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GENERALIZED RADFORD BIPRODUCT HOM-HOPF ALGEBRAS AND RELATED BRAIDED TENSOR CATEGORIES

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Abstract: In this paper, the Hom-type of Radford biproduct is introduced. By combining generalized smash product Hom-algebra and generalized smash coproduct Hom-coalgebra, we derive necessary and suffcient conditions for them to be a Hom-bialgebra, which includes the well-known Radford biproduct.

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1 Introduction

Let H be a bialgebra, $A\#H$ a smash product algebra and $A\times H$ a smash coproduct coalgebra. Radford (see [13]) gave a bialgebra structure on $A\otimes H$ (named Radford biproduct by other researchers) via $A\#H$ and $A\times H$. Later, Majid made the following conclusion: to any Hopf algebra A in the braided category of Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$, one can associate an ordinary Hopf algebra $A\star H$, there called the bosonization of A (i.e., Radford biproduct) (see [8]). While Radford biproduct is one of the celebrated objects in the theory of Hopf algebras, which plays a fundamental role in the classification of finite-dimensional pointed Hopf algebras (see [1]). Other references related to Radford biproduct see [1, 6–8, 13, 14].

The algebra of Hom-type can be found in [2] by Hartwig, Larsson and Silvestrov, where a notion of Hom-Lie algebra in the context of q-deformation theory of Witt and Virasoro algebras (see [3]) was introduced. There are various settings of Hom-structures such as

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algebras, coalgebras, Hopf algebras, see [6, 10–12] and so on. In [15], Yau introduced and characterized the concept of module Hom-algebras as a twisted version of usual module algebras. Based on Yau's definition of module Hom-algebras, Ma, Li and Yang [6] constructed smash product Hom-Hopf algebra $(A\natural H, \alpha\otimes\beta)$ generalizing the Molnar's smash product (see [13]), and gave the cobraided structure (in the sense of Yau's definition in [16]) on $(A\natural H, \alpha\otimes\beta)$. Makhlouf and Panaite defined and studied a class of Yetter-Drinfeld modules over Hom-bialgebras in [9] and derived the constructions of twistors, pseudotwistors, twisted tensor product and smash product in the setting of Hom-case (see [10]). Li and Ma studied the Yetter-Drinfeld category of Hom-type via Radford biproduct (see [5]). Recently, Ma, Liu and Li extend the above results in the monoidal Hom-case.

In this paper, we unify the Makhlouf-Panaite's smash product in [10] and Ma-Li-Yang's in [6], and then extend the Radford biproduct to a more general case. We also construct a class of braided tensor categories (extending the Yetter-Drinfeld category to the Hom-case), and provide a solution to the Hom-quantum Yang-Baxter equation.

2 Preliminaries

Throughout this paper, K will be a field, and all vector spaces, tensor products, and homomorphisms are over K. We use Sweedler's notation for terminologies on coalgebras. For a coalgebra C, we write comultiplication $\Delta(c)=c_1\otimes c_2$ for any $c\in C$. And we denote Id_M for the identity map from M to M. Any unexplained definitions and notations can be found in [4–6, 14]. We now recall some useful definitions.

Definition 2.1 A Hom-algebra is a quadruple $(A,\mu,1_A,\alpha)$ (abbr. (A,α)), where A is a linear space, $\mu: A\otimes A\longrightarrow A$ is a linear map, $1_A\in A$ and α is an automorphism of A, such that

$$(HA1)\quad \alpha(aa')=\alpha(a)\alpha(a');\ \ \alpha(1_A)=1_A,$$
$$(HA2)\quad \alpha(a)(a'a'')=(aa')\alpha(a'');\ \ a1_A=1_Aa=\alpha(a)$$

are satisfied for $a,a',a''\in A$. Here we use the notation $\mu(a\otimes a')=aa'$.

Let (A,α) and (B,β) be two Hom-algebras. Then $(A\otimes B,\alpha\otimes\beta)$ is a Hom-algebra (called tensor product Hom-algebra) with the multiplication $(a\otimes b)(a'\otimes b')=aa'\otimes bb'$ and unit $1_A\otimes 1_B$.

Definition 2.2 A Hom-coalgebra is a quadruple $(C,\Delta,\varepsilon_C,\beta)$ (abbr. (C,β)), where C is a linear space, $\Delta: C\longrightarrow C\otimes C$, $\varepsilon_C: C\longrightarrow K$ are linear maps, and β is an automorphism of C, such that

$$(HC1)\quad \beta(c)_1\otimes\beta(c)_2=\beta(c_1)\otimes\beta(c_2);\ \ \varepsilon_C\circ\beta=\varepsilon_C,$$
$$(HC2)\quad \beta(c_1)\otimes c_{21}\otimes c_{22}=c_{11}\otimes c_{12}\otimes\beta(c_2);\ \ \varepsilon_C(c_1)c_2=c_1\varepsilon_C(c_2)=\beta(c)$$

are satisfied for $c\in A$. Here we use the notation $\Delta(c)=c_1\otimes c_2$ (summation implicitly understood).

Let (C, α) and (D, β) be two Hom-coalgebras. Then $(C \otimes D, \alpha \otimes \beta)$ is a Hom-coalgebra (called tensor product Hom-coalgebra) with the comultiplication $\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$ and counit $\varepsilon_C \otimes \varepsilon_D$.

Definition 2.3 A Hom-bialgebra is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$ (abbr. (H, γ)), where $(H, \mu, 1_H, \gamma)$ is a Hom-algebra and $(H, \Delta, \varepsilon, \gamma)$ is a Hom-coalgebra, such that Δ and ε are morphisms of Hom-algebras, i.e., $\Delta(hh') = \Delta(h)\Delta(h')$; $\Delta(1_H) = 1_H \otimes 1_H$, $\varepsilon(hh') = \varepsilon(h)\varepsilon(h')$; $\varepsilon(1_H) = 1$. Furthermore, if there exists a linear map $S : H \longrightarrow H$ such that

$$S(h_1)h_2 = h_1 S(h_2) = \varepsilon(h)1_H \text{ and } S(\gamma(h)) = \gamma(S(h)),$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)$(abbr. (H, γ, S)) a Hom-Hopf algebra.

Let (H, γ) and (H', γ') be two Hom-bialgebras. The linear map $f : H \longrightarrow H'$ is called a Hom-bialgebra map if $f \circ \gamma = \gamma' \circ f$ and at the same time f is a bialgebra map in the usual sense.

Definition 2.4 Let (A, β) be a Hom-algebra. A left (A, β)-Hom-module is a triple $(M, \triangleright, \alpha)$, where M is a linear space, $\triangleright : A \otimes M \longrightarrow M$ is a linear map, and α is an automorphism of M, such that

$$\begin{aligned} &(HM1) \quad \alpha(a \triangleright m) = \beta(a) \triangleright \alpha(m), \\ &(HM2) \quad \beta(a) \triangleright (a' \triangleright m) = (aa') \triangleright \alpha(m);\ \ 1_A \triangleright m = \alpha(m) \end{aligned}$$

are satisfied for $a, a' \in A$ and $m \in M$.

Let $(M, \triangleright_M, \alpha_M)$ and $(N, \triangleright_N, \alpha_N)$ be two left (A, β)-Hom-modules. Then a linear morphism $f : M \longrightarrow N$ is called a morphism of left (A, β)-Hom-modules if $f(h \triangleright_M m) = h \triangleright_N f(m)$ and $\alpha_N \circ f = f \circ \alpha_M$.

Definition 2.5 Let (H, β) be a Hom-bialgebra and (A, α) a Hom-algebra. If $(A, \triangleright, \alpha)$ is a left (H, β)-Hom-module and for all $h \in H$ and $a, a' \in A$,

$$\begin{aligned} &(HMA1) \quad \beta^2(h) \triangleright (aa') = (h_1 \triangleright a)(h_2 \triangleright a'), \\ &(HMA2) \quad h \triangleright 1_A = \varepsilon_H(h)1_A, \end{aligned}$$

then $(A, \triangleright, \alpha)$ is called an (H, β)-module Hom-algebra.

Definition 2.6 Let (C, β) be a Hom-coalgebra. A left (C, β)-Hom-comodule is a triple (M, ρ, α), where M is a linear space, $\rho : M \longrightarrow C \otimes M$ (write $\rho(m) = m_{-1} \otimes m_0$, $\forall m \in M$) is a linear map, and α is an automorphism of M, such that

$$\begin{aligned} &(HCM1) \quad \alpha(m)_{-1} \otimes \alpha(m)_0 = \beta(m_{-1}) \otimes \alpha(m_0), \\ &(HCM2) \quad \beta(m_{-1}) \otimes m_{0-1} \otimes m_{00} = m_{-11} \otimes m_{-12} \otimes \alpha(m_0);\ \ \varepsilon_C(m_{-1})m_0 = \alpha(m) \end{aligned}$$

are satisfied for all $m \in M$.

Let (M, ρ^M, α_M) and (N, ρ^N, α_N) be two left (C, β)-Hom-comodules. Then a linear map $f : M \longrightarrow N$ is called a map of left (C, β)-Hom-comodules if $f(m)_{-1} \otimes f(m)_0 = m_{-1} \otimes f(m_0)$ and $\alpha_N \circ f = f \circ \alpha_M$.

Definition 2.7 Let (H,β) be a Hom-bialgebra and (C,α) a Hom-coalgebra. If (C,ρ,α) is a left (H,β)-Hom-comodule and for all $c\in C$,

$$(HCMC1)\quad \beta^2(c_{-1})\otimes c_{01}\otimes c_{02}=c_{1-1}c_{2-1}\otimes c_{10}\otimes c_{20},$$
$$(HCMC2)\quad c_{-1}\varepsilon_C(c_0)=1_H\varepsilon_C(c),$$

then (C,ρ,α) is called an (H,β)-comodule Hom-coalgebra.

Definition 2.8 Let (H,β) be a Hom-bialgebra and (C,α) a Hom-coalgebra. If $(C,\rhd,\alpha)$ is a left (H,β)-Hom-module and for all $h\in H$ and $c\in A$,

$$(HMC1)\quad (h\rhd c)_1\otimes(h\rhd c)_2=(h_1\rhd c_1)\otimes(h_2\rhd c_2),$$
$$(HMC2)\quad \varepsilon_C(h\rhd c)=\varepsilon_H(h)\varepsilon_C(c),$$

then $(C,\rhd,\alpha)$ is called an (H,β)-module Hom-coalgebra.

Definition 2.9 Let (H,β) be a Hom-bialgebra and (A,α) a Hom-algebra. If (A,ρ,α) is a left (H,β)-Hom-comodule and for all $a,a'\in A$,

$$(HCMA1)\quad \rho(aa')=a_{-1}a'_{-1}\otimes a_0a'_0,$$
$$(HCMA2)\quad \rho(1_A)=1_H\otimes 1_A,$$

then (A,ρ,α) is called an (H,β)-comodule Hom-algebra.

3 Generalized Radford Biproduct Hom-Hopf Algebra

In this section, we first introduce the notions of generalized smash product Hom-algebra $A\sharp^m H$ and generalized Hom-smash coproduct Hom-coalgebra $A\natural_n H$. Then the necessary and sufficient conditions for $A\sharp^m H$ and $A\natural_n H$ on $A\otimes H$ to be a Hom-bialgebra structure are derived.

Proposition 3.1 Let (H,β) be a Hom-bialgebra, $(A,\triangleright,\alpha)$ an (H,β)-module Hom-algebra and $m\in\mathcal{Z}$. Then $(A\sharp^m H,\alpha\otimes\beta)$ $(A\sharp^m H=A\otimes H$ as a linear space) with the multiplication $(a\otimes h)(a'\otimes h')=a(\beta^m(h_1)\triangleright\alpha^{-1}(a'))\otimes\beta^{-1}(h_2)h'$, where $a,a'\in A,h,h'\in H$, and unit $1_A\otimes 1_H$ is a Hom-algebra. In this case, we call $(A\sharp^m H,\alpha\otimes\beta)$ generalized smash product Hom-algebra.

Proof It is straightforward by the definition of Hom-algebra.

Remarks (1) Noting that $(A\sharp^0 H,\alpha\otimes\beta)$ is exactly the Ma-Li-Yang's Hom-smash product in [5, 6] and $(A\sharp^{-2}H,\alpha\otimes\beta)$ is exactly the Makhlouf-Panaite's Hom-smash product in [10].

(2) If $\alpha=Id_A$ and $\beta=Id_H$ in $(A\sharp^m H,\alpha\otimes\beta)$, then one can obtain the usual smash product $A\#H$ in [13].

(3) Let (H,μ_H,Δ_H) be a bialgebra and (A,α) a left H-module algebra in the usual sense with action denoted by $H\otimes A\to A,h\otimes a\mapsto h\cdot a$. Let $\beta:H\to H$ be a bialgebra endomorphism and $\alpha:A\to A$ an algebra endomorphism, such that $\alpha(h\cdot a)=\beta(h)\cdot\alpha(a)$

for all $h \in H$ and $a \in A$. If we consider the Hom-bialgebra $H_\beta = (H, \beta \circ \mu_H, \Delta_H \circ \beta, \beta)$ and the Hom-associative algebra $A_\alpha = (A, \alpha \circ \mu_H, \alpha)$, then (A_α, α) is a left (H_β, β)-module Hom-algebra with action $H_\beta \otimes A_\alpha \to A_\alpha, h \otimes a \mapsto h \triangleright a := \alpha(h \cdot a) = \beta(h) \cdot \alpha(a)$.

Proof Straightforward.

Proposition 3.2 Let (H, β) be a Hom-bialgebra, (C, ρ, α) an (H, β)-comodule Hom-coalgebra and $n \in \mathcal{Z}$. Then $(C\natural H, \alpha \otimes \beta)$ $(C\natural H = C \otimes H$ as a linear space) with the comultiplication $\Delta_{C\natural H}(c \otimes h) = c_1 \otimes \beta^n(c_{2(-1)})\beta^{-1}(h_1) \otimes \alpha^{-1}(c_{2(0)}) \otimes h_2$, where $c \in C, h \in H$, and counit $\varepsilon_C \otimes \varepsilon_H$ is a Hom-coalgebra. In this case, we call $(A\natural_n H, \alpha \otimes \beta)$ generalized smash coproduct Hom-coalgebra.

Proof Straightforward.

Remarks (1) $(A\natural_0 H, \alpha \otimes \beta)$ is exactly the Li-Ma's Hom-smash coproduct in [5].

(2) $(A\natural_{-2} H, \alpha \otimes \beta)$ is exactly the dual version of the Makhlouf-Panaite's Hom-smash product in [10].

(3) If $\alpha = Id_A$ and $\beta = Id_H$ in $(A\sharp^m H, \alpha \otimes \beta)$, then one can obtain the usual smash coproduct $A \times H$ in [13].

Theorem 3.3 Let (H, β) be a Hom-bialgebra, (A, α) a left (H, β)-module Hom-algebra with module structure $\triangleright : H \otimes A \longrightarrow A$ and a left (H, β)-comodule Hom-coalgebra with comodule structure $\rho : A \longrightarrow H \otimes A$. Then the following are equivalent:

(i) $(A\diamondsuit_n^m H, \mu_{A\sharp H}, 1_A \otimes 1_H, \Delta_{A\natural H}, \varepsilon_A \otimes \varepsilon_H, \alpha \otimes \beta)$ is a Hom-bialgebra, where $(A\sharp^m H, \alpha \otimes \beta)$ is a generalized smash product Hom-algebra and $(A\natural_n H, \alpha \otimes \beta)$ is a generalized smash coproduct Hom-coalgebra.

(ii) The following conditions hold:

(R1) (A, ρ, α) is an (H, β)-comodule Hom-algebra;

(R2) $(A, \triangleright, \alpha)$ is an (H, β)-module Hom-coalgebra;

(R3) ε_A is a Hom-algebra map and $\Delta_A(1_A) = 1_A \otimes 1_A$;

(R4) $\Delta_A(ab) = a_1(\beta^{m+n+2}(a_{2(-1)}) \triangleright \alpha^{-1}(b_1)) \otimes \alpha^{-1}(a_{2(0)})b_2$;

(R5) $\beta^{n+1}((\beta^{m+1}(h_1) \triangleright b)_{-1})h_2 \otimes (\beta^{m+1}(h_1) \triangleright b)_0 = h_1\beta^{n+2}(b_{(-1)}) \otimes \beta^{m+2}(h_2) \triangleright b_{(0)}$,

where $a, b \in B$, $h \in H$ and $m, n \in \mathcal{Z}$. In this case, we call $(A\diamondsuit_n^m H, \alpha \otimes \beta)$ generalized Radford biproduct Hom-bialgebra.

Proof By a tedious computation we can prove it.

Remarks (1) When $m = n = 0$ in Theorem 3.3, we can get [5, Theorem 3.3].

(2) When $\alpha = Id_A$ and $\beta = Id_H$ in Theorem 3.3, then one can obtain [13, Theorem 1].

Proposition 3.4 Let (H, β, S_H) be a Hom-Hopf algebra, and (A, α) ba a Hom-algebra and a Hom-coalgebra. Assume that $(A\diamondsuit_n^m H, \alpha \otimes \beta)$ is a generalized Radford biproduct Hom-bialgebra defined as above, and $S_A : A \to A$ is a linear map such that $S_A(a_1)a_2 = a_1 S_A(a_2) = \varepsilon_A(a)1_A$ and $\alpha \circ S_A = S_A \circ \alpha$ hold. Then $(A\diamondsuit_n^m H, \alpha \otimes \beta, S_{A\diamondsuit_n^m H})$ is a Hom-Hopf algebra, where

$$S_{A\diamondsuit_n^m H}(a \otimes h) = (\beta^m(S_H(\beta^n(a_{(-1)})\beta^{-1}(h))_1) \triangleright S_A(\alpha^{-2}(a_{(0)}))) \otimes \beta^{-1}(S_H(\beta^n(a_{(-1)})\beta^{-1}(h))_2).$$

Proof For all $a\in A, h\in H$, we have

$$
\begin{aligned}
& (S_{A\diamond_n^m H}*Id_{A\diamond_n^m H})(a\otimes h)\\
= & S_{A\diamond_n^m H}(a_1\otimes \beta^n(a_{2(-1)})\beta^{-1}(h_1))(\alpha^{-1}(a_{(0)})\otimes h_2)\\
= & ((\beta^m(S_H(\beta^n(a_{1(-1)})\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_1)\triangleright S_A(\alpha^{-2}(a_{1(0)})))\\
& \otimes\beta^{-1}(S_H(\beta^n(a_{1(-1)})\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_2))(\alpha^{-1}(a_{2(0)})\otimes h_2)\\
= & (\beta^m(S_H(\beta^n(a_{1(-1)})\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_1)\triangleright S_A(\alpha^{-2}(a_{1(0)})))\\
& \times(\beta^m(\beta^{-1}(S_H(\underline{\beta^n(a_{1(-1)})\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1))})_2)_1)\triangleright\alpha^{-2}(a_{2(0)}))\\
& \otimes\beta^{-1}(\beta^{-1}(S_H(\beta^n(a_{1(-1)})\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_2)_2)h_2\\
\overset{(HA2)}{=} & (\beta^m(S_H(\beta^{n-1}(\underline{a_{1(-1)}a_{2(-1)}})\beta^{-1}(h_1))_1)\triangleright S_A(\alpha^{-2}(\underline{a_{1(0)}})))\\
& \times(\beta^m(\beta^{-1}(S_H(\beta^{n-1}(\underline{a_{1(-1)}a_{2(-1)}})\beta^{-1}(h_1))_2)_1)\triangleright\alpha^{-2}(\underline{a_{2(0)}}))\\
& \otimes\beta^{-1}(\beta^{-1}(S_H(\beta^{n-1}(a_{1(-1)}a_{2(-1)})\beta^{-1}(h_1))_2)_2)h_2\\
\overset{(HCMC1)}{=} & (\beta^m(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_1)\triangleright S_A(\alpha^{-2}(a_{(0)1})))\underline{(\beta^m(\beta^{-1}(S_H(\beta^{n+1}(a_{(-1)})}\\
& \underline{\times\beta^{-1}(h_1))_2)_1})\triangleright\alpha^{-2}(a_{(0)2}))\otimes \underline{\beta^{-1}(\beta^{-1}(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_2)_2)}h_2\\
\overset{(HC1)}{=} & (\beta^m(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_1)\triangleright S_A(\alpha^{-2}(a_{(0)1})))(\beta^{m-1}(S_H(\beta^{n+1}(a_{(-1)})\\
& \times\beta^{-1}(h_1))_{21})\triangleright\alpha^{-2}(a_{(0)2}))\otimes\beta^{-2}(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_{22})h_2\\
\overset{(HC2)}{=} & \underline{(\beta^{m-1}(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_{11})\triangleright S_A(\alpha^{-2}(a_{(0)1})))(\beta^{m-1}(S_H(\beta^{n+1}(a_{(-1)})}\\
& \underline{\times\beta^{-1}(h_1))_{12})\triangleright\alpha^{-2}(a_{(0)2}))}\otimes\beta^{-1}(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_2)h_2\\
\overset{(HMA1)}{=} & (\beta^{m+1}(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_1)\triangleright(S_A(\alpha^{-2}(a_{(0)1}))\alpha^{-2}(a_{(0)2}))\\
& \otimes\beta^{-1}(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_2)h_2\\
= & \beta^{m+1}(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_1)\triangleright 1_A\varepsilon_A(a_{(0)})\\
& \otimes\beta^{-1}(S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))_2)h_2\\
= & 1_A\varepsilon_A(a_{(0)})\otimes S_H(\beta^{n+1}(a_{(-1)})\beta^{-1}(h_1))h_2=1_A\varepsilon_A(a)\otimes S_H(h_1)h_2\\
= & (1_A\otimes 1_H)\varepsilon_A(a)\varepsilon_H(h),
\end{aligned}
$$

and the rest is direct.

4 Generalized Hom-Yetter-Drinfeld Category

In this section, we construct a class of braided tensor category, which extends the Yetter-Drinfeld category to the Hom-case. Next we give the concept of Hom-Yetter-Drinfeld module via generalized Radford biproduct Hom-Hopf algebra defined in Theorem 3.3.

Definition 4.1 Let (H,β) be a Hom-bialgebra, $(U,\triangleright_U,\alpha_U)$ a left (H,β)-module with action $\triangleright_U: H\otimes U\to U, h\otimes u\mapsto h\triangleright_U u$ and (U,ρ^U,α_U) a left (H,β)-comodule with coaction $\rho^U: U\to H\otimes U, u\mapsto u_{(-1)}\otimes u_{(0)}$. Then we call $(U,\triangleright_U,\rho^U,\alpha_U)$ a (left-left) Hom-Yetter-Drinfeld module over (H,β) if the following condition holds:

$$h_1\beta^{n+2}(u_{(-1)})\otimes\beta^{m+2}(h_2)\triangleright u_{(0)}=\beta^{n+1}((\beta^{m+1}(h_1)\triangleright u)_{(-1)})h_2\otimes(\beta^{m+1}(h_1)\triangleright u)_{(0)}\quad (HYD)$$

for all $h \in H$ and $u \in U$.

Proposition 4.2 When (H, β) is a Hom-Hopf algebra, (HYD) is equivalent to

$$\rho(\beta^{m+3}(h) \triangleright u) = (\beta^{-n-3}(h_{11})\beta^{-1}(u_{(-1)}))S(\beta^{-n-1}(h_2)) \otimes \beta^{m+2}(h_{12}) \triangleright u_{(0)} \qquad (HYD)'$$

for all $h \in H, u \in U$.

Proof $(HYD) \Longrightarrow (HYD)'$. We have

$$\begin{aligned}
& (\beta^{-n-3}(h_{11})\beta^{-1}(u_{(-1)}))S(\beta^{-n-1}(h_2)) \otimes \beta^{m+2}(h_{12}) \triangleright u_{(0)} \\
= \;& \beta^{-n-1}(\beta^{-2}\underline{(h_{11}\beta^{n+2}(u_{(-1)}))}S(h_2)) \otimes \underline{\beta^{m+2}(h_{12}) \triangleright u_{(0)}} \\
\overset{(HYD)}{=} \;& \beta^{-n-1}(\underline{\beta^{-2}(\beta^{n+1}((\beta^{m+1}(h_{11}) \triangleright u)_{(-1)})S(h_2)))} \otimes (\beta^{m+1}(h_{11}) \triangleright u)_{(0)} \\
\overset{(HA1)}{=} \;& \beta^{-n-1}(\underline{(\beta^{n-1}((\beta^{m+1}(h_{11}) \triangleright u)_{(-1)})\beta^{-2}(h_{12}))S(h_2)}) \otimes (\beta^{m+1}(h_{11}) \triangleright u)_{(0)} \\
\overset{(HA2)}{=} \;& \beta^{-n-1}(\beta^{n}((\beta^{m+1}(h_{11}) \triangleright u)_{(-1)})(\beta^{-2}(h_{12}S(\beta^{-1}(h_2))))) \otimes (\beta^{m+1}(h_{11}) \triangleright u)_{(0)} \\
\overset{(HC1)}{=} \;& \beta^{-n-1}(\beta^{n}((\beta^{m+2}(h_1) \triangleright u)_{(-1)})(\beta^{-2}(h_{21}))S(\beta^{-2}(h_{22}))) \otimes (\beta^{m+2}(h_1) \triangleright u)_{(0)} \\
= \;& (\beta^{m+3}(h) \triangleright u)_{(-1)} \otimes (\beta^{m+3}(h) \triangleright u)_{(0)}.
\end{aligned}$$

$(HYD)' \Longrightarrow (HYD)$ is proved as follows:

$$\begin{aligned}
& \beta^{n+1}((\beta^{m+1}(h_1) \triangleright u)_{(-1)})h_2 \otimes (\beta^{m+1}(h_1) \triangleright u)_{(0)} \\
= \;& \beta^{n+1}(\underline{(\beta^{m+3}(\beta^{-2}(h_1)) \triangleright u)_{(-1)}})h_2 \otimes \underline{(\beta^{m+3}(\beta^{-2}(h_1)) \triangleright u)_{(0)}} \\
\overset{(HYD)'}{=} \;& ((\beta^{-4}(h_{111})\beta^{n}(u_{(-1)}))S(\beta^{-2}(h_{12})))h_2 \otimes \beta^{m}(h_{112}) \triangleright u_{(0)} \\
\overset{(HC2)}{=} \;& \underline{((\beta^{-2}(h_1)\beta^{n}(u_{(-1)}))S(\beta^{-3}(h_{221})))\beta^{-2}(h_{222})} \otimes \beta^{m+1}(h_{21}) \triangleright u_{(0)} \\
\overset{(HA2)}{=} \;& (\beta^{-1}(h_1)\beta^{n+1}(u_{(-1)}))(S(\beta^{-3}(h_{221}))\beta^{-3}(h_{222})) \otimes \beta^{m+1}(h_{21}) \triangleright u_{(0)} \\
= \;& h_1\beta^{n+2}(u_{(-1)}) \otimes \beta^{m+2}(h_2) \triangleright u_{(0)},
\end{aligned}$$

finishing the proof.

Definition 4.3 Let (H, β) be a Hom-bialgebra. We denote by ${}^{H}_{H}\mathbb{YD}$ the category whose objects are Hom-Yetter-Drinfeld modules $(U, \triangleright_U, \rho^U, \alpha_U)$ over (H, β); the morphisms in the category are morphisms of left (H, β)-modules and left (H, β)-comodules.

In the following, we give a solution to the Hom-quantum Yang-Baxter equation introduced and studied by Yau in [16].

Proposition 4.4 Let (H, β) be a Hom-bialgebra and $(U, \triangleright_U, \rho^U, \alpha_U)$, $(V, \triangleright_V, \rho^V, \alpha_V)$ $\in {}^{H}_{H}\mathbb{YD}$. Define the linear map

$$\tau_{U,V} : U \otimes V \to V \otimes U, u \otimes v \mapsto \beta^{m+n+3}(u_{(-1)}) \triangleright_V v \otimes u_{(0)},$$

where $u \in U$ and $v \in V$. Then we have $\tau_{U,V} \circ (\alpha_U \otimes \alpha_V) = (\alpha_V \otimes \alpha_U) \circ \tau_{U,V}$, if $(W, \triangleright_W, \rho^W, \alpha_W) \in {}^{H}_{H}\mathbb{YD}$, the map $\tau_{-,-}$ satisfy the Hom-Yang-Baxter equation

$$(\alpha_W \otimes \tau_{U,V}) \circ (\tau_{U,W} \otimes \alpha_V) \circ (\alpha_U \otimes \tau_{V,W}) = (\tau_{V,W} \otimes \alpha_U) \circ (\alpha_V \otimes \tau_{U,W}) \circ (\tau_{U,V} \otimes \alpha_W).$$

Proof It is easy to prove the first equality, so we only check the second one. For all $u \in U, v \in V$ and $w \in W$, we have

$$\begin{aligned}
& (\alpha_W \otimes \tau_{U,V}) \circ (\tau_{U,W} \otimes \alpha_V) \circ (\alpha_U \otimes \tau_{V,W})(u \otimes v \otimes w) \\
= \ & \alpha_W(\beta^{m+n+3}(\alpha_U(u)_{(-1)} \triangleright_W (\beta^{m+n+3}(v_{(-1)}) \triangleright_W w)) \otimes \beta^{m+n+3}(\alpha_U(u)_{(0)(-1)} \\
& \triangleright_V \alpha_V(v_{(0)}) \otimes \alpha_U(u)_{(0)(0)} \\
= \ & \beta^{m+n+5}(u_{(-1)}) \triangleright_W (\beta^{m+n+4}(v_{(-1)}) \triangleright_W \alpha_W(w)) \otimes \beta^{m+n+4}(u_{(0)(-1)}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u_{(0)(0)}) \\
= \ & \beta^{m+n+4}(u_{(-1)1}) \triangleright_W (\beta^{m+n+4}(v_{(-1)}) \triangleright_W \alpha_W(w)) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U^2(u_{(0)}) \\
= \ & ((\beta^{m+n+3}(u_{(-1)1})\beta^{m+n+4}(v_{(-1)})) \triangleright_W \alpha_W^2(w)) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U^2(u_{(0)}) \\
= \ & (\beta^{m+n+3}(u_{(-1)1}\alpha_V(v)_{(-1)}) \triangleright_W \alpha_W^2(w)) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v)_{(0)} \\
& \otimes \alpha_U^2(u_{(0)}) \\
= \ & (\beta^{m+1}(\underline{\beta^{n+2}(u_{(-1)})_1\beta^{n+2}(\alpha_V(v)_{(-1)})}) \triangleright_W \alpha_W^2(w)) \otimes \underline{\beta^{m+2}(\beta^{n+2}(u_{(-1)})_2)} \\
& \underline{\triangleright_V \alpha_V(v)_{(0)}} \otimes \alpha_U^2(u_{(0)}) \\
= \ & (\beta^{m+1}(\beta^{n+1}((\beta^{m+1}(\beta^{n+2}(u_{(-1)1})) \triangleright_V \alpha_V(v))_{(-1)})\beta^{n+2}(u_{(-1)2})) \triangleright_W \alpha_W^2(w)) \\
& \otimes (\beta^{m+1}(\beta^{n+2}(u_{(-1)1})) \triangleright_V \alpha_V(v))_{(0)} \otimes \alpha_U^2(u_{(0)}) \\
\overset{(HYD)}{=} \ & (\beta^{m+n+2}((\beta^{m+n+3}(u_{(-1)1}) \triangleright_V \alpha_V(v))_{(-1)})\beta^{m+n+3}(u_{(-1)2})) \triangleright_W \alpha_W^2(w) \\
& \otimes (\beta^{m+n+3}(u_{(-1)1}) \triangleright_V \alpha_V(v))_{(0)} \otimes \alpha_U^2(u_{(0)}) \\
= \ & (\beta^{m+n+2}((\beta^{m+n+4}(u_{(-1)}) \triangleright_V \alpha_V(v))_{(-1)})\beta^{m+n+3}(u_{(0)(-1)})) \triangleright_W \alpha_W^2(w)) \\
& \otimes (\beta^{m+n+4}(u_{(-1)}) \triangleright_V \alpha_V(v))_{(0)} \otimes \alpha_U(u_{(0)(0)}) \\
= \ & \beta^{m+n+3}((\beta^{m+n+4}(u_{(-1)}) \triangleright_V \alpha_V(v))_{(-1)}) \triangleright_W (\beta^{m+n+3}(u_{(0)(-1)}) \triangleright_W \alpha_W^2(w)) \\
& \otimes (\beta^{m+n+4}(u_{(-1)}) \triangleright_V \alpha_V(v))_{(0)} \otimes \alpha_U(u_{(0)(0)}) \\
= \ & (\tau_{V,W} \otimes \alpha_U) \circ (\alpha_V \otimes \tau_{U,W}) \circ (\tau_{U,V} \otimes \alpha_W)(u \otimes v \otimes w).
\end{aligned}$$

The proof is completed.

Lemma 4.5 Let (H, β) be a Hom-bialgebra, if $(U, \triangleright_U, \rho^U, \alpha_U), (V, \triangleright_V, \rho^V, \alpha_V)$ are (left-left) Hom-Yetter-Drinfeld modules, then $(U \otimes V, \triangleright_{U\otimes V}, \rho^{U\otimes V}, \alpha_U \otimes \alpha_V)$ is a Hom-Yetter-Drinfeld module with structures

$$\triangleright_{U\otimes V} : H \otimes U \otimes V \to U \otimes V, h \otimes u \otimes v \mapsto (h_1 \triangleright_U u) \otimes (h_2 \triangleright_V v)$$

and

$$\rho^{U\otimes V} : U \otimes V \to H \otimes U \otimes V, u \otimes v \mapsto \beta^{-2}(u_{(-1)}v_{(-1)}) \otimes u_{(0)} \otimes v_{(0)}$$

for all $h \in H, u \in U, v \in V$.

Proof It is easy to check that $(U\otimes V, \triangleright_{U\otimes V}, \alpha_U\otimes\alpha_V)$ is an (H,β)-Hom module and $(U\otimes V, \rho^{U\otimes V}, \alpha_U\otimes\alpha_V)$ is an (H,β)-Hom comodule. Now we check the condition (HYD). For all $h\in H, u\in U, v\in V$, we have

$$
\begin{aligned}
& \beta^{n+1}((\beta^{m+1}(h_1)\triangleright(u\otimes v))_{(-1)})h_2\otimes(\beta^{m+1}(h_1)\triangleright(u\otimes v))_{(0)}\\
= \;& \beta^{n+1}((\beta^{m+1}(h_{11})\triangleright u\otimes\beta^{m+1}(h_{12})\triangleright v)_{(-1)})h_2\otimes(\beta^{m+1}(h_{11})\triangleright u\\
& \otimes\beta^{m+1}(h_{12})\triangleright v)_{(0)}\\
= \;& \beta^{n-1}((\beta^{m+1}(h_{11})\triangleright u)_{(-1)}(\beta^{m+1}(h_{12})\triangleright v)_{(-1)})h_2\otimes(\beta^{m+1}(h_{11})\triangleright u)_{(0)}\\
& \otimes(\beta^{m+1}(h_{12})\triangleright v)_{(0)}\\
= \;& [\beta^{n-1}((\beta^{m+1}(h_{11})\triangleright u)_{(-1)})\beta^{n-1}((\beta^{m+1}(h_{12})\triangleright v)_{(-1)})]h_2\otimes(\beta^{m+1}(h_{11})\triangleright u)_{(0)}\\
& \otimes(\beta^{m+1}(h_{12})\triangleright v)_{(0)}\\
= \;& \beta^{n}((\beta^{m+1}(h_{11})\triangleright u)_{(-1)})[\beta^{n-1}((\beta^{m+1}(h_{12})\triangleright v)_{(-1)})\beta^{-1}(h_2)]\otimes(\beta^{m+1}(h_{11})\triangleright u)_{(0)}\\
& \otimes(\beta^{m+1}(h_{12})\triangleright v)_{(0)}\\
= \;& \beta^{n}((\beta^{m}(h_{1})\triangleright u)_{(-1)})[\beta^{n-1}((\beta^{m+1}(h_{21})\triangleright v)_{(-1)})\beta^{-2}(h_{22})]\otimes(\beta^{m}(h_{1})\triangleright u)_{(0)}\\
& \otimes(\beta^{m+1}(h_{21})\triangleright v)_{(0)}\\
= \;& \beta^{n}((\beta^{m}(h_{1})\triangleright u)_{(-1)})\beta^{-2}[\underline{\beta^{n+1}((\beta^{m+1}(h_{21})\triangleright v)_{(-1)})h_{22}}]\otimes(\beta^{m}(h_{1})\triangleright u)_{(0)}\\
& \otimes\underline{(\beta^{m+1}(h_{21})\triangleright v)_{(0)}}\\
\overset{(HYD)}{=} \;& \beta^{n}((\beta^{m}(h_{1})\triangleright u)_{(-1)})\beta^{-2}(h_{21}\beta^{n+2}(v_{(-1)}))\otimes(\beta^{m}(h_{1})\triangleright u)_{(0)}\otimes\beta^{m+2}(h_{22})\triangleright v_{(0)}\\
= \;& [\beta^{n-1}((\beta^{m}(h_{1})\triangleright u)_{(-1)})\beta^{-2}(h_{21})]\beta^{n+1}(v_{(-1)})\otimes(\beta^{m}(h_{1})\triangleright u)_{(0)}\otimes\beta^{m+2}(h_{22})\triangleright v_{(0)}\\
= \;& [\beta^{n-1}((\beta^{m+1}(h_{11})\triangleright u)_{(-1)})\beta^{-2}(h_{12})]\beta^{n+1}(v_{(-1)})\otimes(\beta^{m+1}(h_{11})\triangleright u)_{(0)}\\
& \otimes\beta^{m+3}(h_2)\triangleright v_{(0)}\\
= \;& \beta^{-2}[\underline{\beta^{n+1}((\beta^{m+1}(h_{11})\triangleright u)_{(-1)})h_{12}}]\beta^{n+1}(v_{(-1)})\otimes\underline{(\beta^{m+1}(h_{11})\triangleright u)_{(0)}}\\
& \otimes\beta^{m+3}(h_2)\triangleright v_{(0)}\\
\overset{(HYD)}{=} \;& (\beta^{-2}(h_{11})\beta^{n}(u_{(-1)}))\beta^{n+1}(v_{(-1)})\otimes\beta^{m+2}(h_{12})\triangleright u_{(0)}\otimes\beta^{m+3}(h_2)\triangleright v_{(0)}\\
= \;& (\beta^{-1}(h_{1})\beta^{n}(u_{(-1)}))\beta^{n+1}(v_{(-1)})\otimes\beta^{m+2}(h_{21})\triangleright u_{(0)}\otimes\beta^{m+2}(h_{22})\triangleright v_{(0)}\\
= \;& h_1(\beta^{n}(u_{(-1)})\beta^{n}(v_{(-1)}))\otimes\beta^{m+2}(h_{21})\triangleright u_{(0)}\otimes\beta^{m+2}(h_{22})\triangleright v_{(0)}\\
= \;& h_1\beta^{n}(u_{(-1)}v_{(-1)})\otimes\beta^{m+2}(h_{2})\triangleright(u_{(0)}\otimes v_{(0)})\\
= \;& h_1\beta^{n+2}((u\otimes v)_{(-1)})\otimes\beta^{m+2}(h_{2})\triangleright(u\otimes v)_{(0)},
\end{aligned}
$$

finishing the proof.

Lemma 4.6 Let (H,β) be a Hom-bialgebra, and

$$(U,\triangleright_U,\rho^U,\alpha_U),\ \ (V,\triangleright_V,\rho^V,\alpha_V),\ \ (W,\triangleright_W,\rho^W,\alpha_W)\in{}^H_H\mathbb{YD}.$$

With notation as above, define the linear map

$$a_{U,V,W}:(U\otimes V)\otimes W\to U\otimes(V\otimes W),(u\otimes v)\otimes w\mapsto\alpha_U^{-1}(u)\otimes(v\otimes\alpha_W(w)),$$

where $u \in U, v \in V$ and $w \in W$. Then $a_{U,V,W}$ is an ismorphism of left (H,β)-Hom-modules and left (H,β)-Hom-comodules.

Proof Same to the proof of [9, Proposition 3.2].

Lemma 4.7 Let (H,β) be a Hom-bialgebra and $(U, \triangleright_U, \rho^U, \alpha_U), (V, \triangleright_V, \rho^V, \alpha_V) \in {}^H_H\mathbb{YD}$. Define the linear map

$$c_{U,V} : U \otimes V \to V \otimes U, u \otimes v \mapsto (\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v)) \otimes \alpha_U^{-1}(u_{(0)}),$$

where $u \in U$ and $v \in V$. Then $c_{U,V}$ is a morphism of left (H,β)-Hom-modules and left (H,β)-Hom-comodules.

Proof For all $h \in H, u \in U$ and $v \in V$, we have

$$\begin{aligned}
& (\alpha_V \otimes \alpha_U) \circ c_{U,V}(u \otimes v) \\
= \ & \alpha_V(\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v)) \otimes u_{(0)} \\
= \ & (\beta^{m+n+3}(u_{(-1)}) \triangleright_V v) \otimes u_{(0)} \\
= \ & \beta^{m+n+2}(\alpha_U(u)_{(-1)}) \triangleright_V \alpha_V^{-1}(\alpha_V(v)) \otimes \alpha_U^{-1}(\alpha_U(u)_{(0)}) \\
= \ & c_{U,V} \circ (\alpha_U \otimes \alpha_V)(u \otimes v), \\
& c_{U,V}(h \triangleright_{U\otimes V} (u \otimes v)) \\
= \ & c_{U,V}((h_1 \triangleright_U u) \otimes (h_2 \triangleright_V v)) \\
= \ & (\beta^{m+n+2}((h_1 \triangleright_U u)_{(-1)}) \triangleright_V \alpha_V^{-1}(h_2 \triangleright_V v)) \otimes \alpha_U^{-1}((h_1 \triangleright_U u)_{(0)}) \\
= \ & (\beta^{m+n+2}((h_1 \triangleright_U u)_{(-1)}) \triangleright_V (\beta^{-1}(h_2) \triangleright_V \alpha_V^{-1}(v))) \otimes \alpha_U^{-1}((h_1 \triangleright_U u)_{(0)}) \\
= \ & ((\underline{\beta^{m+n+1}((h_1 \triangleright_U u)_{(-1)})\beta^{-1}(h_2)}) \triangleright_V v) \otimes \alpha_U^{-1}(\underline{(h_1 \triangleright_U u)_{(0)}}) \\
\overset{(HYD)}{=} \ & (\beta^m(\beta^{n+1}((h_1 \triangleright_U u)_{(-1)})\beta^{-m-1}(h_2)) \triangleright_V v) \otimes \alpha_U^{-1}((h_1 \triangleright_U u)_{(0)}) \\
= \ & (\beta^m(\beta^{-m-1}(h_1)\beta^{n+2}(u_{(-1)})) \triangleright_V v) \otimes \alpha_U^{-1}(\beta^{m+2}(\beta^{-m-1}(h_2)) \triangleright_U u_{(0)}) \\
= \ & ((\beta^{-1}(h_1)\beta^{m+n+2}(u_{(-1)})) \triangleright_V v) \otimes h_2 \triangleright_U \alpha_U^{-1}(u_{(0)}) \\
= \ & (h_1 \triangleright_V (\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v))) \otimes h_2 \triangleright_U \alpha_U^{-1}(u_{(0)}) \\
= \ & h \triangleright_{U\otimes V} ((\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v)) \otimes \alpha_U^{-1}(u_{(0)})) \\
= \ & h \triangleright_{U\otimes V} c_{U,V}(u \otimes v)
\end{aligned}$$

and

$$\begin{aligned}
& (\rho^{V\otimes U} \circ c_{U,V})(u \otimes v) \\
= \ & \rho^{V\otimes U}((\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v)) \otimes \alpha_U^{-1}(u_{(0)})) \\
= \ & \beta^{-2}((\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v))_{(-1)}\alpha_U^{-1}(u_{(0)})_{(-1)}) \\
& \otimes(\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v))_{(0)} \otimes \alpha_U^{-1}(u_{(0)})_{(0)} \\
= \ & \beta^{-2}((\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v))_{(-1)}\beta^{-1}(u_{(0)(-1)})) \otimes (\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v))_{(0)} \\
& \otimes\alpha_U^{-1}(u_{(0)(0)}) \\
= \ & \beta^{-2}((\beta^{m+n+1}(u_{(-1)1}) \triangleright_V \alpha_V^{-1}(v))_{(-1)}\beta^{-1}(u_{(-1)2})) \otimes (\beta^{m+n+1}(u_{(-1)1}) \triangleright_V \alpha_V^{-1}(v))_{(0)} \\
& \otimes u_{(0)}
\end{aligned}$$

$$
\begin{aligned}
&= \beta^{-n-3}(\beta^{n+1}((\beta^{m+1}(\beta^n(u_{(-1)})_1) \triangleright_V \alpha_V^{-1}(v))_{(-1)})\beta^n(u_{(-1)})_2) \otimes (\beta^{m+1}(\beta^n(u_{(-1)})_1) \\
&\quad \triangleright_V \alpha_V^{-1}(v))_{(0)} \otimes u_{(0)} \\
&\overset{(HYD)}{=} \beta^{-n-3}(\beta^n(u_{(-1)1})\beta^{n+2}(\alpha_V^{-1}(v)_{(-1)})) \otimes \beta^{m+2}(\beta^n(u_{(-1)2})) \triangleright_V \alpha_V^{-1}(v)_{(0)} \otimes u_{(0)} \\
&= \beta^{-3}(u_{(-1)1})\beta^{-2}(v_{(-1)}) \otimes \beta^{m+n+2}(u_{(-1)2}) \triangleright_V \alpha_V^{-1}(v_{(0)}) \otimes u_{(0)} \\
&= \beta^{-2}(u_{(-1)}v_{(-1)}) \otimes \beta^{m+n+2}(u_{(0)(-1)}) \triangleright_V \alpha_V^{-1}(v_{(0)}) \otimes \alpha_U^{-1}(u_{(0)(0)}) \\
&= (Id \otimes c_{U,V})(\beta^{-2}(u_{(-1)}v_{(-1)}) \otimes u_{(0)} \otimes v_{(0)}) \\
&= (Id \otimes c_{U,V}) \circ \rho^{U\otimes V}(u \otimes v),
\end{aligned}
$$

finishing the proof.

Theorem 4.8 Let (H, β) be a Hom-bialgebra. Then the Hom-Yetter-Drinfeld category ${}_H^H\mathbb{YD}$ is a pre-braided tensor category, with tensor product, associativity constraints, and pre-braiding in Lemmas 4.5, 4.6 and 4.7, respectively, and the unit $I = (K, Id_K)$.

Proof The proof of the pentagon axiom for $a_{U,V,W}$ is same to the proof of [9, Theorem 3.4]. Next we prove that the hexagonal relation for $c_{U,V}$. Let $(U, \triangleright_U, \rho^U, \alpha_U)$, $(V, \triangleright_V, \rho^V, \alpha_V)$, $(W, \triangleright_W, \rho^W, \alpha_W) \in_H^H \mathbb{YD}$. Then for all $u \in U, v \in V$ and $w \in W$, we have

$$
\begin{aligned}
&((Id_V \otimes c_{U,W}) \circ a_{V,U,W} \circ (c_{U,V} \otimes Id_W))((u \otimes v) \otimes w) \\
&= ((Id_V \otimes c_{U,W}) \circ a_{V,U,W})((\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v)) \otimes \alpha_U^{-1}(u_{(0)}) \otimes w) \\
&= (Id_V \otimes c_{U,W})(\alpha_V^{-1}(\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v)) \otimes (\alpha_U^{-1}(u_{(0)}) \otimes \alpha_W(w))) \\
&= \alpha_V^{-1}(\beta^{m+n+2}(u_{(-1)}) \triangleright_V \alpha_V^{-1}(v)) \otimes \beta^{m+n+1}(u_{(0)(-1)}) \triangleright_W w \otimes \alpha_U^{-2}(u_{(0)(0)}) \\
&= \alpha_V^{-1}(\beta^{m+n+1}(u_{(-1)1}) \triangleright_V \alpha_V^{-1}(v)) \otimes \beta^{m+n+1}(u_{(-1)2}) \triangleright_W w \otimes \alpha_U^{-1}(u_{(0)}) \\
&= a_{V,W,U}(\beta^{m+n+1}(u_{(-1)1}) \triangleright_V \alpha_V^{-1}(v) \otimes \beta^{m+n+1}(u_{(-1)2}) \triangleright_W w \otimes \alpha_U^{-2}(u_{(0)})) \\
&= a_{V,W,U}(\beta^{m+n+2}(\alpha_U^{-1}(u)_{(-1)}) \triangleright_{V\otimes W} (\alpha_V^{-1}(v) \otimes w) \otimes \alpha_U^{-1}(\alpha_U^{-1}(u)_{(0)}) \\
&= (a_{V,W,U} \circ c_{U,V\otimes W})(\alpha_U^{-1}(u) \otimes (v \otimes \alpha_W(w))) \\
&= (a_{V,W,U} \circ c_{U,V\otimes W} \circ a_{U,V,W})((u \otimes v) \otimes w)
\end{aligned}
$$

and

$$
\begin{aligned}
&((c_{U,W} \otimes Id_V) \circ a_{U,W,V}^{-1} \circ (Id_U \otimes c_{V,W}))(u \otimes (v \otimes w)) \\
&= ((c_{U,W} \otimes Id_V) \circ a_{U,W,V}^{-1})(u \otimes (\beta^{m+n+2}(v_{(-1)}) \triangleright_W \alpha_W^{-1}(w)) \otimes \alpha_V^{-1}(v_{(0)})) \\
&= (c_{U,W} \otimes Id_V)(\alpha_U(u) \otimes \beta^{m+n+2}(v_{(-1)}) \triangleright_W \alpha_W^{-1}(w) \otimes \alpha_V^{-2}(v_{(0)})) \\
&= \beta^{m+n+2}(\alpha_U(u)_{(-1)}) \triangleright_W \alpha_W^{-1}(\beta^{m+n+2}(v_{(-1)}) \triangleright_W \alpha_W^{-1}(w)) \otimes \alpha_U^{-1}(\alpha_U(u)_{(0)}) \otimes \alpha_V^{-2}(v_{(0)}) \\
&= \beta^{m+n+3}(u_{(-1)}) \triangleright_W (\beta^{m+n+1}(v_{(-1)}) \triangleright_W \alpha_W^{-2}(w)) \otimes u_{(0)} \otimes \alpha_V^{-2}(v_{(0)}) \\
&= (\beta^{m+n+2}(u_{(-1)})\beta^{m+n+1}(v_{(-1)})) \triangleright_W \alpha_W^{-1}(w) \otimes u_{(0)} \otimes \alpha_V^{-2}(v_{(0)}) \\
&= \beta^{m+n+1}(\alpha_U(u)_{(-1)}v_{(-1)}) \triangleright_W \alpha_W^{-1}(w) \otimes \alpha_U^{-1}(\alpha_U(u)_{(0)}) \otimes \alpha_V^{-2}(v_{(0)}) \\
&= a_{W,U,V}^{-1}(\beta^{m+n}(\alpha_U(u)_{(-1)}v_{(-1)}) \triangleright_W \alpha_W^{-2}(w) \otimes \alpha_U^{-1}(\alpha_U(u)_{(0)}) \otimes \alpha_V^{-1}(v_{(0)})) \\
&= (a_{W,U,V}^{-1} \circ c_{U\otimes V,W})((\alpha_U(u) \otimes v) \otimes \alpha_W^{-1}(w)) \\
&= (a_{W,U,V}^{-1} \circ c_{U\otimes V,W} \circ a_{U,V,W}^{-1})(u \otimes (v \otimes w)),
\end{aligned}
$$

and the rest is obvious. These complete the proof.

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广义Radford双积Hom-Hopf代数和相关辫子张量范畴

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摘要: 本文研究了Radford双积的Hom-型. 通过把广义smash积Hom-代数和广义smash余积Hom-余代数相结合, 得到了他们成为Hom-双代数的充分必要条件, 这一结果推广了著名的Radford双积.

关键词: Radford 双积; 量子Yang-Baxter方程; Yetter-Drinfeld范畴

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