_{t_i}^n(\theta)+\hat{\phi}_{t_i}(\theta)\}_{n=1}^\infty\right) \le M_S\sum_{i=1}^m K_i\sup_{-\infty<\theta\le0}\beta\left(\{y_{t_i}^n(\theta)\}_{n=1}^\infty\right).\end{aligned}$$

Using Lemma 2.2, (H1) (iii), (H2) (iii) and (H3) (ii), we have

$$\begin{aligned}\beta\left(\{N_2y^n(t)\}_{n=1}^\infty\right) \le& G\sup_{-\infty<\theta\le0}\beta\left(y_t^n(\theta)+\hat{\phi}_t(\theta)\}_{n=1}^\infty\right)\\ &+2\frac{\alpha M_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1}g^*(s)\sup_{-\infty<\theta\le0}\beta\left(y_t^n(\theta)+\hat{\phi}_t(\theta)\}_{n=1}^\infty\right)ds\\ &\le (G+2\frac{\alpha M_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta}\|g^*\|_{L^{\frac{1}{\alpha_3}}(J,R^+)})\sup_{-\infty<\theta\le0}\beta\{\{y_t^n(\theta)\}_{n=1}^\infty\},\\ \beta\left(\{N_3y^n(t)\}_{n=1}^\infty\right)\le& 2M_T\int_0^t (t-s)^{\alpha-1}\eta(s)\left[\sup_{-\infty<\theta\le0}\beta\left(\{y_s^n(\theta)\}_{n=1}^\infty\right)\right.\\ &\left.+\beta\left(\{\int_0^s h(s,\tau,y_\tau^n+\hat{\phi}_\tau)d\tau\}_{n=1}^\infty\right)\right]ds\\ &\le 2M_T(1+2\xi^*)\frac{T^\alpha}{\alpha}\|\eta\|_{L^{\frac{1}{\alpha_2}}(J,R^+)}\sup_{-\infty<\theta\le0}\beta\left(\{y_s^n(\theta)\}_{n=1}^\infty\right).\end{aligned}$$

We thus obtain

$$\begin{aligned}\beta\left(\{Ny^n(t)\}_{n=1}^\infty\right)\le&\beta\left(\{N_1y^n(t)\}_{n=1}^\infty\right)+\beta\left(\{N_2y^n(t)\}_{n=1}^\infty\right)+\beta\left(\{N_3y^n(t)\}_{n=1}^\infty\right)\\ \le&\left(M_S\sum_{i=1}^m K_i + G + 2\frac{\alpha M_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta}\|g^*\|_{L^{\frac{1}{\alpha_3}}(J,R^+)}\right.\\ &\left.+2M_T(1+2\xi^*)\frac{T^\alpha}{\alpha}\|\eta\|_{L^{\frac{1}{\alpha_2}}(J,R^+)}\right)\beta\left(\{y^n(t)\}_{n=1}^\infty\right)\\ =&M^*\beta\left(\{y^n(t)\}_{n=1}^\infty\right),\end{aligned}$$

where M^* is defined in assumption (H5). Since W and $N(W)$ are equicountinuous on every J_k, it follows from Lemma 2.2 that the inequality implies $\beta(NW)\le M^*\beta(W)$. Thus from Mönch's condition, we have

$$\beta(W)\le\beta(\overline{\text{co}}(\{0\}\cup N(W)))=\beta(NM)\le M^*\beta(W).$$

Since $M^* < 1$, we get $\beta(W) = 0$. It follows that W is relatively compact. Using Lemma 2.3, we know that N has a fixed point y in W. The proof is completed.

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一类$0 < \alpha < 1$带有无限时滞的中立型脉冲微分方程mild解的存在性

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摘 要: 本文研究了一类$0 < \alpha < 1$ 带有无限时滞的中立型脉冲微分方程mild解的存在性的问题. 利用解算子的相关性质及Mönch不动点理论的方法, 获得了这类方程的mild解并予以证明, 且得到了解的存在性的结果.

关 键 词: 中立型脉冲微分方程; mild解; 不动点定理; 非紧性测度

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