Vol. 38 (2018)
No. 1

数 学 杂 志
J. of Math. (PRC)

A NONTRIVIAL PRODUCT OF $b_0^3\tilde{\delta}_{s+4}$ IN THE COHOMOLOGY OF THE STEENROD ALGEBRA

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Abstract: In this paper, we mainly study the nontriviality of the products in the cohomology of the Steenrod algebra. Let p be a prime greater than five and A be the mod p Steenrod algebra. By using the explicit combinatorial analysis of the May spectral sequence, we prove that the product $b_0^3\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ is nontrivial, where $0 \leqslant s < p-5$, which is helpful for us to study the nontriviality of homotopy elements in the stable homotopy of spheres.

Keywords: Steenrod algebra; cohomology; May spectral sequence
2010 MR Subject Classification: 55Q45
Document code: A **Article ID:** 0255-7797(2018)01-0057-10

1 Introduction

To determine the stable homotopy groups of spheres is one of the most important problems in algebraic topology. So far, several methods were found to determine the stable homotopy groups of spheres. For example, we have the classical Adams spectral sequence (ASS) (see [1]) based on the Eilenberg-MacLane spectrum $K\mathbb{Z}_p$, whose E_2-term is $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ and the Adams differential is given by $\tilde{d}_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$, where A denotes the mod p Steenrod algebra. There are three problems in using the ASS: calculation of E_2-term $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$, computation of the differentials and determination of the nontrivial extensions from E_∞ to the stable homotopy groups of spheres. So, for computing the stable homotopy groups of spheres with the classical ASS, we must compute the E_2-term of the ASS, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$.

Throughout this paper, p denotes an odd prime and $q = 2(p-1)$. The known results on $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ are as follows. $\mathrm{Ext}_A^{0,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ is trivial by its definition. From [2], $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p, \mathbb{Z}_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p, \mathbb{Z}_p)$ for all $i \geqslant 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of α_2, a_0^2, $a_0h_i (i > 0)$, $g_i (i \geqslant 0)$, $k_i (i \geqslant 0)$, $b_i (i \geqslant 0)$, and $h_ih_j (j \geqslant i+2, i \geqslant 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq, respectively. In 1980, Aikawa [3] determined $\mathrm{Ext}_A^{3,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ by λ-algebra.

* **Received date:** 2017-05-06 **Accepted date:** 2017-10-18
Foundation item: Supported by the Youth Foundation of Hebei Educational Committee (QN2017505); NSFC(11571186).

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Studying higher-dimensional cohomology of the mod p Steenrod algebra A was an interesting subject and studied by several authors. For example, Liu and Zhao [4] proved the following theorems, respectively.

Theorem 1.1 For $p \geqslant 11$ and $4 \leqslant s < p$, the product $h_0 b_0 \tilde{\delta}_s \neq 0$ in the classical Adams spectral sequence, where $\tilde{\delta}_s$ was given in [5].

In this paper, our main result can be stated as follows.

Theorem 1.2 Let $p \geqslant 7$, and $0 \leqslant s < p-5$. Then in the cohomology of the mod p Steenrod algebra A, the product $b_0^3 \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$ is nontrivial, where

$$t(s) = q[(s+1) + (s+5)p + (s+3)p^2 + (s+4)p^3] + s.$$

The main method of proof is the (modified) May spectral sequence, so we will recall some knowledge on the May spectral sequence in Section 2. After detecting the generators of some May E_1-terms in Section 3, we will prove Theorem 1.2.

2 The May Spectral Sequence

As we know, the most successful method to compute $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ is the MSS. From [6], there is a May spectral sequence(MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} | m > 0, i \geqslant 0) \otimes P(b_{m,i} | m > 0, i \geqslant 0) \otimes P(a_n | n \geqslant 0), \tag{2.1}$$

where $E(\ )$ is the exterior algebra, $P(\ )$ is the polynomial algebra, and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n \in E_1^{1,2p^n-1,2n+1}.$$

One has

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r} \tag{2.2}$$

and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y). \tag{2.3}$$

In particular, the first May differential d_1 is given by

$$d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j} h_{k,j},\ d_1(a_i) = \sum_{0 \leqslant k < i} h_{i-k,k} a_k,\ d_1(b_{i,j}) = 0. \tag{2.4}$$

There also exists a graded commutativity in the MSS:

$$x \cdot y = (-1)^{ss'+tt'} y \cdot x \ \text{ for } \ x, y = h_{m,i}, b_{m,i} \ \text{ or } \ a_n.$$

For each element $x \in E_1^{s,t,u}$, we define $\dim x = s$, $\deg x = t$, $M(x) = u$. Then we have that

$$\begin{cases} \dim h_{i,j} = \dim a_i = 1, \\ \dim b_{i,j} = 2, \deg a_0 = 1, \\ \deg h_{i,j} = q(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = q(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = q(p^{i-1} + \cdots + 1) + 1, \\ M(h_{i,j}) = M(a_{i-1}) = 2i - 1, \\ M(b_{i,j}) = (2i-1)p, \end{cases} \tag{2.5}$$

where $i \geqslant 1$, $j \geqslant 0$.

Note that by the knowledge on the p-adic expression in number theory, for each integer $t \geqslant 0$, it can be expressed uniquely as $t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e$, where $0 \leqslant c_i < p$ $(0 \leqslant i < n)$, $p > c_n > 0$, $0 \leqslant e < q$.

3 Proof of Theorem 1.2

Before showing Theorem 1.2, we first give some important lemmas which will be used in the proof of it. The first one is a lemma on the representative of $\tilde{\delta}_{s+4}$ in the May spectral sequence.

Lemma 3.1 For $p \geqslant 7$ and $0 \leqslant s < p-4$. Then the fourth Greek letter element $\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+4,t_1(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$ is represented by

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \in E_1^{s+4,t_1(s),*}$$

in the E_1-term of the May spectral sequence, where $\tilde{\delta}_{s+4}$ is actually $\tilde{\alpha}_{s+4}^{(4)}$ described in [6] and $t_1(s) = q[(s+1) + (s+2)p + (s+3)p^2 + (s+4)p^3] + s$.

By (2.2), we know that to prove the non-triviality of $b_0^3\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$, we have to show that the representative of the product cannot be hit by any May differential. For doing it, we give the following two lemmas.

Lemma 3.2 Let $p \geqslant 7$, $0 \leqslant s < p-5$. Then we have the May E_1-term

$$E_1^{s+9,t(s),*} = \mathbb{Z}_p \left\{ G_1, G_2, \cdots, G_{11} \right\},$$

where $t(s,n) = q[(s+1) + (s+5)p + (s+3)p^2 + (s+4)p^3] + s$, and

$$\begin{aligned} &G_1 = a_4^{s-1} a_2 b_{3,0}^2 b_{1,0} h_{4,0} h_{3,1} h_{1,3}, \ \ G_2 = a_4^{s-1} a_2 b_{3,0}^3 h_{4,0} h_{1,3} h_{1,1}, \\ &G_3 = a_4^{s-1} a_2 b_{3,0}^2 b_{1,2} h_{4,0} h_{3,1} h_{1,1}, \ \ G_4 = a_4^s b_{3,0} b_{1,0}^2 h_{4,0} h_{3,1} h_{1,3}, \\ &G_5 = a_4^s b_{3,0}^2 b_{1,0} h_{4,0} h_{1,3} h_{1,1}, \ \ G_6 = a_4^s b_{3,0} b_{1,2} b_{1,0} h_{4,0} h_{3,1} h_{1,1}, \\ &G_7 = a_4^{s-2} a_2^2 b_{3,0}^2 h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,1}, \ \ G_8 = a_4^{s-1} a_2 b_{3,0} b_{1,0} h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,1}, \\ &G_9 = a_4^{s-1} a_2 b_{3,0} b_{1,2} h_{4,0} h_{3,1} h_{2,1} h_{1,3} h_{1,1}, \ \ G_{10} = a_4^s b_{1,0}^2 h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,1}, \\ &G_{11} = a_4^s b_{1,2} b_{1,0} h_{4,0} h_{3,1} h_{2,1} h_{1,3} h_{1,1}. \end{aligned}$$

For the convenience of writing,we make the following rules:

(i) if $i>j,$ we put a_i on the left side of a_j;

(ii) if $j<k,$ we put $h_{i,j}$ on the left side of $h_{w,k}$;

(iii) if $i>w,$ we put $h_{i,j}$ on the left side of $h_{w,j}$;

(iv) apply the rules (ii) and (iii) to $b_{i,j}$.

Now we give the proof of the above lemma.

Proof The proof of this lemma is divided into the following six cases. Consider

$$h=x_1x_2\cdots x_m\in E_1^{s+9,t(s),*}$$

in the MSS, where x_i is one of a_k, $h_{r,j}$ or $b_{u,z}$, $0\leqslant k\leqslant 4$, $0\leqslant r+j\leqslant 4$, $0\leqslant u+z\leqslant 3$, $r>0$, $j\geqslant 0$, $u>0$, $z\geqslant 0$. By (2.5), we can assume that $\deg x_i=q(c_{i,3}p^3+c_{i,2}p^2+c_{i,1}p+c_{i,0})+e_i$, where $c_{i,j}=0$ or 1, $e_i=1$ if $x_i=a_{k_i}$ or $e_i=0$. It follows that $\dim h=\sum\limits_{i=1}^{m}\dim x_i=s+9$ and

$$\begin{aligned}\deg h &= \sum_{i=1}^{m}\deg x_i=q[(\sum_{i=1}^{m}c_{i,3})p^3+(\sum_{i=1}^{m}c_{i,2})p^2+(\sum_{i=1}^{m}c_{i,1})p+(\sum_{i=1}^{m}c_{i,0})]+(\sum_{i=1}^{m}e_i)\\ &= q[(s+4)p^3+(s+3)p^2+(s+5)p+(s+1)]+s.\end{aligned}$$

Note that

$$\dim h_{i,j}=\dim a_i=1,\ \ \dim b_{i,j}=2\ \text{ and }\ 0\leqslant s<p-5.$$

From $\dim h=\sum\limits_{i=1}^{m}\dim x_i=s+9$, we can have $m\leqslant s+9\leqslant p+3$.

Using $0\leqslant s+5,s+4,s+3,s+1,s<p$ and the knowledge on the p-adic expression in number theory, we have that

$$\begin{cases}\sum\limits_{i=1}^{m}e_i=s;\\ \sum\limits_{i=1}^{m}c_{i,0}=s+1;\\ \sum\limits_{i=1}^{m}c_{i,1}=s+5;\\ \sum\limits_{i=1}^{m}c_{i,2}=s+3;\\ \sum\limits_{i=1}^{m}c_{i,3}=s+4.\end{cases}\tag{3.1}$$

By $c_{i,2}=0$ or 1, one has $m\geqslant s+4$ from $\sum\limits_{i=1}^{m}c_{i,3}=s+4$. Note that $m\leqslant s+9$. Thus m may equal $s+4$, $s+5$, $s+6$, $s+7$, $s+8$ or $s+9$. Since $\sum\limits_{i=1}^{m}e_i=s$, $\deg h_{i,j}\equiv 0(\bmod\ q)$ $(i>0,j\geqslant 0)$, $\deg a_i\equiv 1(\bmod\ q)$ $(i\geqslant 0)$ and $\deg b_{i,j}\equiv 0(\bmod\ q)$ $(i>0,j\geqslant 0)$, then by the graded commutativity of $E_1^{*,*,*}$ and degree reasons, we can assume that $h=a_0^xa_1^ya_2^za_3^ka_4^lh'$ with $h'=x_{s+1}x_{s+2}\cdots x_m$, where $0\leqslant x,y,z,k,l\leqslant s$, $x+y+z+k+l=s$. Consequently, we have

$$h'=x_{s+1}x_{s+2}\cdots x_m\in E_1^{9,t_2(s),*},$$

where $t_2(s) = q[(s+4-l)p^3 + (s+3-l-k)p^2 + (s+5-l-k-z)p + (s+1-l-k-z-y)]$. From (3.1), we have

$$\begin{cases} \sum\limits_{i=s+1}^{m} e_i = 0; \\ \sum\limits_{i=s+1}^{m} c_{i,0} = s+1-l-k-z-y; \\ \sum\limits_{i=s+1}^{m} c_{i,1} = s+5-l-k-z; \\ \sum\limits_{i=s+1}^{m} c_{i,2} = s+3-l-k; \\ \sum\limits_{i=s+1}^{m} c_{i,3} = s+4-l. \end{cases} \tag{3.2}$$

By the reason of dimension, all the possibilities of h' can be listed as

$$y_1z_1\cdots z_4,\ \ y_1y_2y_3z_1z_2z_3,\ \ y_1\cdots y_5z_1z_2,\ \ y_1\cdots y_7z_1,\ \ y_1\cdots y_9,$$

where y_i is in the form of $h_{r,j}$with $0 \leqslant r+j \leqslant 4$, $r > 0$, $j \geqslant 0$ and $z_i$is in the form of $b_{u,z}$ with $0 \leqslant u+z \leqslant 3$, $u > 0$, $z \geqslant 0$.

Case 1 $m = s+4$. So $h' = x_{s+1}x_{s+2}x_{s+3}x_{s+4} \in E_1^{9,q(4p^3+3p^2+5p+1),*}$ and it is impossible to exist. Then h doesn't exist either.

Case 2 $m = s+5$. From $\sum\limits_{i=s+1}^{s+5} c_{i,3} = s+4-l$ in (3.2), we have that $l = s+4-\sum\limits_{i=s+1}^{s+5} c_{i,3} \geqslant s-1$. Thus $l = s-1$ or s and $h' = y_1z_1\cdots z_4 \in E_1^{9,t_2(s),*}$. We list all the possibilities in Table 1.

Table 1: for Case 2

The possibility	l	x	y	z	k	$E_1^{9,t_2(s),*}$	The existence of $h' = x_{s+1}\cdots x_m$
The 1st	$s-1$	1	0	0	0	$E_1^{9,q(5p^3+4p^2+6p+2),*} = 0$	Nonexistence
The 2nd	$s-1$	0	1	0	0	$E_1^{9,q(5p^3+4p^2+6p+1),*} = 0$	Nonexistence
The 3rd	$s-1$	0	0	1	0	$E_1^{9,q(5p^3+4p^2+5p+1),*} = 0$	Nonexistence
The 4th	$s-1$	0	0	0	1	$E_1^{9,q(5p^3+3p^2+5p+1),*} = 0$	Nonexistence
The 5th	s	0	0	0	0	$E_1^{9,q(4p^3+3p^2+5p+1),*} = 0$	Nonexistence

Case 3 $m = s+6$. From $\sum\limits_{i=s+1}^{s+6} c_{i,3} = s+4-l$ in (3.2), we have that $l = s+4-\sum\limits_{i=s+1}^{s+6} c_{i,3} \geqslant s-2$. Thus $l = s-2$,$s-1$ or s and $h' = y_1y_2y_3z_1z_2z_3 \in E_1^{9,t_2(s),*}$. We list all the possibilities in Table 2.

In the table, $b_{3,0}^2b_{1,0}h_{4,0}h_{3,1}h_{1,3}$, $b_{3,0}^3h_{4,0}h_{1,3}h_{1,1}$, $b_{3,0}^2b_{1,2}h_{4,0}h_{3,1}h_{1,1}$, $b_{3,0}b_{1,0}^2h_{4,0}h_{3,1}h_{1,3}$, $b_{3,0}^2b_{1,0}h_{4,0}h_{1,3}h_{1,1}$, $b_{3,0}b_{1,2}b_{1,0}h_{4,0}h_{3,1}h_{1,1}$, denoted by $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$, $\mathbf{g}_4$, $\mathbf{g}_5$, $\mathbf{g}_6$, respectively. Consequently, in this case up to sign $h = a_4^{s-1}a_2\mathbf{g}_1$, $a_4^{s-1}a_2\mathbf{g}_2$, $a_4^{s-1}a_2\mathbf{g}_3$, $a_4^s\mathbf{g}_4$,$a_4^s\mathbf{g}_5$,$a_4^s\mathbf{g}_6$denoted by $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$, $\mathbf{G}_4$, $\mathbf{G}_5$, $\mathbf{G}_6$, respectively.

Case 4 $m = s+7$. From $\sum\limits_{i=s+1}^{s+7} c_{i,3} = s+4-l$ in (3.2), we have that $l = s+4-\sum\limits_{i=s+1}^{s+7} c_{i,3} \geqslant s-3$. Thus $l = s-3$, $s-2$, $s-1$ or s, and $h' = y_1\cdots y_5z_1z_2 \in E_1^{9,t_2(s),*}$. When $l = s-3$, we

Table 2: for Case 3

The possibility	l	x	y	z	k	$E_1^{9,t_2(s),*}$	The existence of $h'=x_{s+1}\cdots x_m$
The 1st	$s-2$	2	0	0	0	$E_1^{9,q(6p^3+5p^2+7p+3),*}=0$	Nonexistence
The 2nd	$s-2$	0	2	0	0	$E_1^{9,q(6p^3+5p^2+7p+1),*}=0$	Nonexistence
The 3rd	$s-2$	0	0	2	0	$E_1^{9,q(6p^3+5p^2+5p+1),*}=0$	Nonexistence
The 4th	$s-2$	0	0	0	2	$E_1^{9,q(6p^3+3p^2+5p+1),*}=0$	Nonexistence
The 5th	$s-2$	1	1	0	0	$E_1^{9,q(6p^3+5p^2+7p+2),*}=0$	Nonexistence
The 6th	$s-2$	1	0	1	0	$E_1^{9,q(6p^3+5p^2+6p+2),*}=0$	Nonexistence
The 7th	$s-2$	1	0	0	1	$E_1^{9,q(6p^3+4p^2+6p+2),*}=0$	Nonexistence
The 8th	$s-2$	0	1	1	0	$E_1^{9,q(6p^3+5p^2+6p+1),*}=0$	Nonexistence
The 9th	$s-2$	0	1	0	1	$E_1^{9,q(6p^3+4p^2+6p+1),*}=0$	Nonexistence
The 10th	$s-2$	0	0	1	1	$E_1^{9,q(6p^3+4p^2+5p+1),*}=0$	Nonexistence
The 11th	$s-1$	1	0	0	0	$E_1^{9,q(5p^3+4p^2+6p+2),*}=0$	Nonexistence
The 12th	$s-1$	0	1	0	0	$E_1^{9,q(5p^3+4p^2+6p+1),*}=0$	Nonexistence
The 13th	$s-1$	0	0	1	0	$E_1^{9,q(5p^3+4p^2+5p+1),*}$ $=\mathbb{Z}_p\{\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3\}$	$h'=\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3$ up to sign
The 14th	$s-1$	0	0	0	1	$E_1^{9,q(5p^3+3p^2+5p+1),*}=0$	Nonexistence
The 15th	s	0	0	0	0	$E_1^{9,q(4p^3+3p^2+5p+1),*}$ $=\mathbb{Z}_p\{\mathbf{g}_4,\mathbf{g}_5,\mathbf{g}_6\}$	$h'=\mathbf{g}_4,\mathbf{g}_5,\mathbf{g}_6$ up to sign

Table 3: for Case 4

The possibility	l	x	y	z	k	$E_1^{9,t_2(s),*}$	The existence of $h'=x_{s+1}\cdots x_m$
The 1st	$s-2$	2	0	0	0	$E_1^{9,q(6p^3+5p^2+7p+3),*}=0$	Nonexistence
The 2nd	$s-2$	0	2	0	0	$E_1^{9,q(6p^3+5p^2+7p+1),*}=0$	Nonexistence
The 3rd	$s-2$	0	0	2	0	$E_1^{9,q(6p^3+5p^2+5p+1),*}$ $=\mathbb{Z}_p\{\mathbf{g}_7\}$	$h'=\mathbf{g}_7$ up to sign
The 4th	$s-2$	0	0	0	2	$E_1^{9,q(6p^3+3p^2+5p+1),*}=0$	Nonexistence
The 5th	$s-2$	1	1	0	0	$E_1^{9,q(6p^3+5p^2+7p+2),*}=0$	Nonexistence
The 6th	$s-2$	1	0	1	0	$E_1^{9,q(6p^3+5p^2+6p+2),*}=0$	Nonexistence
The 7th	$s-2$	1	0	0	1	$E_1^{9,q(6p^3+4p^2+6p+2),*}=0$	Nonexistence
The 8th	$s-2$	0	1	1	0	$E_1^{9,q(6p^3+5p^2+6p+1),*}=0$	Nonexistence
The 9th	$s-2$	0	1	0	1	$E_1^{9,q(6p^3+4p^2+6p+1),*}=0$	Nonexistence
The 10th	$s-2$	0	0	1	1	$E_1^{9,q(6p^3+4p^2+5p+1),*}=0$	Nonexistence
The 11th	$s-1$	1	0	0	0	$E_1^{9,q(5p^3+4p^2+6p+2),*}=0$	Nonexistence
The 12th	$s-1$	0	1	0	0	$E_1^{9,q(5p^3+4p^2+6p+1),*}=0$	Nonexistence
The 13th	$s-1$	0	0	1	0	$E_1^{9,q(5p^3+4p^2+5p+1),*}$ $=\mathbb{Z}_p\{\mathbf{g}_8,\mathbf{g}_9\}$	$h'=\mathbf{g}_8,\mathbf{g}_9$ up to sign
The 14th	$s-1$	0	0	0	1	$E_1^{9,q(5p^3+3p^2+5p+1),*}=0$	Nonexistence
The 15th	s	0	0	0	0	$E_1^{9,q(4p^3+3p^2+5p+1),*}$ $=\mathbb{Z}_p\{\mathbf{g}_{10},\mathbf{g}_{11}\}$	$h'=\mathbf{g}_{10},\mathbf{g}_{11}$ up to sign

have that $t_2(s) = q[7p^3 + \cdots]$. In this case, h' is impossible to exist. Then h doesn't exist either. Next we list all the rest of possibilities in Table 3.

In the table, $b_{3,0}^2 h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,1}$, $b_{3,0}b_{1,0}h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,1}$, $b_{3,0}b_{1,2}h_{4,0}h_{3,1}h_{2,1}h_{1,3}h_{1,1}$, $b_{1,0}^2 h_{4,0}h_{3,1}h_{2,2}h_{1,3}h_{1,1}$, $b_{1,2}b_{1,0}h_{4,0}h_{3,1}h_{2,1}h_{1,3}h_{1,1}$, denoted by $\mathbf{g}_7$, $\mathbf{g}_8$, $\mathbf{g}_9$, $\mathbf{g}_{10}$, $\mathbf{g}_{11}$, respectively. Consequently, in this case up to sign $h = a_4^{s-2}a_2^2\mathbf{g}_7$, $a_4^{s-1}a_2\mathbf{g}_8$, $a_4^{s-1}a_2\mathbf{g}_9$, $a_4^s\mathbf{g}_{10}$,$a_4^s\mathbf{g}_{11}$ denoted by $\mathbf{G}_7$, $\mathbf{G}_8$, $\mathbf{G}_9$, $\mathbf{G}_{10}$, $\mathbf{G}_{11}$, respectively.

Case 5 $m = s+8$. From $\sum\limits_{i=s+1}^{s+8} c_{i,3} = s+4-l$ in (3.2), we have that $l = s+4-\sum\limits_{i=s+1}^{s+8} c_{i,3} \geqslant s-4$. Thus $l = s-4$, $s-3$, $s-2$, $s-1$ or s, and $h' = y_1\cdots y_7z_1 \in E_1^{9,t_2(s),*}$. When $l \leqslant s-2$, the coefficient of $P^3 \in t_2(s)$is > 5. In these cases, h' is impossible to exist. Then h doesn't exist either. Next we list all the other possibilities in Table 4.

Table 4: for Case 5

The possibility	l	x	y	z	k	$E_1^{9,t_2(s),*}$	The existence of $h' = x_{s+1}\cdots x_m$
The 1st	$s-1$	1	0	0	0	$E_1^{9,q(5p^3+4p^2+6p+2),*} = 0$	Nonexistence
The 2nd	$s-1$	0	1	0	0	$E_1^{9,q(5p^3+4p^2+6p+1),*} = 0$	Nonexistence
The 3rd	$s-1$	0	0	1	0	$E_1^{9,q(5p^3+4p^2+5p+1),*} = 0$	Nonexistence
The 4th	$s-1$	0	0	0	1	$E_1^{9,q(5p^3+3p^2+5p+1),*} = 0$	Nonexistence
The 5th	s	0	0	0	0	$E_1^{9,q(4p^3+3p^2+5p+1),*} = 0$	Nonexistence

Case 6 $m = s+9$. From $\sum\limits_{i=s+1}^{s+9} c_{i,3} = s+4-l$ in (3.2), we have that $l = s+4-\sum\limits_{i=s+1}^{s+9} c_{i,3} \geqslant s-5$. Thus $l = s-5$, $s-4$, $s-3$, $s-2$, $s-1$ or s, and $h' = y_1\cdots y_9 \in E_1^{9,t_2(s),*}$. When $l \leqslant s-1$, the coefficient of $P^3 \in t_2(s)$ is $\geqslant 5$. In these cases, h' is impossible to exist. Then h doesn't exist either. In the last possibility, $t_2(s) = 4p^3+3p^2+5p+1$, so $h_{4,0}, h_{3,1}, h_{2,2}, h_{1,3} \in h'$, $h'$is impossible to exist in this case by the reason of dimension. Then h doesn't exist either.

Combining Cases 1–6 above, we obtain that $E_1^{s+9,t(s),*} = \mathbb{Z}_p\{\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_{11}\}$. This completes the proof of Lemma 3.2.

Lemma 3.3 (1) $b_0^3\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$ is represented by $b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+10,t(s),*}$ in the MSS, where $t(s) = q[(s+4)p^3 + (s+3)p^2 + (s+5)p + (s+1)] + s$.

(2) For the eleven generators of $E_1^{s+9,t(s),*}$, we have that

$$
\begin{aligned}
&M(\mathbf{G}_1) = M(\mathbf{G}_3) = M(\mathbf{G}_5) = 11p + 9s + 9,\\
&M(\mathbf{G}_2) = 15p + 9s + 5,\ \ M(\mathbf{G}_4) = M(\mathbf{G}_6) = 7p + 9s + 13,\\
&M(\mathbf{G}_7) = 10p + 9s + +9,\ \ M(\mathbf{G}_8) = M(\mathbf{G}_9) = 6p + 9s + 13,\\
&M(\mathbf{G}_{10}) = M(\mathbf{G}_{11}) = 2p + 9s + 17.
\end{aligned}
$$

Moreover, we have that $M(b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}) = 3p + 9s + 16$.

Proof (1) Since it is known that $b_{1,i}$ and $a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{*,*,*}$ are all permanent cycles in the MSS as [7] and converge nontrivially to b_i, $\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ for $0 \leqslant s < p-5$

and $i \geqslant 0$, respectively (cf. Lemma 3.1), then $b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \in E_1^{s+10,t(s),3p+9s+16}$ is a permanent cycle in the MSS and converges to $b_0^3 \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$.

(2) From (2.5), the result follows by direct calculation.

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2 From Lemma 3.3 (1), $b_0^3 \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$ is represented by $b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \in E_1^{s+10,t(s),3p+9s+16}$ in the MSS. Now we will show that nothing hits the permanent cycle $b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3}$ under the May differential d_r for $r \geqslant 1$. From Lemma 3.2, we have $E_1^{s+9,t(s),*} = \mathbb{Z}_p\{\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_1 1\}$.

For the generators $\mathbf{G}_1$, $\mathbf{G}_3$ and $\mathbf{G}_5$ whose May filtration are

$$M(\mathbf{G}_1) = M(\mathbf{G}_3) = M(\mathbf{G}_5) = 11p + 9s + 9$$

(see Lemma 3.3), by the reason of May filtration, from (2.2) we see that

$$b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \in E_1^{s+10,t(s),3p+9s+16},$$

which represents $b_0^3 \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p, \mathbb{Z}_p)$ in the MSS is not in $d_1(E_1^{s+9,t(s),11p+9s+9})$. Now we will show $E_r^{s+9,t(s),11p+9s+9} = 0$ for $r \geqslant 2$. By an easy calculation, from (2.3) and (2.4), one can have the first May differentials of $\mathbf{G}_1$, $\mathbf{G}_3$ and $\mathbf{G}_5$ as follows

$$\begin{aligned}
d_1(\mathbf{G}_1) &= (-1)^{s+8} a_4^{s-1} a_2 b_{3,0}^2 b_{1,0} h_{3,1} h_{2,2} h_{2,0} h_{1,3} + \cdots \neq 0, \\
d_1(\mathbf{G}_3) &= (-1)^{s+8} a_4^{s-1} a_2 b_{3,0}^2 b_{1,2} h_{3,1} h_{2,2} h_{2,0} h_{1,1} + \cdots \neq 0, \\
d_1(\mathbf{G}_5) &= (-1)^{s+8} a_4^{s} b_{3,0}^2 b_{1,0} h_{2,2} h_{2,0} h_{1,3} h_{1,1} + \cdots \neq 0.
\end{aligned}$$

It is easy to see that the first May differentials of $\mathbf{G}_1$,$\mathbf{G}_3$and $\mathbf{G}_5$ are linearly independent.Consequently, the cocycle of $E_1^{s+9,t(s),11p+9s+9}$ must be zero. This means that $E_r^{s+9,t(s),11p+9s+9} = 0$ for $r \geqslant 2$, from which we have that

$$b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \notin d_r(E_r^{s+9,t(s),11p+9s+9})$$

for $r \geqslant 2$. In all,$b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \notin d_r(E_r^{s+9,t(s),11p+9s+9})$ for $r \geqslant 1$.

For the generator $\mathbf{G}_2$ with May filtration $M(\mathbf{G}_2) = 15p + 9s + 5$ (see Lemma 3.3), by an easy calculation, from (2.3) and (2.4), we have the first May differentials of $\mathbf{G}_2$ as follows

$$d_1(\mathbf{G}_2) = (-1)^{s+8} a_4^{s-1} a_2 b_{3,0}^2 h_{3,1} h_{1,3} h_{1,1} h_{1,0} + \cdots \neq 0.$$

Thus $E_r^{s+9,t(s),15p+9s+5} = 0$ for $r \geqslant 2$. At the same time, we also have that up to nonzero scalar $d_1(\mathbf{G}_2) \neq b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3}$.

In summary, $b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \notin d_r(E_r^{s+9,t(s),15p+9s+5})$ for $r \geqslant 1$.

For the generators $\mathbf{G}_4$ and $\mathbf{G}_6$ whose May filtration are $M(\mathbf{G}_4) = M(\mathbf{G}_6) = 7p+9s+13$ (see Lemma 3.3), by the reason of May filtration, from (2.2) we see that

$$b_{1,0}^3 a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \in E_1^{s+10,t(s),3p+9s+16},$$

which represents $b_0^3\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p) \notin d_1(E_1^{s+9,t(s),7p+9s+13})$. Now we will show $E_r^{s+9,t(s),7p+9s+13} = 0$ for $r \geqslant 2$. By an easy calculation, from (2.3) and (2.4) one can have the first May differentials of $\mathbf{G}_4$ and $\mathbf{G}_6$ as follows

$$d_1(\mathbf{G}_4) = (-1)^{s+8}a_4^s b_{3,0}b_{1,0}^2 h_{3,1}h_{2,2}h_{2,0}h_{1,3} + \cdots \neq 0,$$
$$d_1(\mathbf{G}_6) = (-1)^{s+8}a_4^s b_{3,0}b_{1,2}b_{1,0}h_{3,1}h_{2,2}h_{2,0}h_{1,1} + \cdots \neq 0.$$

It is easy to see that the first May differentials of $\mathbf{G}_4$ and $\mathbf{G}_6$ are linearly independent. Consequently, the cocycle of $E_1^{s+9,t(s),7p+9s+13}$ must be zero. This means that

$$E_r^{s+9,t(s),7p+9s+13} = 0$$

for $r \geqslant 2$, from which we have that $b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \notin d_r(E_r^{s+9,t(s),7p+9s+13})$ for $r \geqslant 2$. In all, $b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \notin d_r(E_r^{s+9,t(s),7p+9s+13})$ for $r \geqslant 1$.

For the generator $\mathbf{G}_7$ with May filtration $M(\mathbf{G}_7) = 10p + 9s + 9$ (see Lemma 3.3), by an easy calculation, from (2.3) and (2.4) we have the first May differentials of $\mathbf{G}_7$ as follows

$$d_1(\mathbf{G}_7) = (-1)^{s+8}a_4^{s-2}a_2a_0b_{3,0}^2 h_{4,0}h_{3,1}h_{2,2}h_{2,0}h_{1,3}h_{1,1} + \cdots \neq 0.$$

Thus $E_r^{s+9,t(s),10p+9s+9} = 0$ for $r \geqslant 2$. At the same time, we also have that up to nonzero scalar $d_1(\mathbf{G}_7) \neq b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$.

In summary, $b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \notin d_r(E_r^{s+9,t(s),10p+9s+9})$ for $r \geqslant 1$.

Finally,for the generators $\mathbf{G}_8$ and $\mathbf{G}_9$ whose May filtration are $M(\mathbf{G}_8) = M(\mathbf{G}_9) = 6p + 9s + 13$ (see Lemma 3.3), by the reason of May filtration, from (2.2) we see that $b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+10,t(s),3p+9s+16}$, which represents $b_0^3\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the MSS is not in $d_r(E_1^{s+9,t(s),6p+9s+13})$ for $r \geqslant 1$.

The discussion of $\mathbf{G}_{10}$ and $\mathbf{G}_{11}$ whose May filtration are $M(\mathbf{G}_{10}) = M(\mathbf{G}_{11}) = 2p + 9s + 17$ is just like the analysis about $\mathbf{G}_{10}$ and $\mathbf{G}_{11}$.

From the above discussion, we see the permanent cycle $b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential in the MSS. Thus, $b_{1,0}^3 a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+10,t(s),3p+9s+16}$ converges nontrivially to $b_0^3\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the MSS. Consequently, $b_0^3\tilde{\delta}_{s+4} \neq 0$. This finishes the proof of Theorem 1.2.

Remark For further study on the typesetting based on English-Chinese LaTeX and some special techniques, we may refer to [1–7].

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Steenrod代数上同调中的一个非平凡乘积元$b_0^3\tilde{\delta}_{s+4}$

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摘要: 本文主要研究了Steenrod代数上同调非平凡乘积元问题. 设p为大于5的素数, A代表模p的Steenrod代数. 通过对May谱序列的详尽组合分析, 证明了古典Admas谱序列中乘积元$-b_0^3\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+10,t(s)}(\mathbb{Z}_p,\mathbb{Z}_p)$的非平凡性, 其中$p \geqslant 7$, $0 \leqslant s < p-5$, $t(s) = 2(p-1)[(s+4)p^3+(s+3)p^2+(s+5)p+(s+1)]+s$. 这有助于对球面稳定同伦群中同伦元素非平凡性进行进一步研究.

关键词: Steenrod代数; 上同调; May谱序列

MR(2010)主题分类号: 55Q45 中图分类号: O152.4