Vol. 37 (2017)
No. 6

数 学 杂 志
J. of Math. (PRC)

QUASISYMMETRICALLY PACKING-MINIMAL MORAN SETS

LI Yan-zhe, HE Qi-han
(*College of Mathematics and Information Science, Guangxi University, Nanning 530004, China*)

Abstract: In this paper, we study the problem of packing-minimality of 1-dimensional Moran sets. By using the principle of mass distribution, we obtain that a large class of Moran sets on the line with packing dimension 1 is quasisymmetrically packing-minimal, which extends a known result of quasisymmetrically packing-minimality.

Keywords: quasisymmetric mapping; packing dimension; Moran set
2010 MR Subject Classification: 28A75; 28A78; 28A80
Document code: A **Article ID:** 0255-7797(2017)06-1125-09

1 Introduction

A homeomorphism mapping $f\colon X \to Y$, where X and Y are two metric spaces, is said to be quasisymmetric if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{\big|f(x)-f(a)\big|}{\big|f(x)-f(b)\big|} \le \eta\left(\frac{|x-a|}{|x-b|}\right)$$

for all triples a, b, x of distinct points in X. Here we follow the notation in Heinonen [1] by using $|x-y|$ to denote the distance between the two points x and y in every metric space. In particular, we also say that f is an n-dimensional quasisymmetric mapping when $X = Y = \mathbb{R}^n$.

Definition 1 We call a set $E \subset \mathbb{R}^n$ quasisymmetrically packing-minimal, if $\dim_P f(E) \ge \dim_P E$ for any n-dimensional quasisymmetric mapping f.

In this paper, we will show that a large class of Moran sets in $\mathbb{R}^1$ of packing dimension 1 have quasisymmetric packing-minimality.

Similarly, we call a set $E \subset \mathbb{R}^n$ is quasisymmetrically Hausdorff-minimal, if $\dim_H f(E) \ge \dim_H E$ for any n-dimensional quasisymmetric mapping f. Recall some results on the Hausdorff dimensions of quasisymmetric images. First, n-dimensional quasisymmetric mappings are locally Hölder continuous [2], so if $\dim_H E = 0$, then $\dim_H f(E) = 0$ and E is quasisymmetrically Hausdorff-minimal. In Euclidean space $\mathbb{R}^n$ with $n \ge 2$, Gehring [3, 4] obtained

* **Received date:** 2016-08-15 **Accepted date:** 2016-11-09
Foundation item: Supported by NSFC (11626069); Guangxi Natural Science Foundation (2016GXNSFAA380003); Science Foundation of Guangxi University (XJZ150827).
Biography: Li Yanzhe (1986–), male, born at Guilin, Guangxi, lecturer, major in fractal geometry.
Corresponding author: He Qihan.

that for any subset $E \subset \mathbb{R}^n$ of Hausdorff dimension n, its quasisymmetric image also has Hausdorff dimension n, so E is quasisymmetrically Hausdorff-minimal. If $0 < \dim_H E < 1$, there are 1-dimensional quasisymmetric mappings f_ε and F_ε such that $\dim_H f_\varepsilon(E) < \varepsilon$ (see [5]) and $\dim_H F_\varepsilon(E) > 1 - \varepsilon$ (see [6]), that is any $E \subset \mathbb{R}^1$ satisfies $0 < \dim_H E < 1$ is not quasisymmetrically Hausdorff-minimal.

For $\mathbb{R}^1$, Tukia [7] found an interesting fact, quite different from Gehring's result for $\mathbb{R}^n$ with $n \geq 2$, that there exists $E \subset \mathbb{R}^1$ such that $\dim_H E = 1$ and $\dim_H f(E) < 1$ for some 1-dimensional quasisymmetric mapping f, so E is not quasisymmetrically Hausdorff-minimal.

There is a question: what kinds of sets in $\mathbb{R}^1$ are quasisymmetrically Hausdorff-minimal?

For $\mathbb{R}^1$, many works were devoted to the quasisymmetrically Hausdorff-minimal set, i.e., the subset $E \subset \mathbb{R}^1$ satisfying $\dim_H f(E) \geq \dim_H E$ for any 1-dimensional quasisymmetric mapping f.

Kovalev [5] showed that any quasisymmetrically Hausdorff-minimal set in $\mathbb{R}^1$ with $\dim_H E > 0$ has full Hausdorff dimension 1. Hakobyan [8] proved that middle interval Cantor sets of Hausdorff dimension 1 are all quasisymmetrically Hausdorff-minimal. Hu and Wen [9] obtained that some uniform Cantor sets of Hausdorff dimension 1 are quasisymmetrically Hausdorff-minimal. Dai, Wen, Xi and Xiong [10] found a large class of Moran sets of Hausdorff dimension 1 which are quasisymmetrically Hausdorff-minimal.

Compared with quasisymmetric Hausdorff-minimality, there are few results on quasisymmetric packing-minimality.

Kovalev [5] showed that any quasisymmetrically packing-minimal set in $\mathbb{R}^1$ with $\dim_p E > 0$ has packing dimension 1. Li, Wu and Xi [11] find two classes of Moran sets of packing dimension 1 which are quasisymmetrically packing-minimal. Wang and Wen [12] obtained that the uniform Cantor sets of packing dimension 1 are quasisymmetrically packing-minimal.

In this paper, we will show that a result of [11] is not accidents. In fact, a larger class of Moran sets on the line with packing dimension 1 is quasisymmetrically packing-minimal (Theorem 1).

This paper is organized as follows. In Section 2, we state our main results and give the introduction to the Moran sets. Some preliminaries are given in Section 3, including quasisymmetric mappings, Moran sets and certain probability measure supported on the quasisymmetric image. The key of this paper is to get the estimate in Lemma 1 for the above measure. Section 4 is the proof of Theorem 1.

2 Definition and Main Results

2.1 Definition of Moran Sets

Before the statement of theorems, we introduce the notion of Moran setsin $\mathbb{R}^1$. Let $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ and $\{c_{k,j}\}_{1\leq j\leq n_k} \subset \mathbb{R}^+$ be sequences satisfying $n_k \geq 2$ and $\sum\limits_{j=1}^{n_k} c_{k,j} < 1$ for any $k \geq 1$, set $\Omega_k = \big\{\sigma = \sigma_1 \cdots \sigma_k\colon\ \sigma_j \in [1, n_j] \cap \mathbb{N} \text{ for all } 1 \leq j \leq k\big\}$ and $\Omega_0 = \{\emptyset\}$ with empty word $\emptyset$. Write $\Omega = \bigcup\limits_{k\geq 0} \Omega_k$ and $(\sigma_1 \cdots \sigma_k) * \sigma_{k+1} = \sigma_1 \cdots \sigma_k \sigma_{k+1}$. Let $I \subset \mathbb{R}^1$ be a closed interval. Denote by $|A|$ the diameter of $A \subset \mathbb{R}^n$. We say that $\mathcal{F} = \{I_\sigma\colon\ \sigma \in \Omega\}$, which is a collection of closed intervals, has Moran structure $(I, \{n_k\}, \{c_{k,j}\})$, if $I_\emptyset = I$ and for any $\sigma \in \Omega_{k-1}$, $I_{\sigma*1}, \cdots, I_{\sigma*n_k}$, whose interiors are pairwise disjoint, are subintervals of I_σ such that

$$|I_{\sigma*j}|/|I_\sigma| = c_{k,j} \text{ for all } j. \tag{2.1}$$

Then a Moran set determined by $\mathcal{F}$ is defined by

$$E(\mathcal{F}) = \bigcap_{k\geq 1} \bigcup_{\sigma\in\Omega_k} I_\sigma, \tag{2.2}$$

where any I_σ in $\mathcal{F}$ is called a basic interval of rank k if $\sigma \in \Omega_k$. Denote by $\mathcal{M}(I, \{n_k\}, \{c_{k,j}\})$ the class of all Moran sets associated with I, $\{n_k\}$ and $\{c_{k,j}\}$.

For the class $\mathcal{M}(I, \{n_k\}, \{c_{k,j}\})$, we write

$$D_k = \max_{1\leq j\leq n_k} c_{k,j}, \quad c_* = \inf_{k,j} c_{k,j}$$

and

$$s_* = \liminf_{k\to\infty} s_k \quad \text{and} \quad s^* = \limsup_{k\to\infty} s_k,$$

where s_k is defined by the equation

$$\prod_{i=1}^{k} \sum_{j=1}^{n_i} (c_{i,j})^{s_k} = 1.$$

If $\sigma \in \Omega_{k-1}, k \geq 1$, let I_σ^L (or I_σ^R) be the most left (or the most right) one of $I_{\sigma*1}, \cdots, I_{\sigma*n_k}$.Write $r_* = \inf\limits_{\sigma\in\Omega} \min\{\frac{|I_\sigma^L|}{|I_\sigma|}, \frac{|I_\sigma^R|}{|I_\sigma|}\}$.

Some probability of quasisymmetric mappings and Moran sets can be seen in [13] and [14].

2.2 Main Results

The main result of paper are stated as follows.

Theorem 1 Suppose $E \in \mathcal{M}(I, \{n_k\}, \{c_{k,j}\})$, $r_* > 0$ and $\sup_k n_k < \infty$, and there exist a costant $l > 1$ such that $\sum\limits_{i=1}^{n_k} (|\ I_{\sigma*i}\ |) \geq l(|\ I_\sigma^L\ | + |\ I_\sigma^R\ |)$ for any $\sigma \in \Omega_{k-1}$ and $k \geq 2$. If $\dim_P E = 1$, then $\dim_P f(E) = 1$ for any 1-dimensional quasisymmetric mappings f.

Remark 1 Without loss of generality, suppose $\frac{|I_\sigma^L|}{|I_\sigma|} = c_{k,1}$ and $\frac{|I_\sigma^R|}{|I_\sigma|} = c_{k,n_k}$ for $\sigma \in \Omega_k$, $k \geq 1$, the conditions of Theorem 1 implies $c_{k,1}$ and c_{k,n_k} is neither too "large" nor too "small", and for $2 \leq i \leq n_k - 1, c_{k,i}$ may be very "small", even $c_* = 0$, but $\sum\limits_{i=2}^{n_k-1} c_{k,i}$ is not too "small".

Remark 2 Notice that the condition "and there exist a costant $l > 1$ such that $\sum\limits_{i=1}^{n_k} (|\ I_{\sigma*i}\ |) \geq l(|\ I_\sigma^L\ | + |\ I_\sigma^R\ |)$ for any $\sigma \in \Omega_{k-1}$ and $k \geq 2$" implies $n_k \geq 3$ for all $k \geq 1$.

Notice that If $E \in \mathcal{M}(I, \{n_k\}, \{c_{k,j}\})$, then $E \in \mathcal{M}(I, \{N_k\}, \{C_{k,q}\})$, where $N_k = n_{2k-1} \cdot n_{2k} \geq 3$ and $C_{k,(i-1)n_{2k-1}+j} = c_{2k-1,i} \cdot c_{2k,j}$ for $1 \leq i \leq n_{2k-1}, 1 \leq j \leq n_{2k}$, so without loss of generality, we always assume that $n_k \geq 3$ for all $k \geq 1$ in this paper.

Example 1 Let $E \in \mathcal{M}(I, \{n_k\}, \{c_{k,j}\})$ with $c_* > 0$. If $n_k \geq 3$ for all $k \geq 1$, then $r_* \geq c_* > 0$, $\sup_k n_k < \infty$ and $\sum\limits_{i=1}^{n_k}(\mid I_{\sigma * i} \mid) \geq (1 + c_*)(\mid I_\sigma^L \mid + \mid I_\sigma^R \mid)$ for any $\sigma \in \Omega_{k-1}$ and $k \geq 2$; if $\inf_k n_k = 2$, then $E \in \mathcal{M}(I, \{N_k\}, \{C_{k,q}\})$, where N_k and $C_{k,q}$ are defined the same as the above remark ($N_k \geq 3$), it is easy to obtain that $E \in \mathcal{M}(I, \{N_k\}, \{C_{k,q}\})$ satisfies the conditions of Theorem 1. Then by Theorem 1, if $\dim_P E = 1$, we have $\dim_P f(E) = 1$ for any 1-dimensional quasisymmetric mapping f.

Therefore Theorem 1 extends the results of Theorem 2 in [11].

Example 2 Let E be an uniform Cantor set (see [12]) with $c_* > 0$. If $n_k \geq 3$, then $r_* = c_* > 0$, $\sup_k n_k < \infty$ and $\sum\limits_{i=1}^{n_k}(\mid I_{\sigma * i} \mid) \geq (1 + c_*)(\mid I_\sigma^L \mid + \mid I_\sigma^R \mid)$ for any $\sigma \in \Omega_{k-1}$ and $k \geq 2$; if $\inf_k n_k = 2$, then $E \in \mathcal{M}(I, \{N_k\}, \{C_{k,q}\})$, where N_k and $C_{k,q}$ are defined the same as the above remark($N_k \geq 3$), it is easy to obtain that $E \in \mathcal{M}(I, \{N_k\}, \{C_{k,q}\})$ satisfies the conditions of Theorem 1. Then by Theorem 1, if $\dim_P E = 1$, we have $\dim_P f(E) = 1$ for any 1-dimensional quasisymmetric mapping f.

Therefore Theorem 1 extends the results of Theorem 1.2 in [12] when $c_* > 0$.

3 Preliminaries

Before the proofs of the two theorems, we give some preliminaries.

The following fact on packing dimension can be found in Proposition 2.3 of [15].

Lemma 1 Let $E \subset \mathbb{R}^n$ be a Borel set, and μ a probability measure supported on E. If there exists $E' \subset E$ with $\mu(E') > 0$ and a constant $c > 0$ such that

$$\liminf_{r \to 0} \frac{\mu(B(x,r))}{r^s} \leq c \text{ for all } x \in E',$$

then $\dim_P E \geq s$.

We need some properties on quasisymmetry. For closed interval I, set ρI be a closed interval with a length of $\rho|I|$ and with the same center with I.

From [16], it is easy to check the following lemma.

Lemma 2 Suppose $f : \mathbb{R}^1 \to \mathbb{R}^1$ is quasisymmetric, there exist constants λ, $K_\rho > 0$, $q \geq 1$ and $p \in (0, 1]$ such that

$$\frac{\mid f(\rho I) \mid}{\mid f(I) \mid} \leq K_\rho \tag{3.1}$$

and

$$\lambda(\frac{\mid I \mid}{\mid I' \mid})^q \leq \frac{\mid f(I) \mid}{\mid f(I') \mid} \leq 4(\frac{\mid I \mid}{\mid I' \mid})^p, \tag{3.2}$$

whenever closed interval I, I' satisfying $I \subset I'$

The following lemma comes from [17].

Lemma 3 Suppose E is a Moran set satisfying the following conditions

(1) $\sup_k n_k < \infty$;

(2) $0 < \inf_k D_k \leq \sup_k D_k < 1$.

Then we have $\dim_P E = s^*$.

It is easy to verify that if for a Moran set $E \in \mathcal{M}(I, \{n_k\}, \{c_{k,j}\})$, the conditions of Theorem 1 hold, then E satisfies $\sup_k n_k < \infty$ and $0 < \inf_k D_k \leq \sup_k D_k < 1$, by Lemma 3, $\dim_P E = s^*$.

The length of $\sigma \in \Omega_k$ will be denoted by $|\sigma|(= k)$.

Fix a 1-dimensional quasisymmetric mapping $f : \mathbb{R}^1 \to \mathbb{R}^1$. Given a Moran set E and its basic interval I_σ of E with rank k, we also call $f(I_\sigma)$ a basic interval of $f(E)$ with rank k for convenience. Let $J_\sigma = f(I_\sigma)$.

3.1 The Measure μ_d Supported on $f(E)$

Fix $d \in (0, 1)$. We will define a probability measure μ_d on $f(E)$ as follows.

Without loss of generality, we set $I = [0, 1]$ the initial interval of E.

Let $\mu_d(f([0, 1])) = 1$, for every $k \geq 1$, and for every basic interval J_σ of rank $k - 1$, we define

$$\mu_d(J_{\sigma * j'}) = \frac{|J_{\sigma * j'}|^d}{\sum_{j=1}^{n_k} |J_{\sigma * j}|^d} \mu_d(J_\sigma)$$

for $1 \leq j' \leq n_k$.

3.2 Estimate of $\mu_d(J_\sigma)$

The next proposition can be found in [11].

Proposition 1 Suppose E is the Moran set satisfying $\sup_k n_k < \infty$ and

$$\lim_{k \to \infty} \frac{\mathrm{card}\{1 \leq i \leq k \colon D_i \leq \alpha\}}{k} = 1 \tag{3.3}$$

for some constant $\alpha \in (0, 1)$. If s$^* = 1$, then there exists a subsequence $\{k_t\}_t$ and a constant $c > 0$ such that

$$\mu_d(J_\sigma) \leq c\, |J_\sigma|^d \tag{3.4}$$

for any basic interval J_σ of $f(E)$ with $|\sigma| \in \{k_t\}_t$.

By Proposition 1, we have the corollary below.

Corollary 1 Suppose E is the Moran satisfies the conditions of Theorem 1. If $\dim_P E = 1$, then there exists a subsequence $\{k_t\}_t$ and a constant $c > 0$ such that

$$\mu_d(J_\sigma) \leq c\, |J_\sigma|^d$$

for any basic interval J_σ of $f(E)$ with $|\sigma| \in \{k_t\}_t$.

Proof Since $n_k \geq 2$, $r_* > 0$, we have $D_k \leq 1 - r_* < 1$. Take $\alpha = 1 - r_*$, we have

$$\lim_{k \to \infty} \frac{\mathrm{card}\{1 \leq i \leq k \colon D_k \leq 1 - r_*\}}{k} = 1,$$

notice that $\dim_p E = s^*$ by Lemma 3 and Proposition 1, the corollary follows.

4 Proof of Theorem 1

Let $\{k_t\}_t$ be the subsequence in Proposition 1. Let

$$B_t = \left\{ x \in f(E) : f^{-1}(x) \in I_\sigma^L \cup I_\sigma^R \text{ for some } |\sigma| = k_t - 1 \right\}$$

and $B = \bigcup_{s=1}^{\infty} \bigcap_{t \geq s} B_t$.

Lemma 4 Suppose that $n_k \geq 3$ and $c_* > 0$. Then there exists a constant $\epsilon > 0$ such that

$$\frac{\mu_d(J_\sigma^L) + \mu_d(J_\sigma^R)}{\mu_d(J_\sigma)} \leq 1 - \epsilon$$

for all σ with $|\sigma| = k - 1$ and $1 \leq j_1, j_2 \leq n_k$.

Proof With out loss of generality, we let $J_\sigma^L = J_{\sigma*1}, J_\sigma^R = J_{\sigma*n_k}$, since $n_k \geq 3$. Take $i_0 (1 \leq i_0 \leq n_k)$ as follows

Case 1 If $\max\limits_{1 \leq j \leq n_k} |J_{\sigma*i}| \neq \max\big\{|J_{\sigma*1}|, |J_{\sigma*n_k}|\big\}$, pick i_0 such that $\max\limits_{1 \leq i \leq n_k} |J_{\sigma*i}| = |J_{\sigma*i_0}|$, then $2 \leq i_0 \leq n_k - 1$, we have

$$\begin{aligned} \frac{\mu_d(J_\sigma^L) + \mu_d(J_\sigma^R)}{\mu_d(J_\sigma)} &= \frac{\mu_d(J_{\sigma*1}) + \mu_d(J_{\sigma*n_k})}{\mu_d(J_\sigma)} = \frac{|J_{\sigma*1}|^d + |J_{\sigma*n_k}|^d}{\sum\limits_{i=1}^{n_k} |J_{\sigma*i}|^d} \\ \leq \quad & 1 - \frac{|J_{\sigma*i_0}|^d}{\sum\limits_{i=1}^{n_k} |J_{\sigma*i}|^d} \leq 1 - \frac{|J_{\sigma*i_0}|^d}{(\sup\limits_k n_k)(\max\limits_{1 \leq j \leq n_k} |J_{\sigma*i}|)^d} = 1 - \frac{1}{\sup\limits_k n_k}. \end{aligned}$$

Pick $\epsilon = \frac{1}{\sup\limits_k n_k}$, we have

$$\frac{\mu_d(J_\sigma^L) + \mu_d(J_\sigma^R)}{\mu_d(J_\sigma)} \leq 1 - \epsilon.$$

Case 2 If $\max\limits_{1 \leq j \leq n_k} |J_{\sigma*i}| = \max\big\{|J_{\sigma*1}|, |J_{\sigma*n_k}|\big\}$, pick i_0 such that $|I_{\sigma*i_0}| = \max\limits_{2 \leq i \leq n_k - 1} |I_{\sigma*i}|$, we have

$$\begin{aligned} \frac{\mu_d(J_\sigma^L) + \mu_d(J_\sigma^R)}{\mu_d(J_\sigma)} &= \frac{\mu_d(J_{\sigma*1}) + \mu_d(J_{\sigma*n_k})}{\mu_d(J_\sigma)} = \frac{|J_{\sigma*1}|^d + |J_{\sigma*n_k}|^d}{\sum\limits_{i=1}^{n_k} |J_{\sigma*i}|^d} \\ \leq \quad & 1 - \frac{|J_{\sigma*i_0}|^d}{\sum\limits_{i=1}^{n_k} |J_{\sigma*i}|^d} \leq 1 - \frac{|J_{\sigma*i_0}|^d}{(\sup\limits_k n_k)(\max\limits_{1 \leq j \leq n_k} |J_{\sigma*i}|)^d}. \end{aligned}$$

Assume $|J_{\sigma*1}| = \max\limits_i |J_{\sigma*i}|$. Since

$$\begin{aligned} \sup_k n_k |I_{\sigma*i_0}| &\geq \sum_{2 \leq i \leq n_k - 1} |I_{\sigma*i}| = \sum_{1 \leq i \leq n_k} |I_{\sigma*i}| - |I_{\sigma*1}| - |I_{\sigma*n_k}| \\ \geq \quad & (l-1)(|I_{\sigma*1}| + |I_{\sigma*n_k}|) \geq (l-1)(1 + \frac{1}{r_*})|I_{\sigma*1}| \end{aligned}$$

and $\frac{|I_{\sigma*1}|}{|I_\sigma|} \geq r_*$, where l and r_* are obtained in Theorem 1, then there exists constants $\delta_1 > 0$ and $\delta_2 > 0$, such that $\frac{|I_{\sigma*i_0}|}{|I_\sigma|} \geq \delta_1$ and $\frac{|I_{\sigma*1}|}{|I_{\sigma*i_0}|} \leq \delta_2$.

By Lemma 2, we have

$$\begin{aligned} \lambda(\frac{|I_{\sigma*i_0}|}{|I_\sigma|})^q &\leq \frac{|J_{\sigma*i_0}|}{|J_\sigma|} \leq 4(\frac{|I_{\sigma*i_0}|}{|I_\sigma|})^p, \\ \lambda(\frac{|I_{\sigma*1}|}{|I_\sigma|})^q &\leq \frac{|J_{\sigma*1}|}{|J_\sigma|} \leq 4(\frac{|I_{\sigma*1}|}{|I_\sigma|})^p \leq 4(\frac{\delta_2\, |I_{\sigma*i_0}|}{|I_\sigma|})^p, \end{aligned}$$

which imply

$$\frac{|J_{\sigma*i_0}|}{|J_{\sigma*1}|} \geq \frac{\lambda}{4\delta_2^p}(\frac{|I_{\sigma*i_0}|}{|I_\sigma|})^{q-p} \geq \frac{\lambda}{4\delta_2^p}(\delta_1)^{q-p}.$$

Therefore

$$\frac{\mu_d(J_\sigma^L) + \mu_d(J_\sigma^R)}{\mu_d(J_\sigma)} \leq 1 - \frac{|J_{\sigma*i_0}|^d}{(\sup\limits_k n_k)(|J_{\sigma*1}|)^d} \leq 1 - \epsilon,$$

where

$$\epsilon = \frac{\lambda^d}{4^d\delta_2^{pd}}(\delta_1)^{(q-p)d}/\sup_k n_k.$$

Proposition 2 $\mu_d(B) = 0$ for $d \in (0, 1)$.

Proof It suffices to prove $\mu_d(\bigcap\limits_{t\geq s} B_t) = 0$ for any s.

Let

$$E_k = \bigcup_{\substack{|\sigma|=k,\\ (\bigcap\limits_{t\geq s} B_t)\bigcap J_\sigma\neq\varnothing}} J_\sigma.$$

Notice that $\bigcap\limits_{t\geq s} B_t = \bigcap\limits_{k=1}^{\infty} E_k$ and $E_{k+1} \subset E_k \subset E_{k-1} \subset \cdots$, where

$$\frac{\mu_d(E_{k+1})}{\mu_d(E_k)} \leq 1. \tag{4.1}$$

For any σ, let $J_\sigma^L = f(I_\sigma^L)$, $J_\sigma^R = f(I_\sigma^R)$. Therefore, for $t \geq s$,

$$E_{k_t} = \bigcup_{\substack{|\sigma|=k_t-1,\\ (\bigcap\limits_{t\geq s} B_t)\bigcap J_\sigma\neq\varnothing}} \left(J_\sigma^L \bigcup J_\sigma^R\right).$$

By Lemma 4, we have

$$\frac{\mu_d(E_{k_t})}{\mu_d(E_{k_t-1})} \leq \sup_{|\sigma|=K_t-1} \frac{\mu_d(J_\sigma^L) + \mu_d(J_\sigma^R)}{\mu_d(J_\sigma)} \leq 1 - \epsilon, \tag{4.2}$$

it follows from (4.1) and (4.2) that

$$\frac{\mu_d(E_{k_t})}{\mu_d(E_{k_{t-1}})} = \frac{\mu_d(E_{k_t})}{\mu_d(E_{k_t-1})} \cdot \frac{\mu_d(E_{k_t-1})}{\mu_d(E_{k_t-2})} \cdots \frac{\mu_d(E_{k_{t-1}+1})}{\mu_d(E_{k_{t-1}})} \leq 1 - \epsilon,$$

which implies

$$\mu_d(\bigcap_{t\geq s} B_t) = \lim_{t\to\infty} \mu_d(E_{k_t}) \leq \lim_{t\to\infty} (1-\epsilon)^{t-s}\mu_d(E_{k_s}) = 0.$$

Next we finish the proof of Theorem 1.

From the proposition above, we have $\mu_d(f(E)\backslash B) = 1 > 0$. Fix $x \in (f(E)\backslash B)$, then we can pick $t_n \uparrow \infty$ satisfies $f^{-1}(x) \in I_\sigma \backslash (I_\sigma^L \bigcup I_\sigma^R)$ with some $|\sigma| = k_{t_n} - 1$.

Notice that $g_x(\alpha) = |f^{-1}(B(x,\alpha))|$ is continuous. We can pick r_n such that

$$\left|f^{-1}(B(x,r_n))\right| = \min\left\{|I_\sigma^L|, |I_\sigma^R|\right\}, \tag{4.3}$$

notice that $r_n \to 0$ when $n \to \infty$.

Since $f^{-1}(x) \in I_\sigma \backslash (I_\sigma^L \bigcup I_\sigma^R)$ and $|f^{-1}(B(x,r_n))| = \min\left\{|I_\sigma^L|, |I_\sigma^R|\right\}$, we have

$$f^{-1}(B(x,r_n)) \subset I_\sigma,$$

which implies $B(x,r_n) \subset J_\sigma$.

Let $J_{\sigma * j_1}$, $J_{\sigma * j_2}$,$\cdots$,$J_{\sigma * j_{l'}}$ $(1 \leq l' \leq n_{k_{tn}})$ be the basic intervals of rank k_{t_n} meeting $B(x,r_n)$. Then $(B(x,r_n) \bigcap f(E)) \subset \bigcup\limits_{i=1}^{l'} J_{\sigma * j_i}$.

Using the conclusion of Corollary 2, we get

$$\begin{aligned} \mu_d(B(x,r_n)) &= \mu_d(B(x,r_n)\bigcap f(E)) \\ \leq \quad & \mu_d(\bigcup_{i=1}^{l'} J_{\sigma*j_i}) \leq \sum_{i=1}^{l'} \mu_d(J_{\sigma*j_i}) \leq c\sum_{i=1}^{l'} |J_{\sigma*j_i}|^d. \end{aligned}$$

Since $r_* > 0$, for any i, there exists a constant $\delta \geq 1$,

$$|I_{\sigma*j_i}| \leq \delta \min\left\{|I_\sigma^L|, |I_\sigma^R|\right\} = \delta \left|f^{-1}(B(x,r_n))\right|,$$

hence $|I_{\sigma*j_i}| \subset (3\delta)f^{-1}(B(x,r_n))$, where $\delta \geq 1$. By Lemma 2, we have

$$|J_{\sigma*j_i}| = |f(I_{\sigma*j_i})| \leq \left|f((3\delta)f^{-1}(B(x,r_n)))\right| \leq K_{3\delta}\left|(B(x,r_n))\right| = (2K_{3\delta})r_n,$$

where $K_{3\delta} > 0$ is a constant. This together with (4.4) gives

$$\mu_d(B(x,r_n) \leq [2^d(\sup_i n_i)c(K_{3\delta})^d] \cdot r_n^d.$$

Let $n \to \infty$, then for any $x \in f(E)\backslash B$, there exists a constant $C' > 0$, such that

$$\liminf_{r\to 0} \frac{\mu_d(B(x,r))}{r^d} \leq C',$$

it follows from $\mu_d(f(E)\backslash B) > 0$ and Lemma 1 that $\dim_p f(E) \geq d$. Let $d \to 1$, we have

$$\dim_p f(E) = 1.$$

References

[1] Heinonen J. Lectures on analysis on metric spaces[M]. New York: Springer-Verlag, 2001.

[2] Ahlfors L V. Lectures on quasiconformal mappings (2nd ed.)[M]. Unversity Lecture Series, Vol. 38, Maryland: American Mathematical Society, 2006.

[3] Gehring F W. The L^p-integrability of the partial derivatives of a quasiconformal mapping[J]. Acta Math., 1973, 130: 265–277.

[4] Gehring F W, Väisälä J. Hausdorff dimension and quasiconformal mappings[J]. J. London Math. Soc., 1973, 6(2): 504–512.

[5] Kovalev L V. Conformal dimension does not assume values between zero and one[J]. Duke Math. J., 134(1): 1–13, 2006.

[6] Bishop C J. Quasiconformal mappings which increase dimension[J]. Ann. Acad. Sci. Fenn. Math., 1999, 24(2): 397–407.

[7] Tukia P. Hausdorff dimension and quasisymmetric mappings[J]. Math. Scand., 1989, 65(1): 152–160.

[8] Hakobyan H. Cantor sets that are minimal for quasisymmetric mappings[J]. J. Contemp. Math. Anal., 2006, 41(2): 13–21.

[9] Hu Meidan, Wen Shengyou. Quasisymmetrically minimal uniform Cantor sets[J]. Topology. Appl., 2008, 155(6): 515–521.

[10] Dai Yuxia, Wen Zhixiong, Xi Lifeng, Xiong Ying. Quasisymmetrically minimal Moran sets and Hausdorff dimension[J]. Ann. Acad. Sci. Fenn. Math., 2011, 36: 139–151.

[11] Li Yanzhe, Wu Min, Xi Lifeng. Quasisymmetric minimality on packing dimension for Moran sets[J]. J. Math. Anal. Appl., 2013, 408: 324–334.

[12] Wang Wen, Wen Shengyou. Quasisymmetric minimality of Cantor sets[J]. Topology Appl., 2014, 178: 300–314.

[13] Lou Manli. Gap sequence and quasisymmetric mapping[J]. J. Math., 2015, 35: 705–708.

[14] Liu Xiaoli, Liu Weibin. The Hausdorff dimension of a class of Moran sets[J]. J. Math., 2016, 36: 100–104.

[15] Falconer K. Techniques in fractal geometry[M]. Chichester: John Wiley Sons Ltd, 1997.

[16] Wu J M. Null sets for doubling and dyadic doubling measures[J]. Ann. Acad. Sci. Fenn. Ser. A. Math., 1993, 18(1): 77–91.

[17] Hua Su, Rao Hui, Wen Zhiying, Wu Jun. On the structures and dimensions of Moran sets[J]. Sci. China Ser. A., 2000, 43(8): 836–852.

拟对称packing极小Moran集

李彦哲, 何其涵

(广西大学数学与信息科学学院, 广西 南宁 530004)

摘要: 本文研究了一维Moran集的拟对称packing极小性的问题. 利用质量分布原理的方法, 获得了直线上一类packing维数为1的Moran集为拟对称packing极小集的结果, 推广了参考文献中关于拟对称packing极小性的已知结果.

关键词: 拟对称映射; packing维数; Moran集

MR(2010)**主题分类号**: 28A75; 28A78; 28A80 **中图分类号**: O174.12