GENERALIZED RADFORD BIPRODUCT HOM-HOPF ALGEBRAS AND RELATED BRAIDED TENSOR CATEGORIES

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Abstract: In this paper, the Hom-type of Radford biproduct is introduced. By combining generalized smash product Hom-algebra and generalized smash coproduct Hom-coalgebra, we derive necessary and sufficient conditions for them to be a Hom-bialgebra, which includes the well-known Radford biproduct.

Keywords: Radford biproduct; quantum Yang-Baxter equation; Yetter-Drinfeld category

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1 Introduction

Let $H$ be a bialgebra, $A\#H$ a smash product algebra and $A \times H$ a smash coproduct coalgebra. Radford (see [13]) gave a bialgebra structure on $A \otimes H$ (named Radford biproduct by other researchers) via $A\#H$ and $A \times H$. Later, Majid made the following conclusion: to any Hopf algebra $A$ in the braided category of Yetter-Drinfeld modules $H^{H}_{H}YD$, one can associate an ordinary Hopf algebra $A \star H$, there called the bosonization of $A$ (i.e., Radford biproduct) (see [8]). While Radford biproduct is one of the celebrated objects in the theory of Hopf algebras, which plays a fundamental role in the classification of finite-dimensional pointed Hopf algebras (see [1]). Other references related to Radford biproduct see [1, 6–8, 13, 14].

The algebra of Hom-type can be found in [2] by Hartwig, Larsson and Silvestrov, where a notion of Hom-Lie algebra in the context of $q$-deformation theory of Witt and Virasoro algebras (see [3]) was introduced. There are various settings of Hom-structures such as

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algebras, coalgebras, Hopf algebras, see [6, 10–12] and so on. In [15], Yau introduced and characterized the concept of module Hom-algebras as a twisted version of usual module algebras. Based on Yau’s definition of module Hom-algebras, Ma, Li and Yang [6] constructed smash product Hom-Hopf algebra \((A^\sharp H, \alpha \otimes \beta)\) generalizing the Molnar’s smash product (see [13]), and gave the cobraided structure (in the sense of Yau’s definition in [16]) on \((A^\sharp H, \alpha \otimes \beta)\). Makhlouf and Panaite defined and studied a class of Yetter-Drinfeld modules over Hom-bialgebras in [9] and derived the constructions of twistors, pseudotwistors, twisted tensor product and smash product in the setting of Hom-case (see [10]). Li and Ma studied the Yetter-Drinfeld category of Hom-type via Radford biproduct (see [5]). Recently, Ma, Liu and Li extend the above results in the monoidal Hom-case.

In this paper, we unify the Makhlouf-Panaite’s smash product in [10] and Ma-Li-Yang’s in [6], and then extend the Radford biproduct to a more general case. We also construct a class of braided tensor categories (extending the Yetter-Drinfeld category to the Hom-case), and provide a solution to the Hom-quantum Yang-Baxter equation.

2 Preliminaries

Throughout this paper, \(K\) will be a field, and all vector spaces, tensor products, and homomorphisms are over \(K\). We use Sweedler’s notation for terminologies on coalgebras. For a coalgebra \(C\), we write comultiplication \(\Delta(c) = c_1 \otimes c_2\) for any \(c \in C\). And we denote \(\text{Id}_M\) for the identity map from \(M\) to \(M\). Any unexplained definitions and notations can be found in [4–6, 14]. We now recall some useful definitions.

**Definition 2.1** A Hom-algebra is a quadruple \((A, \mu, 1_A, \alpha)\) (abbr. \((A, \alpha)\)), where \(A\) is a linear space, \(\mu: A \otimes A \to A\) is a linear map, \(1_A \in A\) and \(\alpha\) is an automorphism of \(A\), such that

\[
\begin{align*}
(HA1) & \quad \alpha(aa') = \alpha(a)\alpha(a'); \quad \alpha(1_A) = 1_A, \\
(HA2) & \quad \alpha(a)(a'a'') = (aa')\alpha(a''); \quad a1_A = 1_Aa = \alpha(a)
\end{align*}
\]

are satisfied for \(a, a', a'' \in A\). Here we use the notation \(\mu(a \otimes a') = aa'\).

Let \((A, \alpha)\) and \((B, \beta)\) be two Hom-algebras. Then \((A \otimes B, \alpha \otimes \beta)\) is a Hom-algebra (called tensor product Hom-algebra) with the multiplication \((a \otimes b)(a' \otimes b') = aa' \otimes bb'\) and unit \(1_A \otimes 1_B\).

**Definition 2.2** A Hom-coalgebra is a quadruple \((C, \Delta, \varepsilon_C, \beta)\) (abbr. \((C, \beta)\)), where \(C\) is a linear space, \(\Delta: C \to C \otimes C, \varepsilon_C: C \to K\) are linear maps, and \(\beta\) is an automorphism of \(C\), such that

\[
\begin{align*}
(HC1) & \quad \beta(c_1) \otimes \beta(c_2) = \beta(c_1) \otimes \beta(c_2); \quad \varepsilon_C \circ \beta = \varepsilon_C, \\
(HC2) & \quad \beta(c_1 \otimes c_2) = c_11 \otimes c_2 \otimes \beta(c_2); \quad \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c)
\end{align*}
\]

are satisfied for \(c \in A\). Here we use the notation \(\Delta(c) = c_1 \otimes c_2\) (summation implicitly understood).
Let \((C, \alpha)\) and \((D, \beta)\) be two Hom-coalgebras. Then \((C \otimes D, \alpha \otimes \beta)\) is a Hom-coalgebra (called tensor product Hom-coalgebra) with the comultiplication \(\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2\) and counit \(\varepsilon_C \otimes \varepsilon_D\).

**Definition 2.3** A Hom-bialgebra is a sextuple \((H, \mu, 1_H, \Delta, \varepsilon, \gamma)\) (abbr. \((H, \gamma)\)), where \((H, \mu, 1_H, \gamma)\) is a Hom-algebra and \((H, \Delta, \varepsilon, \gamma)\) is a Hom-coalgebra, such that \(\Delta\) and \(\varepsilon\) are morphisms of Hom-algebras, i.e., \(\Delta(hh') = \Delta(h)\Delta(h')\); \(\Delta(1_H) = 1_H \otimes 1_H\), \(\varepsilon(hh') = \varepsilon(h)\varepsilon(h')\); \(\varepsilon(1_H) = 1\). Furthermore, if there exists a linear map \(S : H \rightarrow H\) such that

\[
S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H \quad \text{and} \quad S(\gamma(h)) = \gamma(S(h)),
\]

then we call \((H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)\) (abbr. \((H, \gamma, S)\)) a Hom-Hopf algebra.

Let \((H, \gamma)\) and \((H', \gamma')\) be two Hom-bialgebras. The linear map \(f : H \rightarrow H'\) is called a Hom-bialgebra map if \(f \circ \gamma = \gamma' \circ f\) and at the same time \(f\) is a bialgebra map in the usual sense.

**Definition 2.4** Let \((A, \beta)\) be a Hom-algebra. A left \((A, \beta)\)-Hom-module is a triple \((M, \triangleright, \alpha)\), where \(M\) is a linear space, \(\triangleright : A \otimes M \rightarrow M\) is a linear map, and \(\alpha\) is an automorphism of \(M\), such that

\[
(HM1) \quad \alpha(a \triangleright m) = \beta(a) \triangleright \alpha(m),
\]

\[
(HM2) \quad \beta(a) \triangleright (a' \triangleright m) = (aa') \triangleright \alpha(m); \quad 1_A \triangleright m = \alpha(m)
\]

are satisfied for \(a, a' \in A\) and \(m \in M\).

Let \((M, \triangleright_M, \alpha_M)\) and \((N, \triangleright_N, \alpha_N)\) be two left \((A, \beta)\)-Hom-modules. Then a linear morphism \(f : M \rightarrow N\) is called a morphism of left \((A, \beta)\)-Hom-modules if \(f(h \triangleright_M m) = h \triangleright_N f(m)\) and \(\alpha_N \circ f = f \circ \alpha_M\).

**Definition 2.5** Let \((H, \beta)\) be a Hom-bialgebra and \((A, \alpha)\) a Hom-algebra. If \((A, \triangleright, \alpha)\) is a left \((H, \beta)\)-Hom-module and for all \(h \in H\) and \(a, a' \in A\),

\[
(HMA1) \quad \beta^2(h) \triangleright (aa') = (h_1 \triangleright a)(h_2 \triangleright a'),
\]

\[
(HMA2) \quad h \triangleright 1_A = \varepsilon_H(h)1_A,
\]

then \((A, \triangleright, \alpha)\) is called an \((H, \beta)\)-module Hom-algebra.

**Definition 2.6** Let \((C, \alpha)\) be a Hom-coalgebra. A left \((C, \beta)\)-Hom-comodule is a triple \((M, \triangleright, \alpha)\), where \(M\) is a linear space, \(\triangleright : C \otimes M \rightarrow M\) (write \(\rho(m) = m_{-1} \otimes m_0\), \(\forall m \in M\)) is a linear map, and \(\alpha\) is an automorphism of \(M\), such that

\[
(HCM1) \quad \alpha(m_{-1}) \triangleright \alpha(m_0) = \beta(m_{-1}) \otimes \alpha(m_0),
\]

\[
(HCM2) \quad \beta(m_{-1}) \otimes m_0 \otimes m_0 = m_{-11} \otimes m_{-12} \otimes \alpha(m_0); \quad \varepsilon_C(m_{-1})m_0 = \alpha(m)
\]

are satisfied for all \(m \in M\).

Let \((M, \rho_M, \alpha_M)\) and \((N, \rho_N, \alpha_N)\) be two left \((C, \beta)\)-Hom-comodules. Then a linear map \(f : M \rightarrow N\) is called a map of left \((C, \beta)\)-Hom-comodules if \(f(m_{-1}) \otimes f(m_0) = m_{-1} \otimes f(m_0)\) and \(\alpha_N \circ f = f \circ \alpha_M\).
Definition 2.7 Let \((H, \beta)\) be a Hom-bialgebra and \((C, \alpha)\) a Hom-coalgebra. If \((C, \rho, \alpha)\) is a left \((H, \beta)\)-Hom-comodule and for all \(c \in C\),
\[
(HMC1) \quad \beta^2(c_{-1}) \otimes c_{01} \otimes c_{02} = c_{-1}c_{2-1} \otimes c_{10} \otimes c_{20},
\]
\[
(HMC2) \quad c_{-1} \varepsilon_C(c_0) = 1_H \varepsilon_C(c),
\]
then \((C, \rho, \alpha)\) is called an \((H, \beta)\)-comodule Hom-coalgebra.

Definition 2.8 Let \((H, \beta)\) be a Hom-bialgebra and \((C, \alpha)\) a Hom-coalgebra. If \((C, \triangleright, \alpha)\) is a left \((H, \beta)\)-Hom-module and for all \(h \in H\) and \(c \in A\),
\[
(HMC1) \quad (h \triangleright c)_1 \otimes (h \triangleright c)_2 = (h_1 \triangleright c_1) \otimes (h_2 \triangleright c_2),
\]
\[
(HMC2) \quad \varepsilon_C(h \triangleright c) = \varepsilon_H(h) \varepsilon_C(c),
\]
then \((C, \triangleright, \alpha)\) is called an \((H, \beta)\)-module Hom-coalgebra.

Definition 2.9 Let \((H, \beta)\) be a Hom-bialgebra and \((A, \alpha)\) a Hom-algebra. If \((A, \rho, \alpha)\) is a left \((H, \beta)\)-Hom-comodule and for all \(a, a' \in A\),
\[
(HCMA1) \quad \rho(aa') = a_{-1}a'_{-1} \otimes a_0a'_0,
\]
\[
(HCMA2) \quad \rho(1_A) = 1_H \otimes 1_A,
\]
then \((A, \rho, \alpha)\) is called an \((H, \beta)\)-comodule Hom-algebra.

3 Generalized Radford Biproduct Hom-Hopf Algebra

In this section, we first introduce the notions of generalized smash product Hom-algebra \(A^m_H\) and generalized Hom-smash coproduct Hom-coalgebra \(A^n_H\). Then the necessary and sufficient conditions for \(A^m_H\) and \(A^n_H\) on \(A \otimes H\) to be a Hom-bialgebra structure are derived.

Proposition 3.1 Let \((H, \beta)\) be a Hom-bialgebra, \((A, \triangleright, \alpha)\) an \((H, \beta)\)-module Hom-algebra and \(m \in \mathbb{Z}\). Then \((A^m_H, \alpha \otimes \beta)\) \((A^m_H = A \otimes H\) as a linear space\) with the multiplication \((a \otimes h)(a' \otimes h') = a(\beta^m(h_1) \triangleright \alpha^{-1}(a')) \otimes \beta^{-1}(h_2)h', \) where \(a, a' \in A, h, h' \in H,\) and unit \(1_A \otimes 1_H\) is a Hom-algebra. In this case, we call \((A^m_H, \alpha \otimes \beta)\) generalized smash product Hom-algebra.

Proof It is straightforward by the definition of Hom-algebra.

Remarks (1) Noting that \((A^0_H, \alpha \otimes \beta)\) is exactly the Ma-Li-Yang’s Hom-smash product in [5, 6] and \((A^{-2}_H, \alpha \otimes \beta)\) is exactly the Makhlouf-Panaite’s Hom-smash product in [10].

(2) If \(\alpha = \text{Id}_A\) and \(\beta = Id_H\) in \((A^m_H, \alpha \otimes \beta)\), then one can obtain the usual smash product \(A\#H\) in [13].

(3) Let \((H, \mu_H, \Delta_H)\) be a bialgebra and \((A, \alpha)\) a left \(H\)-module algebra in the usual sense with action denoted by \(H \otimes A \rightarrow A, h \otimes a \mapsto h \cdot a.\) Let \(\beta : H \rightarrow H\) be a bialgebra endomorphism and \(\alpha : A \rightarrow A\) an algebra endomorphism, such that \(\alpha(h \cdot a) = \beta(h) \cdot \alpha(a)\)
for all \( h \in H \) and \( a \in A \). If we consider the Hom-bialgebra \( H_\beta = (H, \beta \circ \mu_H, \Delta_H \circ \beta, \beta) \)
and the Hom-associative algebra \( A_\alpha = (A, \alpha \circ \mu_H, \alpha) \), then \((A_\alpha, \alpha)\) is a left \((H_\beta, \beta)\)-module Hom-algebra with action \( H_\beta \otimes A_\alpha \to A_\alpha, h \otimes a \mapsto h \triangleright a := \alpha(h \cdot a) = \beta(h) \cdot \alpha(a) \).

**Proof** Straightforward.

**Proposition 3.2** Let \((H, \beta)\) be a Hom-bialgebra, \((C, \rho, \alpha)\) an \((H, \beta)\)-comodule Hom-coalgebra and \( n \in \mathbb{Z} \). Then \((C \otimes H, \alpha \otimes \beta)\) \((C \otimes H = C \otimes H \) as a linear space\) with the comultiplication \( \Delta_{C \otimes H}(c \otimes h) = c_1 \otimes \beta^n(c_2(-1)) \beta^{-1}(h_1) \otimes \alpha^{-1}(c_2(0)) \otimes h_2 \), where \( c \in C, h \in H \), and counit \( \varepsilon_C \otimes \varepsilon_H \) is a Hom-coalgebra. In this case, we call \((A \otimes H, \alpha \otimes \beta)\) generalized smash coproduct Hom-coalgebra.

**Proof** Straightforward.

**Remarks**

(1) \((A \otimes H, \alpha \otimes \beta)\) is exactly the Li-Ma’s Hom-smash coproduct in [5].

(2) \((A \otimes H, \alpha \otimes \beta)\) is exactly the dual version of the Makhlfou-Panaite’s Hom-smash product in [10].

(3) If \( \alpha = \text{Id}_A \) and \( \beta = \text{Id}_H \) in \((A \otimes H, \alpha \otimes \beta)\), then one can obtain the usual smash coproduct \( A \times H \) in [13].

**Theorem 3.3** Let \((H, \beta)\) be a Hom-bialgebra, \((A, \alpha)\) a left \((H, \beta)\)-module Hom-algebra with module structure \( \triangleright : H \otimes A \to A \) and a left \((H, \beta)\)-comodule Hom-coalgebra with comodule structure \( \rho : A \to H \otimes A \). Then the following are equivalent:

(i) \((A \otimes H, \mu_{A \otimes H}, 1_A \otimes 1_{H}, \Delta_{A \otimes H}, \varepsilon_A \otimes \varepsilon_H, \alpha \otimes \beta)\) is a Hom-bialgebra, where \((A \otimes H, \alpha \otimes \beta)\) is a generalized smash product Hom-algebra and \((A \otimes H, \alpha \otimes \beta)\) is a generalized smash coproduct Hom-coalgebra.

(ii) The following conditions hold:

(R1) \((A, \rho, \alpha)\) is an \((H, \beta)\)-comodule Hom-algebra;

(R2) \((A, \rho, \alpha)\) is an \((H, \beta)\)-module Hom-coalgebra;

(R3) \( \varepsilon_A \) is a Hom-algebra map and \( \Delta_A(1_A) = 1_A \otimes 1_A \);

(R4) \( \Delta_A(ab) = a_1(\beta^{m+n+2}(a_2(-1)) \triangleright \alpha^{-1}(b_1)) \otimes \alpha^{-1}(a_2(0)) b_2; \)

(R5) \( \beta^{n+1}((\beta^{m+1}(h_1) \triangleright b_{-1}) h_2 \otimes (\beta^{m+1}(h_1) \triangleright b_0) = h_1 \beta^{n+2}(b_{-1}) \otimes \beta^{m+2}(h_2) \triangleright b_0), \)

where \( a, b \in B, h \in H \) and \( m, n \in \mathbb{Z} \). In this case, we call \((A \otimes H, \alpha \otimes \beta)\) generalized Radford biproduct Hom-bialgebra.

**Proof** By a tedious computation we can prove it.

**Remarks**

(1) When \( m = n = 0 \) in Theorem 3.3, we can get [5, Theorem 3.3].

(2) When \( \alpha = \text{Id}_A \) and \( \beta = \text{Id}_H \) in Theorem 3.3, then one can obtain [13, Theorem 1].

**Proposition 3.4** Let \((H, \beta, S_H)\) be a Hom-Hopf algebra, and \((A, \alpha)\) a Hom-algebra and a Hom-coalgebra. Assume that \((A \otimes H, \alpha \otimes \beta)\) is a generalized Radford biproduct Hom-bialgebra defined as above, and \( S_A : A \to A \) is a linear map such that \( S_A(a_1) a_2 = a_1 S_A(a_2) = \varepsilon_A(a) 1_A \) and \( \alpha \circ S_A = S_A \circ \alpha \) hold. Then \((A \otimes H, \alpha \otimes \beta, S_A \otimes H)\) is a Hom-Hopf algebra, where

\[
S_{A \otimes H}(a \otimes h) = (\beta^m(S_H(\beta^n(a_{-1})) \beta^{-1}(h))_1 \triangleright S_A(\alpha^{-2}(a_{0})) ) \otimes \beta^{-1} (S_H(\beta^n(a_{-1})) \beta^{-1}(h))_2).
\]
**Proof** For all $a \in A, h \in H$, we have

\[
(S_{A \otimes_H H} \ast \text{Id}_{A \otimes_H H})(a \otimes h) = S_{A \otimes_H H}(a_1 \otimes \beta^n(a_{2(-1)})\beta^{-1}(h_1))(\alpha^{-1}(a_{0(0)}) \otimes h_2)
\]

\[
= (\beta^m(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_1) \triangleright S_A(\alpha^2(a_{0(0)}))
\]

\[
\otimes \beta^{-1}(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_{2(1)}(\alpha^{-1}(a_{2(0)}) \otimes h_2)
\]

\[
= (\beta^m(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_1) \triangleright S_A(\alpha^2(a_{0(0)}))
\]

\[
\otimes \beta^{-1}(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_{2(1)}(\alpha^{-1}(a_{2(0)}) \otimes h_2)
\]

\[
(H_{A2}) = (\beta^m(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1)))_1) \triangleright S_A(\alpha^2(a_{0(0)}))
\]

\[
\otimes \beta^{-1}(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_{2(1)}(\alpha^{-1}(a_{2(0)}) \otimes h_2)
\]

\[
(H_{CM1}) = (\beta^m(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1)))_1) \triangleright S_A(\alpha^2(a_{0(0)}))
\]

\[
\otimes \beta^{-1}(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_{2(1)}(\alpha^{-1}(a_{2(0)}) \otimes h_2)
\]

\[
(H_{C1}) = (\beta^m(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1)))_1) \triangleright S_A(\alpha^2(a_{0(0)}))
\]

\[
\otimes \beta^{-1}(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_{2(1)}(\alpha^{-1}(a_{2(0)}) \otimes h_2)
\]

\[
(H_{C2}) = (\beta^m(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1)))_1) \triangleright S_A(\alpha^2(a_{0(0)}))
\]

\[
\otimes \beta^{-1}(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_{2(1)}(\alpha^{-1}(a_{2(0)}) \otimes h_2)
\]

\[
(H_{MA1}) = (\beta^m(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1)))_1) \triangleright (S_A(\alpha^{-2}(a_{0(0)})) \alpha^{-2}(a_{0(0)})
\]

\[
\otimes \beta^{-1}(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_{2(1)}h_2
\]

\[
(H_{MA1}) = (\beta^m(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1)))_1) \triangleright (S_A(\alpha^{-2}(a_{0(0)})) \alpha^{-2}(a_{0(0)})
\]

\[
\otimes \beta^{-1}(S_H(\beta^n(a_{1(-1)}))\beta^{-1}(\beta^n(a_{2(-1)})\beta^{-1}(h_1)))_{2(1)}h_2
\]

\[
= 1_A \otimes h \in H \in A \in A \in H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1))h_2 = 1_A \otimes h \in H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1))h_2
\]

\[
= (1_A \otimes h) \in H(\beta^n(a_{1(-1)}))\beta^{-1}(h_1))h_2
\]

and the rest is direct.

### 4 Generalized Hom-Yetter-Drinfeld Category

In this section, we construct a class of braided tensor category, which extends the Yetter-Drinfeld category to the Hom-case. Next we give the concept of Hom-Yetter-Drinfeld module via generalized Radford biproduct Hom-Hopf algebra defined in Theorem 3.3.

**Definition 4.1** Let $(H, \beta)$ be a Hom-bialgebra, $(U, \triangleright_U, \alpha_U)$ a left $(H, \beta)$-module with action $\triangleright_U : H \otimes U \rightarrow U, h \otimes u \mapsto h \triangleright_U u$ and $(U, \rho_U, \alpha_U)$ a left $(H, \beta)$-comodule with coaction $\rho_U : U \rightarrow H \otimes U, u \mapsto u_{(1)} \otimes u_{(0)}$. Then we call $(U, \triangleright_U, \rho_U, \alpha_U)$ a left-left Hom-Yetter-Drinfeld module over $(H, \beta)$ if the following condition holds:

$$h_1 \beta^n(u_{(-1)}) \otimes \beta^{m+2}(h_2) \triangleright u_{(0)} = \beta^{n+1}((\beta^{m+1}(h_1) \triangleright u_{(-1)})h_2 \otimes (\beta^{n+1}(h_1) \triangleright u_{(0)})(HYD)$$
for all \( h \in H \) and \( u \in U \).

**Proposition 4.2** When \((H, \beta)\) is a Hom-Hopf algebra, \((HYD)\) is equivalent to

\[
\rho(\beta^{m+3}(h) \triangleright u) = (\beta^{-n-3}(h_{11})\beta^{-1}(u_{(-1)}))S(\beta^{-n-1}(h_{2})) \otimes \beta^{m+2}(h_{12}) \triangleright u_{(0)} \tag{HYD}'
\]

for all \( h \in H, u \in U \).

**Proof** \((HYD) \implies (HYD)'\). We have

\[
(HYD) \implies \beta^{-n-1}(\beta^{-2}(h_{11})\beta^{-1}(u_{(-1)}))S(h_{2}) \otimes \beta^{m+2}(h_{12}) \triangleright u_{(0)}
\]

\[
(HA1) \beta^{-n-1}((\beta^{m+1}(h_{11}) \triangleright u)_{(-1)})\beta^{-2}(h_{12})S(h_{2}) \otimes \beta^{m+1}(h_{11}) \triangleright u_{(0)}
\]

\[
(HA2) \beta^{-n-1}((\beta^{m+1}(h_{11}) \triangleright u)_{(-1)})\beta^{-2}(h_{12})S(h_{2}) \otimes \beta^{m+1}(h_{11}) \triangleright u_{(0)}
\]

\[
(HC1) \beta^{-n-1}(\beta^{m+2}(h_{1}) \triangleright u_{(-1)})(\beta^{-2}(h_{21}))S(\beta^{-2}(h_{22})) \otimes \beta^{m+2}(h_{1}) \triangleright u_{(0)}
\]

\[
(HD) (\beta^{m+3}(h) \triangleright u)_{(-1)} \otimes (\beta^{m+3}(h) \triangleright u)_{(0)}
\]

finishing the proof.

**Definition 4.3** Let \((H, \beta)\) be a Hom-bialgebra. We denote by \(H^H_H \otimes \mathcal{D}\) the category whose objects are Hom-Yetter-Drinfeld modules \((U, \triangleright_U, \rho_U, \alpha_U)\) over \((H, \beta)\); the morphisms in the category are morphisms of left \((H, \beta)\)-modules and left \((H, \beta)\)-comodules.

In the following, we give a solution to the Hom-quantum Yang-Baxter equation introduced and studied by Yau in [16].

**Proposition 4.4** Let \((H, \beta)\) be a Hom-bialgebra and \((U, \triangleright_U, \rho_U, \alpha_U), (V, \triangleright_V, \rho_V, \alpha_V) \in H^H_H \otimes \mathcal{D}\). Define the linear map

\[
\tau_{U,V} : U \otimes V \rightarrow V \otimes U, \ u \otimes v \mapsto \beta^{m+n+3}(u_{(-1)}) \triangleright_U v \otimes u_{(0)},
\]

where \( u \in U \) and \( v \in V \). Then we have \( \tau_{U,V} \circ (\alpha_U \otimes \alpha_V) = (\alpha_V \otimes \alpha_U) \circ \tau_{U,V} \), if \((W, \triangleright_W, \rho_W, \alpha_W) \in H^H_H \otimes \mathcal{D}\), the map \(\tau_{-,-}\) satisfy the Hom-Yang-Baxter equation

\[
(\alpha_W \otimes \tau_{U,V}) \circ (\tau_{U,W} \otimes \alpha_U) \circ (\alpha_U \otimes \tau_{V,W}) = (\tau_{V,W} \otimes \alpha_U) \circ (\alpha_V \otimes \tau_{U,W}) \circ (\tau_{U,V} \otimes \alpha_W).
\]
It is easy to prove the first equality, so we only check the second one. For all $u \in U, v \in V$ and $w \in W$, we have

\[
\begin{align*}
& (\alpha_W \otimes \tau_{U,V}) \circ (\tau_{U,W} \otimes \alpha_V) \circ (\alpha_U \otimes \tau_{V,W})(u \otimes v \otimes w) \\
= & \alpha_W(\beta^{m+n+3}(\alpha_U(u_{(-1)}) \triangleright_W (\beta^{m+n+3}(v_{(-1)}) \triangleright_W w)) \otimes \beta^{m+n+3}(\alpha_U(u)_{(0)(-1)}) \\
& \triangleright_V \alpha_V(v_{(0)}) \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+5}(u_{(-1)}) \triangleright_W (\beta^{m+n+4}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(0)}_{(-1)}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+4}(u_{(-1)}) \triangleright_W (\beta^{m+n+4}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+3}(u_{(-1)1}) \triangleright_W (\beta^{m+n+4}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+3}(u_{(-1)1}) \triangleright_W (\beta^{m+n+4}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+3}(u_{(-1)1}) \triangleright_W (\beta^{m+n+3}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+3}(u_{(-1)1}) \triangleright_W (\beta^{m+n+4}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
= & \beta^{m+n+3}(u_{(-1)1}) \triangleright_W (\beta^{m+n+4}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+3}(u_{(-1)1}) \triangleright_W (\beta^{m+n+3}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+3}(u_{(-1)}) \triangleright_W (\beta^{m+n+3}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
= & \beta^{m+n+3}(u_{(-1)}) \triangleright_W (\beta^{m+n+3}(v_{(-1)}) \triangleright_W w) \triangleright_V \alpha_W(w) \otimes \beta^{m+n+4}(u_{(-1)2}) \triangleright_V \alpha_V(v_{(0)}) \\
& \otimes \alpha_U(u)_{(0)(0)} \\
& \circ \tau_{V,W} \circ \alpha_V \circ \tau_{U,V} \circ \alpha_U \circ \tau_{W,U} \circ \alpha_W \circ \alpha_V \circ \tau_{U,V} \circ \alpha_U \circ \tau_{V,W})(u \otimes v \otimes w).
\end{align*}
\]

The proof is completed.

**Lemma 4.5** Let $(H, \beta)$ be a Hom-bialgebra, if $(U, \triangleright_U, \rho^U, \alpha_U), (V, \triangleright_V, \rho^V, \alpha_V)$ are (left-left) Hom-Yetter-Drinfeld modules, then $(U \otimes V, \triangleright_{U \otimes V}, \rho^{U \otimes V}, \alpha_U \otimes \alpha_V)$ is a Hom-Yetter-Drinfeld module with structures

\[
\triangleright_{U \otimes V} : H \otimes U \otimes V \to U \otimes V, h \otimes u \otimes v \mapsto (h_1 \triangleright_U u) \otimes (h_2 \triangleright_V v)
\]

and

\[
\rho^{U \otimes V} : U \otimes V \to H \otimes U \otimes V, u \otimes v \mapsto \beta^{-2}(u_{(-1)}v_{(-1)}) \otimes u_{(0)} \otimes v_{(0)}
\]

for all $h \in H, u \in U, v \in V$. 
Proof It is easy to check that \((U \otimes V, \varphi_{U \otimes V}, \alpha_U \otimes \alpha_V)\) is an \((H, \beta)\)-Hom module and 
\((U \otimes V, \rho_{U \otimes V}, \alpha_U \otimes \alpha_V)\) is an \((H, \beta)\)-Hom comodule. Now we check the condition \((HYD)\). 

For all \(h \in H, u \in U, v \in V\), we have 

\[
\beta^{n+1}((\beta^{n+1}(h_1) \triangleright (u \otimes v))(-1))h_2 \otimes (\beta^{n+1}(h_1) \triangleright (u \otimes v))(0) = 
\beta^{n+1}((\beta^{n+1}(h_{11}) \triangleright u \otimes \beta^{n+1}(h_{12}) \triangleright v)(-1))h_2 \otimes (\beta^{n+1}(h_{11}) \triangleright u) \otimes (\beta^{n+1}(h_{12}) \triangleright v)(0) \]

\[
= \beta^{n-1}((\beta^{n+1}(h_{11}) \triangleright u)(-1))(\beta^{n+1}(h_{12}) \triangleright v)(-1))h_2 \otimes (\beta^{n+1}(h_{11}) \triangleright u)(0) \otimes (\beta^{n+1}(h_{12}) \triangleright v)(0) \]

\[
= \beta^{n}((\beta^{n+1}(h_{11}) \triangleright u)(-1)) \beta^{n-1}((\beta^{n+1}(h_{12}) \triangleright v)(-1)) \beta^{-1}(h_{2}) \otimes (\beta^{n+1}(h_{11}) \triangleright u)(0) \otimes (\beta^{n+1}(h_{12}) \triangleright v)(0) \]

\[
= \beta^{n}((\beta^{n+1}(h_{11}) \triangleright u)(-1)) \beta^{-2}(h_{21}) \otimes (\beta^{n+1}(h_{11}) \triangleright u)(0) \otimes (\beta^{n+1}(h_{12}) \triangleright v)(0) \]

\[
= \beta^{-2}(\beta^{n+1}(h_{11}) \triangleright u)(-1)h_{12} \otimes (\beta^{n+1}(h_{11}) \triangleright u)(0) \otimes (\beta^{n+1}(h_{12}) \triangleright v)(0) \]

\[
\otimes (\beta^{n+3}(h_{2}) \triangleright v)(0) \]

\[
= \beta^{-2}(\beta^{n+1}(u_{(-1)})) \beta^{n+1}(v_{(-1)}) \otimes (\beta^{n+1}(h_{1}) \triangleright u)(0) \otimes (\beta^{n+2}(h_{2}) \triangleright v)(0) \]

\[
= \beta^{-1}(h_{1}) \beta^{n+1}(v_{(-1)}) \otimes (\beta^{n+1}(h_{1}) \triangleright u)(0) \otimes (\beta^{n+2}(h_{2}) \triangleright v)(0) \]

\[
= h_{1}(\beta^{n}(u_{(-1)})) \beta^{n+2}(h_{21}) \triangleright u)(0) \otimes (\beta^{n+2}(h_{22}) \triangleright v)(0) \]

\[
= h_{1}(\beta^{n}(u_{(-1)})) \beta^{n+2}(h_{21}) \triangleright u)(0) \otimes (\beta^{n+2}(h_{22}) \triangleright v)(0) \]

\[
= h_{1}(\beta^{n+2}(u \otimes v)(-1)) \otimes (\beta^{n+2}(h_{2}) \triangleright (u \otimes v)(0)) \]

finishing the proof.

Lemma 4.6 Let \((H, \beta)\) be a Hom-bialgebra, and 

\((U, \varphi_V, \rho^V, \alpha_U), (V, \varphi_V, \rho^V, \alpha_V), (W, \varphi_W, \rho^W, \alpha_W) \in H \text{YD} \).

With notation as above, define the linear map 

\[
a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), (u \otimes v) \otimes w \mapsto \alpha_U^{-1}(u) \otimes \alpha_V^{-1}(v) \otimes \alpha_W(w), \]
where \( u \in U, v \in V \) and \( w \in W \). Then \( \alpha_{U,V,W} \) is an isomorphism of left \((H, \beta)\)-Hom-modules and left \((H, \beta)\)-Hom-comodules.

**Proof** Same to the proof of [9, Proposition 3.2].

**Lemma 4.7** Let \((H, \beta)\) be a Hom-bialgebra and \((U, \rho^U, \alpha_U), (V, \rho^V, \alpha_V) \in \mathcal{H}_{HYD}^\dagger\). Define the linear map

\[
c_{U,V} : U \otimes V \to V \otimes U, u \otimes v \mapsto (\beta^{m+n+2}(u_{(-1)}) \circ V \alpha_V^{-1}(u_{(1)})) \otimes \alpha_U^{-1}(u_{(0)}),
\]

where \( u \in U \) and \( v \in V \). Then \( c_{U,V} \) is a morphism of left \((H, \beta)\)-Hom-modules and left \((H, \beta)\)-Hom-comodules.

**Proof** For all \( h \in H, u \in U \) and \( v \in V \), we have

\[
(\alpha_V \otimes \alpha_U) \circ c_{U,V}(u \otimes v) = \alpha_V((\beta^{m+n+2}(u_{(-1)}) \circ V \alpha_V^{-1}(v))) \otimes u_{(0)}
\]

\[
= (\beta^{m+n+3}(u_{(-1)}) \circ V v) \otimes u_{(0)}
\]

\[
= \beta^{m+n+2}(\alpha_U(u)_{(-1)}) \circ V \alpha_V^{-1}(\alpha_V(v)) \otimes \alpha_U^{-1}(\alpha_U(u)_{(0)})
\]

\[
c_{U,V} \circ (\alpha_U \otimes \alpha_V)(u \otimes v),
\]

\[
c_{U,V}(h \circ V \alpha_U(u) \otimes (h_2 \circ V v)) = (\beta^{m+n+2}(h_1 \circ V u)_{(-1)}) \circ V \alpha_V^{-1}(h_2 \circ V v)) \otimes \alpha_U^{-1}((h_1 \circ V u)_{(0)})
\]

\[
= \beta^{m+n+2}((h_1 \circ V u)_{(-1)}) \circ V (\beta^{-1}(h_2) \circ V \alpha_V^{-1}(v))) \otimes \alpha_U^{-1}((h_1 \circ V u)_{(0)})
\]

\[
= (\beta^{m+n+1}((h_1 \circ V u)_{(-1)}) \beta^{-1}(h_2) \circ V v) \otimes \alpha_U^{-1}((h_1 \circ V u)_{(0)})
\]

\[
(HYD)
\]

\[
(\rho \otimes U) \circ c_{U,V}(u \otimes v) = \rho \circ U((\beta^{m+n+2}(u_{(-1)}) \circ V \alpha_V^{-1}(v))) \otimes \alpha_U^{-1}(u_{(0)})
\]

\[
= \beta^{-2}((\beta^{m+n+2}(u_{(-1)}) \circ V \alpha_V^{-1}(v))_{(-1)} \alpha_U^{-1}(u_{(0)})_{(-1)})
\]

\[
\otimes (\beta^{m+n+2}(u_{(-1)}) \circ V \alpha_V^{-1}(v))_{(0)} \otimes \alpha_U^{-1}(u_{(0)}_{(0)})
\]

\[
= \beta^{-2}((\beta^{m+n+2}(u_{(-1)}) \circ V \alpha_V^{-1}(v))_{(-1)} \beta^{-1}(u_{(0)}_{(-1)})) \otimes (\beta^{m+n+2}(u_{(-1)}) \circ V \alpha_V^{-1}(v))_{(0)}
\]

\[
\otimes \alpha_U^{-1}(u_{(0)}_{(0)})
\]

\[
= \beta^{-2}((\beta^{m+n+1}(u_{(-1)}_{(1)}) \circ V \alpha_V^{-1}(v))_{(-1)} \beta^{-1}(u_{(0)}_{(-1)})) \otimes (\beta^{m+n+1}(u_{(-1)}_{(1)}) \circ V \alpha_V^{-1}(v))_{(0)}
\]

\[
\otimes u_{(0)}
\]
Theorem 4.8 Let \((H, \beta)\) be a Hom-bialgebra. Then the Hom-Yetter-Drinfeld category \(H \otimes YD\) is a pre-braided tensor category, with tensor product, associativity constraints, and pre-braiding in Lemmas 4.5, 4.6 and 4.7, respectively, and the unit \(I = (K, Id_K)\).

Proof The proof of the pentagon axiom for \(a_{U \otimes V, W}\) is same to the proof of [9, Theorem 3.4]. Next we prove that the hexagonal relation for \(c_{U \otimes V}\). Let \((U, \rho_U, \rho^U, \alpha_U), (V, \rho_V, \rho^V, \alpha_V), (W, \rho_W, \rho^W, \alpha_W) \in H \otimes YD\). Then for all \(u, v \in U, v \in V\) and \(w \in W\), we have

\[
((Id_U \otimes c_{U, W}) \circ a_{U \otimes V, W} \circ (c_{U, V} \otimes I_d_W))(u \otimes v \otimes w)
\]

and

\[
((c_{U, W} \otimes I_d_V) \circ a_{U, W, V} \circ (Id_U \otimes c_{V, W}))(u \otimes (v \otimes w))
\]
and the rest is obvious. These complete the proof.

References


广义Radford双积Hom-Hopf代数和相关辫子张量范畴

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摘要: 本文研究了Radford双积的Hom-型, 通过把广义smash积Hom-代数和广义smash积的余代数相结合, 得到了它们成为Hom-双代数的充要条件. 这一结果推广了著名的Radford双积.

关键词: Radford 双积; 量子Yang-Baxter方程; Yetter-Drinfeld范畴