EXISTENCE OF SOLUTIONS TO THE INITIAL VALUE PROBLEM OF SEMI-LINEAR GENERALIZED TRICOMI EQUATION

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Abstract: In this paper, we study the existence of solutions to the initial value problem of semi-linear generalized Tricomi equation with characteristic families coincided on \( t = 0 \). Based on the inequality of \( \dot{H}^s \) estimates of two Fourier integral operators, we establish the local and global existence of solution for the semi-linear equation in hyperbolic half space. Meanwhile, we give the loss of regularities on degenerate domain and the decay rates when time tends to infinity in the weighted estimates.

Keywords: generalized Tricomi equation; semi-linear; weighted estimate; existence

2010 MR Subject Classification: 35L15; 35L71; 35L80

1 Introduction and Main Results

In this paper we study the local and global existence of solution semi-linear generalized Tricomi equation in \( \mathbb{R}^+ \times \mathbb{R}^n, \ n \geq 2 \),

\[ \partial_t^2 u - t^m \Delta u = t^\alpha u|u|^\beta \]  

(1.1)

with the initial value

\[ u(t, x)|_{t=0} = \varphi_1(x), \partial_t u(t, x)|_{t=0} = \varphi_2(x), \]  

(1.2)

where \( \Delta = \sum_{i=1}^{n} \partial_{x_i}^2 \) is the Laplace operator, \( m > 0, \beta > 0 \) and some constant \( \alpha \).

Equation (1.1) is a semi-linear hyperbolic equation with variable coefficient and characteristic families coincide on \( t = 0 \). There exist three fundamental features. First, variable coefficient with \( t \) makes the well known Duhamel’s principle of wave equation do not work. Then, we have to estimate the solution to the linear equation with source term. Second, its multiple characteristics cause some loss of regularities on degenerate domain. This leads to
the difficulty of local solvability. Third, the unbounded speed of propagation is an obstacle in establishing the global solution for the needed uniform decay estimate.

For $m = 1$ and $n = 1$, the homogeneous linear equation is the classic Tricomi equation, which was extensively investigated with suitable boundary value conditions from various viewpoints, such as [1–3] and the references therein. For $n > 1$, the local existence of solution to the equation $\partial_t^2 u - t^m \Delta u = f(t, x, u)$ with one initial datum $u(0, x) \in H^s(\mathbb{R}^n)$ ($s > \frac{2}{n}$) was established in mixed-type domain in [4, 5]. Meanwhile, the propagation of weakly singularity along characteristics was studied. The low regularity solution problem of the equation with two initial data was considered in [6–7] and the positive answers were obtained. In [6], the local existence of solution was established for the given discontinuous initial data and nonlinear source term with some restrictions. In [7], the global existence and nonexistence of solutions were considered in $L^q(\mathbb{R}^n)$ under some conditions and the decay of solutions when time trends to infinity was given. With respect to other specific cases, so far there were existence results of solution such as [8–10]. In this note, we focus on the local and global existence of weak solutions to the initial value problem of semi-linear generalized Tricomi equation under some general conditions compared with the result given in [7], and establish a uniform weighted estimate in homogeneous Sobolev space $\dot{H}_q^s(\mathbb{R}^n)$,

$$\dot{H}_q^s(\mathbb{R}^n) = \{ f(x) : \|f\|_{\dot{H}_q^s(\mathbb{R}^n)} = \|(-\nabla)^{\frac{s}{2}} f(x)\|_{L^q(\mathbb{R}^n)} = \|(|\xi|^s f^\wedge(\xi))'(x)\|_{L^q(\mathbb{R}^n)} < \infty\}, \quad (1.3)$$

where $f^\wedge(\xi) = \int_{\mathbb{R}^n} e^{-i(x, \xi)} f(x) dx$ and $g^\wedge(x)$ is the corresponding inverse Fourier transformation.

Now we state our main results.

**Theorem 1.1** Assume $\frac{1}{p} + \frac{1}{q} = 1$, $p = \frac{\beta + 2}{\beta + 1}$, $\frac{2}{m+2} \geq \frac{n\beta}{2+\beta}$ and $n - (\beta + 1)p(\frac{2}{2+\beta} - \frac{n\beta}{2+\beta}) = 1$. Given initial data $\phi_i(x) \in C_c^\infty(\mathbb{R}^n)$, $i = 1, 2$, then there exists a unique solution $u \in C([0, T]; \dot{H}_q^\frac{n\beta}{2+\beta}(\mathbb{R}^n)) \cap C^1([0, T] : \dot{H}_q^\frac{n\beta}{2+\beta}(\mathbb{R}^n))$ to problem (1.1) with $\alpha = 0$ and some positive $T$, which satisfies

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{\dot{H}_q^\frac{n\beta}{2+\beta}(\mathbb{R}^n)} + \|\partial_t u(t, \cdot)\|_{\dot{H}_q^\frac{n\beta}{2+\beta}(\mathbb{R}^n)} \leq C(\|\varphi_1\|_{\dot{H}_q^\frac{n\beta}{2+\beta}(\mathbb{R}^n)} + \|\varphi_2\|_{L^p(\mathbb{R}^n)}).$$

**Theorem 1.2** Assume $\frac{1}{p} + \frac{1}{q} = 1$, $p = \frac{\beta + 2}{\beta + 1}$, $\frac{m+4}{m+2} \geq \frac{(n+1)\beta}{2(2+\beta)}$, $n - (\beta + 1)p(\frac{2}{2+\beta} - \frac{n\beta}{2+\beta}) = 1$ and

$$(\beta + 1)(\frac{m}{4} + \frac{(m+2)(n-1)\beta}{4(2+\beta)}) - \alpha = 1.$$ 

Given initial data $\phi_i(x) \in C_c^\infty(\mathbb{R}^n)$, $i = 1, 2$ with

$$\|\varphi_1\|_{\dot{H}_q^\frac{n\beta}{2+\beta}(\mathbb{R}^n)} + \|\varphi_2\|_{L^p(\mathbb{R}^n)} < \epsilon$$

for small $\epsilon$, then there exists a unique solution

$$u \in C((0, \infty); \dot{H}_q^\frac{m+4}{m+2}(\mathbb{R}^n)) \cap C^1((0, \infty); \dot{H}_q^\frac{(n+1)\beta}{2(2+\beta)}(\mathbb{R}^n))$$
to problem (1.1), which satisfies
\[
\begin{align*}
\sup_{t \in (0, \infty)} t^{\frac{(m+2)(n-1)\beta}{4+4\beta}} \| t^\alpha H_\gamma \partial_t u(t, \cdot) \|_{L^p(\mathbb{R}^n)} + \| t^\alpha \partial_t u(t, \cdot) \|_{L^p(\mathbb{R}^n)} & \\
\leq C \| \varphi_1 \|_{H^m_{\delta}(\mathbb{R}^n)} + \| \varphi_2 \|_{L^p(\mathbb{R}^n)}.
\end{align*}
\] (1.4)

**Remark 1.1** Generalized Tricomi equation has the property of “smooth effect” as we pointed out in [5]. In this note, by compared with the regularity given in Theorem 1.1 and Theorem 1.2 near \( t = 0 \), we note that the solution has a higher regularity away from the degenerate domain.

**Remark 1.2** In homogeneous Sobolev space \( H^m_{\delta}(\mathbb{R}^n) \), the weights in the uniform estimate (1.4) of global solution in time describe the degeneracy near \( t = 0 \) and the decay as \( t \to +\infty \). However unlike in inhomogeneous Sobolev space \( H^m(\mathbb{R}^n) \) [5, 7], the author could not obtain those at the same time for the nonlinear problem.

**Remark 1.3** In fact, the embedding theorem in homogeneous Sobolev space in page 119 [11, p.119, Theorem 2] and inequality (1.4) yield \( u \in C([0, \infty); L^{n-\frac{m+1}{2(m+2)}}(\mathbb{R}^n)) \) such that
\[
\begin{align*}
\sup_{t \in (0, \infty)} t^{\frac{(m+2)(n-1)\beta}{4+4\beta}} \| u(t, \cdot) \|_{L^{n-\frac{m+1}{2(m+2)}}(\mathbb{R}^n)} & \\
\leq C \| \varphi_1 \|_{H^m_{\delta}(\mathbb{R}^n)} + \| \varphi_2 \|_{L^p(\mathbb{R}^n)}. 
\end{align*}
\]

Hence, for \( \varphi_1(x) \in C^\infty_0(\mathbb{R}^n) \), \( \varphi_2(x) \in C^\infty(\mathbb{R}^n) \), set \( \frac{(n-1)\beta}{2(n+2)} \geq \frac{m}{2(n+2)} \), then in terms of Theorem 1.1, the conditions in Theorem 1.2 imply that \( u \in C([0, \infty); L^{n-\frac{m+1}{2(m+2)}}(\mathbb{R}^n)) \).

This paper is organized as follows. In Section 2, for the later uses, we recite some preliminary results as our lemmas. In Section 3, by use of confluent hypergeometric functions, we derive some weighted homogeneous Sobolev regularity estimates for the corresponding inhomogeneous equation. Based on these estimates, we establish the local and global existence of solutions in Section 4 by constructing a contraction map.

## 2 Preliminaries

In this section, for reader’s convenience, we will recall some fundamental results of confluent hypergeometric functions and useful estimates, which will be used in Section 3 below.

The confluent hypergeometric equation is \( zw''(z) + (c-z)w'(z) - aw(z) = 0 \), where \( z \in \mathbb{C} \), \( a \) and \( c \) are constants. When \( c \) is not an integer, (2.1) has one pair of linearly independent solutions
\[
w_1(z) = \Phi(a, c; z), \quad w_2(z) = z^{1-c} \Phi(a-c + 1, 2 - c; z),
\] (2.1)

where \( \Phi(a, c; z) \) is Humbert’s symbol and \( w_i \) is called the confluent hypergeometric function.

The Wronskian determinant for the system \( w_1 \) and \( w_2 \) is
\[
W(w_1, w_2) = (1-c)z^{-c}e^z.
\] (2.2)
Next, we list some basic properties of the confluent hypergeometric functions which can be found in [12].

**Lemma 2.1**  
1) For $-\pi < \arg z < \pi$ and large $|z|$, then  
\[
\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(c - a)} (e^{\pi \tau} z^{-1})^a \sum_{n=0}^{M} \frac{(a - c + 1)_n (-z)^{-n}}{n!} + O(|z|^{-a-M-1})
\]
\[
+ \frac{\Gamma(c - a)}{\Gamma(a)} e^{z} z^{a-c} \sum_{n=0}^{N} \frac{(c - a)_n (1 - a)_n z^{-n}}{n!} + O(|z|^{-a-N-1}),
\]
(2.3)

where $\epsilon = 1$ if $\text{Im} z > 0$, $\epsilon = -1$ if $\text{Im} z < 0$, and $M, N = 0, 1, 2, 3, \cdots$.

2)  
\[
\frac{d}{dz} \Phi(a, c; z) = \frac{a}{c} \Phi(a + 1, c + 1; z).
\]
(2.4)

Then by (2.3) in Lemma 2.1, we obtain
\[
\left| \Phi\left(\frac{m}{2(m + 2)}, \frac{m}{m + 2}; |\xi|\right) \right| = C(|\xi|)^{-\frac{m}{m+2}} \left(1 + O\left(|\xi|^{-1}\right)\right), |\xi| \gg 1
\]
(2.5)
and
\[
\left| \Phi\left(\frac{m + 4}{2(m + 2)}, \frac{m + 4}{m + 2}; |\xi|\right) \right| = C(|\xi|)^{-\frac{m+4}{m+2}} \left(1 + O\left(|\xi|^{-1}\right)\right), |\xi| \gg 1.
\]
(2.6)

Next, we list Theorem 1.11 in [13] and Lemma 4 in [14] as our Lemma 2.2 and Lemma 2.3.

**Lemma 2.2** Let $f$ be a measurable function such that, with $1 < r < +\infty$, we have for some constant $C$,
\[
\text{meas}\{\xi : |f(\xi)| \geq k\} \leq Ck^{-r},
\]
(2.7)
then $f \in M^q_r$ if $1 < p \leq 2 \leq q < +\infty$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{r}$.

**Lemma 2.3** Let $P$ be real, $C^\infty$ in neighborhood of the support of $v \in C^\infty$. Assume that the rank of $H_\rho(y) = \partial^2_{y/y^m} P(y)$ is at least $\rho$ on the support of $v$. Then for some integer $M$,
\[
\|(e^{itP}v)^{\wedge}\|^\infty \leq C(1 + |t|)^{-\frac{2}{2}} \sum_{|\alpha| \leq M} \|D^\alpha v\|_1,
\]
(2.8)
where $C$ depends on bounds of the derivatives of $P$ on supp$(v)$ and on a lower bound of the maximum of the absolute values of the minors of order $\rho$ of $H_\rho$ on supp$(v)$, and on supp$(v)$.

### 3 Weighted Estimates of Solutions to the Linear Equation

In this section, we first take partial Fourier-transformation on $x \in \mathbb{R}^n$, then the linear equation $\partial^2_t v - t^m \Delta v = 0$ becomes
\[
y''(t, \xi) + t^m |\xi|^2 y(t, \xi) = 0,
\]
(3.1)

where $y(t, \xi) = \int_{\mathbb{R}^n} v(t, x) e^{-ix \cdot \xi} dx$ with $\xi \in \mathbb{R}^n$ and $y''(t, \xi) \equiv \partial^2_t y(t, \xi)$. 

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Set $z = z^{m} \frac{2 + m}{2(m + 2)} |\xi|$ and $w(z) = y(z) e \hat{z}$, then if $t \neq 0$ and $|\xi| \neq 0$, the equation (3.1) can be written as

$$zw''(z) + \left( \frac{m}{2 + m} - z \right) w'(z) - \frac{m}{2(2 + m)} w(z) = 0.$$  \hfill (3.2)

Thus (3.2) is a special case of the confluent hypergeometric equation with $c = \frac{m}{2 + m}$ and $a = \frac{m}{2(2 + m)}$.

Next, we recite the result in the reference [15].

**Lemma 3.1** The functions

\[ V_1(t, |\xi|) = e^{-\frac{1}{2} t \Phi(\frac{m}{2(2 + m)}, \frac{m}{2 + m}; z)}, \quad V_2(t, |\xi|) = t e^{-\frac{1}{2} t \Phi(\frac{4 + m}{2(2 + m)}, \frac{4 + m}{2 + m}; z)} \]

on $\mathbb{R} \times \mathbb{R}^n$, where $z = \frac{2 + m}{2(m + 2)} |\xi|$, form the fundamental solution system of (3.2), which satisfy

\[ V_1(0, |\xi|) = 1, \quad \partial_t V_1(0, |\xi|) = 0; \quad V_2(0, |\xi|) = 0, \quad \partial_t V_2(0, |\xi|) = 1. \]

Unlike the method of using maximum principle to solve the degenerate parabolic equation [16], here the hypergeometric functions play an important role. According to Lemma 2.1, it is easy to verify that $(V_1(1, |\xi|))$ and $(V_2(1, |\xi|))$ are two Fourier integral operators with symbols of order $-\frac{m}{2(m + 2)}$ or $-\frac{m + 4}{2(m + 2)}$. Then they are bounded in $L^p(\mathbb{R}^n)$. Moreover, the following results were obtained by taking a similar procedure in Theorem 3.1 [7] in terms of some micro-local representations.

**Lemma 3.2** For $\phi \in \dot{H}^s_p(\mathbb{R}^n)$, $n > 1$, set $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

\[ \| (V_1(t, |\xi|) \phi, \xi) \|^2 \leq C t^{- \frac{(2 + m)(s + n)(\frac{1}{p} - \frac{1}{q})}{2}} \| \phi \|_{H^s_p(\mathbb{R}^n)} \] \hfill (3.3)

with $-n(\frac{1}{p} - \frac{1}{q}) < s \leq \frac{m}{2(2 + m)} - \frac{1}{2}(n + 1)(\frac{1}{p} - \frac{1}{q})$ and

\[ \| (\partial_t V_1(t, |\xi|) \phi, \xi) \|^2 \leq C t^{- \frac{(2 + m)(s + n)(\frac{1}{p} - \frac{1}{q})}{2}} \| \phi \|_{H^s_p(\mathbb{R}^n)} \] \hfill (3.4)

with $-1 - n(\frac{1}{p} - \frac{1}{q}) < s < -\frac{m + 4}{2(2 + m)} - \frac{1}{2}(n + 1)(\frac{1}{p} - \frac{1}{q})$;

\[ \| (V_2(t, |\xi|) \phi, \xi) \|^2 \leq C t^{- \frac{(2 + m)(s + n)(\frac{1}{p} - \frac{1}{q})}{2}} \| \phi \|_{H^s_p(\mathbb{R}^n)} \] \hfill (3.5)

with $-n(\frac{1}{p} - \frac{1}{q}) < s \leq \frac{m + 4}{2(2 + m)} - \frac{1}{2}(n + 1)(\frac{1}{p} - \frac{1}{q})$ and

\[ \| (\partial_t V_2(t, |\xi|) \phi, \xi) \|^2 \leq C t^{- \frac{(2 + m)(s + n)(\frac{1}{p} - \frac{1}{q})}{2}} \| \phi \|_{H^s_p(\mathbb{R}^n)} \] \hfill (3.6)

with $-n(\frac{1}{p} - \frac{1}{q}) < s < -\frac{m + 4}{2(2 + m)} - \frac{1}{2}(n + 1)(\frac{1}{p} - \frac{1}{q})$.

**Remark 3.1** Inequalities (3.3)–(3.6) take effect for $p < 2 < q$ and $n \geq 2$.

Next, we consider the initial value problems of homogeneous equation

\[
\begin{cases}
\partial_t^2 u_0 - t^m \Delta u_0 = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^n, \\
u_0(0, x) = \varphi_1(x), \quad \partial_t u_0(0, x) = \varphi_2(x).
\end{cases}
\]  \hfill (3.7)
As direct results of Lemma 3.1 and Lemma 3.2, we obtain the representation of distributional solution
\[ u_0^s(t, \xi) = V_1(t, |\xi|) \phi_1^s(\xi) + V_2(t, |\xi|) \phi_2^s(\xi), \] (3.8)
and the following estimates.

**Lemma 3.3** Assume \( \phi_1(x) \in \dot{H}_p^{s \infty -\frac{n+1}{q} - \frac{1}{2}}(\mathbb{R}^n) \), \( \phi_2(x) \in \dot{H}_p^s(\mathbb{R}^n) \), then \( u_0(t, x) = (V_1(t, |\xi|) \nu_1^s(\xi) + V_2(t, |\xi|) \nu_2^s(\xi)) \) solves (3.7) and satisfies

\[
\begin{align*}
\| t^{-\frac{n+1}{q} - \frac{1}{2}} \|_{\dot{H}_p^{s \infty -\frac{n+1}{q} - \frac{1}{2}}} & + \| t^{-\frac{n+1}{q} - \frac{1}{2}} \|_{\dot{H}_p^s(\mathbb{R}^n)}; \\
\| u_0(t, \cdot) \|_{\dot{H}_p^{s \infty -\frac{n+1}{q} - \frac{1}{2}}(\mathbb{R}^n)} & + \| \tilde{\partial}_t u_0(t, \cdot) \|_{\dot{H}_p^s(\mathbb{R}^n)}; \\
\leq C(\| \phi_1 \|_{\dot{H}_p^{s \infty -\frac{n+1}{q} - \frac{1}{2}}(\mathbb{R}^n)} & + \| \phi_2 \|_{\dot{H}_p^s(\mathbb{R}^n)})
\end{align*}
\] (3.9)

for \( 1 < p < 2, \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof** In terms of the representation of \( u_0(t, x) \) defined in (3.8), by use of the inequalities (3.3)–(3.6) with the value of \( s \) fixed on the right endpoint number separately, it is easy to verify that (3.9) holds. And (3.10) can be derived by choosing the value of \( s \) with the corresponding left endpoint number.

Last, we consider the inhomogeneous problem

\[
\begin{cases}
\partial_t^2 v - t^m \triangle v = f(t, x) & \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \\
v(0, x) = 0, \quad \partial_t v(0, x) = 0.
\end{cases}
\] (3.11)

In terms of Lemma 3.1 and a direct computation with (2.2), the distribution solution of (3.11) can be expressed as

\[ v^\wedge(t, \xi) \equiv (Ef)^\wedge(t, \xi) = \int_0^t (V_2(t, |\xi|)V_1(\tau, |\xi|) - V_1(t, |\xi|)V_2(\tau, |\xi|)) f^\wedge(\tau, \xi)d\tau. \] (3.12)

Then we conclude

**Lemma 3.4** If \( f(t, x) \in C([0, \infty), \dot{H}_p^s(\mathbb{R}^n)) \), then

\[
\begin{align*}
\| t^{-\frac{n+1}{q} - \frac{1}{2}} Ef(t, \cdot) \|_{\dot{H}_p^{s \infty -\frac{n+1}{q} - \frac{1}{2}}} & + \| t^{-\frac{n+1}{q} - \frac{1}{2}} \|_{\dot{H}_p^s(\mathbb{R}^n)}; \\
\| Ef(t, \cdot) \|_{\dot{H}_p^{s \infty -\frac{n+1}{q} - \frac{1}{2}}(\mathbb{R}^n)} & + \| \tilde{\partial}_t Ef(t, \cdot) \|_{\dot{H}_p^s(\mathbb{R}^n)}; \\
\leq C \int_0^t \| f(\tau, \cdot) \|_{\dot{H}_p^s(\mathbb{R}^n)}d\tau \\
\leq C \int_0^t \| f(\tau, \cdot) \|_{\dot{H}_p^s(\mathbb{R}^n)}d\tau
\end{align*}
\] (3.13) (3.14)

for \( 1 < p < 2, \frac{1}{p} + \frac{1}{q} = 1 \).
Hence, by use of (17) in Lemma 3.2, we derive
\[
(\partial_t E)(t, \xi) = \int_0^t (\partial_t V_2(t, |\xi|)V_1(\tau, |\xi|) - \partial_t V_1(t, |\xi|)V_2(\tau, |\xi|))f^\wedge(\tau, \xi)d\tau, \tag{3.15}
\]
then it is easy to verify that
\[
\|(-\Delta)^\frac{s}{2} E(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C \int_0^t \|(-\Delta)^\frac{s}{2} V_2(t, |\xi|)V_1(\tau, |\xi|)f^\wedge(\tau, \xi)\|_{L^2(\mathbb{R}^n)}
+ \|(-\Delta)^\frac{s}{2} V_1(t, |\xi|)V_2(\tau, |\xi|)f^\wedge(\tau, \xi)\|_{L^2(\mathbb{R}^n)}d\tau \tag{3.16}
\]
and
\[
\|(-\Delta)^\frac{s}{2} \partial_t E(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C \int_0^t \|(-\Delta)^\frac{s}{2} \partial_t V_2(t, |\xi|)V_1(\tau, |\xi|)f^\wedge(\tau, \xi)\|_{L^2(\mathbb{R}^n)}
+ \|(-\Delta)^\frac{s}{2} \partial_t V_1(t, |\xi|)V_2(\tau, |\xi|)f^\wedge(\tau, \xi)\|_{L^2(\mathbb{R}^n)}d\tau. \tag{3.17}
\]

In terms of (2.4) in Lemma 2.1 or (3.11)–(3.12) as given in [12], we obtain
\[
\partial_t V_1(t, \xi) = i\left(\frac{2m+4}{4i}\right)\pi^{\frac{n+1}{2}}e^{-\frac{t}{2}}\left[\Phi(\frac{4m+4}{2(2+m)}, \frac{2(1+m)}{2+m}; z) - \Phi(\frac{m}{2(2+m)}, \frac{2m}{2+m}; z)\right]
\]
and
\[
\partial_t V_2(t, \xi) = e^{-\frac{t}{2}}\left[\Phi(\frac{4m+4}{2(2+m)}, \frac{2(1+m)}{2+m}; z) + \frac{2m}{4}e^{-\frac{t}{2}}\Phi(\frac{4m+4}{2(2+m)}, \frac{4m}{2+m}; z)\right]. \tag{3.18}
\]

Then for the similarity of the procedure on proof of above estimate, so we only consider the term \(V_2(t, |\xi|)V_1(\tau, |\xi|)f^\wedge(\tau, \xi)\) with \(\tau \in (0, t)\).

Since \(V_i^\wedge(t, |\xi|), i = 1, 2\) is the Fourier integral operator with the symbol of order \(-\frac{m}{m+2}\) or \(-\frac{m+4}{2(m+2)}\). Then we obtain
\[
\|V_i^\wedge(\tau, |\xi|)f(\tau, x)\|_{L^p(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}}\|f(\tau, x)\|_{L^p(\mathbb{R}^n)}, \quad i = 1, 2. \tag{3.19}
\]

Hence, by use of (17) in Lemma 3.2, we derive
\[
\|(-\Delta)^\frac{s}{2} \left(V_2(t, |\xi|)V_1(\tau, |\xi|)f^\wedge(\tau, \xi)\right)\|_{L^2(\mathbb{R}^n)} \leq C t^{-\frac{(2m+4)(n+1)}{4}+\frac{1}{2}+\frac{1}{4}}\|f(\tau, x)\|_{L^p(\mathbb{R}^n)} \tag{3.19}
\]
Choose \(s = \frac{m+4}{2(m+2)} - \frac{1}{2}(n+1)(\frac{1}{2} - \frac{1}{4})\), inequality (3.19) becomes
\[
\|\left(V_2(t, |\xi|)V_1(\tau, |\xi|)f^\wedge(\tau, \xi)\right)\|_{L^p(\mathbb{R}^n)} \leq C t^{-\frac{(m+4)(n+1)}{4}+\frac{1}{2}+\frac{1}{4}}\|f(\tau, x)\|_{L^p(\mathbb{R}^n)} \tag{3.20}
\]
On the other hand, by a similar method, we obtain
\[
\| (V_1(t, |\xi|) V_2(\tau, |\xi|) f^\wedge(\tau, |\xi|)) \|_{H^{\frac{m+4}{4}}_q - n\beta(\frac{1}{2} - \frac{1}{4}) (R^n)}
\leq Ct^{-\frac{m+2}{4}(\beta-1)\left(\frac{1}{2} - \frac{1}{4}\right)} \| f(\tau, x) \|_{L^p(R^n)}. \quad (3.21)
\]

Combining (3.17) with inequalities (3.20)–(3.21), we conclude
\[
\| t^\frac{n}{2} E f(t, \cdot) \|_{H^{\frac{m+4}{4}}_q - n\beta(\frac{1}{2} - \frac{1}{4}) (R^n)} \leq Ct^{\frac{(m+2)(1-n)}{2}(\beta-1)} \int_0^t \| f(\tau, \cdot) \|_{L^p(R^n)} d\tau. \quad (3.22)
\]

Then, by taking a similar procedure, we obtain (3.13)–(3.14).

4 Existence of Solutions to the Semi-Linear Problem

In this section, we construct a contraction map on a defined closed set separately, then use fixed point theorem to prove the local and global existence of solution to initial value problem (1.1).

Proof of Theorem 1.1 Define a closed set
\[
S_M = \{ u \in C([0, T]; H^{\frac{m+4}{4}}_q - n\beta(\frac{1}{2} - \frac{1}{4}) (R^n)) \cap C^1([0, T]; H^{\frac{m+4}{4}}_q - n\beta(\frac{1}{2} - \frac{1}{4}) (R^n)) : \| u \|_M = \sup_{t \in [0, T]} \| u \|_{H^{\frac{m+4}{4}}_q - n\beta(\frac{1}{2} - \frac{1}{4}) (R^n)} + \| \partial_t u \|_{H^{\frac{m+4}{4}}_q - n\beta(\frac{1}{2} - \frac{1}{4}) (R^n)} \leq M \} \quad (4.1)
\]

for a constant $M > 0$ and the distant of elements in $S_M$, $d(u, v) = \sup_{t \in [0, T]} \| u - v \|_M$.

Consider the map $F$ defined on $S_M$,
\[
F(u) = u_0 + E(u|u|^\beta), \quad (4.2)
\]

where $u_0$ is defined by (3.8) in Lemma 3.3 and $E(u|u|^\beta)$ is defined by (3.12) in Lemma 3.4. Set $q = (\beta + 1)p$, then in terms of the embedding theorem in homogeneous Sobolev space under the condition $n - (\beta + 1)p(\frac{2}{2+\beta} - \frac{n\beta}{2+\beta}) = 1$, we obtain
\[
\| F(u) \|_M \leq \| u_0 \|_M + \| E(u|u|^\beta) \|_M \leq \| u_0 \|_M + C t \int_0^1 \| u(t\tau, \cdot) \|_{L^p(R^n)}^{\beta+1} d\tau \\
\leq \| u_0 \|_M + C t \sup_{0 \leq \tau \leq t} (\| u(\tau, \cdot) \|_{L^p(R^n)})^{\beta+1} \leq \| u_0 \|_M + C t \sup_{0 \leq \tau \leq t} (\| u(\tau, \cdot) \|_M)^{\beta+1} \\
\leq C(\| \varphi_1 \|_{H^{\frac{m+4}{4}}_p (R^n)} + \| \varphi_2 \|_{L^p(R^n)}) + CtM^{\beta+1}. \quad (4.3)
\]

This implies that $F(u) \in S_M$ for smallness of $T$. 
Meanwhile, \( q = \frac{np}{q-p} \), \( \beta < q \), then for any \( u \in S_M \) and \( v \in S_M \), we obtain

\[
\|F(u) - F(v)\|_M \leq C \int_0^1 \|u|^{\beta} - v|^{\beta}\|_{L_p} \, d\tau
\]

\[
\leq C \int_0^1 \|((\beta+1)(u-v) \int_0^1 \|\theta(u-v) + v|^{\beta} \|_{L_p} \, d\tau
\]

\[
\leq C \int_0^1 \|u - v\|_{L^q} \|\theta(u-v) + v|^{\beta} \|_{L_p} \, d\tau
\]

\[
\leq C \int_0^1 \|u - v\|_{L^q} \|\theta(u-v) + v|^{\beta} \|_{L_p} \, d\tau
\]

\[
\leq C \|u - v\|_M \|u\|_M^\alpha \|v\|_M^\alpha \, d\tau
\]

This yields the contraction property of map \( F \) on \( S_M \) for smallness of \( T \).

Note that \( p = 1 + \frac{1}{\beta+1} \), then in terms of (4.3) and (4.4), we establish Theorem 1.1.

**Proof of Theorem 1.2** Define a closed set

\[
S_\epsilon = \{ u \in C((-\infty, \infty); \mathcal{H}_{\frac{m+1}{2}}^{\frac{m+1}{2}}) \cap C^1((-\infty, \infty); \mathcal{H}_{\frac{m+1}{2}}^{\frac{m+1}{2}}) \}
\]

\[
\|u\|_\epsilon = \sup_{t \in (0, \infty)} \left( t^{\frac{1}{\beta+1}} \int \|t^\beta u\|_{\mathcal{H}_{\frac{m+1}{2}}^{\frac{m+1}{2}}} \, d\tau \right)
\]

(4.5)

with the distant of elements in \( S_\epsilon \), \( d(u, v) = \sup_{t \in (0, \infty)} \|u - v\|_\epsilon \).

Consider the map \( F \) defined on \( S_\epsilon \), \( F(u) = u_0 + E(t^\alpha u|u|^{\beta}) \), where \( u_0 \) is defined by (3.8) in Lemma 3.3 and \( E(u|u|^{\beta}) \) is defined by (3.12) in Lemma 3.4. Set \( q = (\beta + 1)p \), \( \gamma = \frac{m}{4} + \frac{(m+2)(n-1)}{4} \left( \frac{1}{p} - \frac{1}{q} \right) \), in terms of the embedding theorem in homogeneous Sobolev space under the condition \( \alpha - (\beta + 1)p(\frac{2}{2p+1}) = 1 \), we obtain

\[
\|F(u)\|_\epsilon \leq \|u_0\|_\epsilon + \|E(t^\alpha u|u|^{\beta})\|_\epsilon \leq \|u_0\|_\epsilon + C \int_0^1 \|t^\beta u\|_{\mathcal{H}_{\frac{m+1}{2}}^{\frac{m+1}{2}}} \, d\tau
\]

\[
\leq \|u_0\|_\epsilon + C \int_0^1 \|t^\beta u\|_{\mathcal{H}_{\frac{m+1}{2}}^{\frac{m+1}{2}}} \, d\tau
\]

\[
\leq \|u_0\|_\epsilon + C \int_0^1 \|t^\beta u\|_{\mathcal{H}_{\frac{m+1}{2}}^{\frac{m+1}{2}}} \, d\tau
\]

(4.6)

Then, in terms of the condition

\[
1 + \alpha - (\beta + 1)\left( \frac{m}{4} + \frac{(m+2)(n-1)}{4} \left( \frac{1}{p} - \frac{1}{q} \right) \right) = 0,
\]
this implies that $F(u) \in S_\epsilon$ for the smallness of $\epsilon$.

Meanwhile, $q = \frac{\nu q}{q+\nu}$, $\beta < q$, then for any $u \in S_\epsilon$ and $v \in S_\epsilon$, we obtain

$$\|F(u) - F(v)\|_\epsilon = \|E(\tau^\alpha(u|u|^\beta - v|v|^\beta))\|_\epsilon$$

$$\leq Ct \int_0^1 \|((\beta + 1)(u - v)\int_0^1 \|\theta(u - v) + v|\theta|^p d\theta(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} d\tau$$

$$\leq Ct^{1+\alpha} \int_0^1 \|((\beta + 1)(u - v)\int_0^1 \|\theta(u - v) + v|\theta|^p d\theta(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} d\tau$$

$$\leq Ct^{1+\alpha-\gamma(\beta+1)} \int_0^1 \|((\beta + 1)(u - v)\int_0^1 \|\theta(u - v) + v|\theta|^p d\theta(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} d\tau$$

$$\leq Ct^{1+\alpha-\gamma(\beta+1)}(u, v)e^\beta \leq C d(u, v)e^\beta.$$  \hspace{1cm} (4.7)

This yields the contraction property of map $F$ on $S_\epsilon$.

Finally, in terms of (4.6)–(4.7) and $p = 1 + \frac{1}{\beta+1}$, we complete the proof of Theorem 1.2.

References


半线性广义Tricomi方程初值问题的解的存在性

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摘要: 本文研究了半线性广义Tricomi方程初值问题解的存在性. 基于两个傅立叶积分算子的$\tilde{H}^s_\sigma$-$\tilde{H}^{2\sigma}$加权估计不等式, 建立了半线性广义Tricomi方程在双曲平面上的局部和全局解, 同时, 给出了了解在退化区域附近的正侧性性定理和在无穷远处衰减律.

关键词: 广义Tricomi方程; 半线性; 加权估计; 存在性

MR(2010)主题分类号: 35L15; 35L71; 35L80 中图分类号: O175.21