Vol. 37 (2017)
No. 6

数 学 杂 志
J. of Math. (PRC)

KILLING VECTOR FIELDS ON COMPACT RIEMANNIAN MANIFOLDS WITH NEGATIVE SCALAR CURVATURE

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Abstract: In this paper, we investigate killing vector fields on compact Riemannian manifolds with negative scalar curvature. By using the Bochner method, we obtain a necessary condition of the existence of non-trivial killing vector fields on these manifolds, which extends Theorem 1 due to [6].

Keywords: killing vector field; negative scalar curvature; trace-free Ricci curvature tensor

2010 MR Subject Classification: 53C20; 53C24

Document code: A **Article ID:** 0255-7797(2017)06-1118-07

1 Introduction

A vector field V on a Riemannian manifold (M, g) is Killing if the Lie derivative of the metric with respect to V vanishes as follows

$$L_V g = 0, \tag{1.1}$$

which is equivalent to

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0, \tag{1.2}$$

where ∇ denotes the covariant differential operator of (M, g) and $X, Y \in TM$. This is equivalent to the fact that the one-parameter group of diffeomorphisms associated to V consists in isometries. Therefore, the space of the non-trivial Killing vector fields for (M, g) in some sense measures the size of the isometry group of (M, g).

The study of killing vector field has a long time. In 1946, Bochner [2] proved that when (M, g) is compact and has negative Ricci curvature, every Killing vector field must vanish. Later, Bochner's result was extended by Yano to include conformal vector fields [10]. It is well known that the existence of non-trivial closed conformal vector fields also imposes many restrictions on a compact Riemannian manifold (see [9]). The Killing vector fields

* **Received date:** 2016-09-02 **Accepted date:** 2016-11-17

Foundation item: Supported by the National Natural Science Foundations of China (11261038; 11361041).

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were generalized to the Killing p-form and conformal Killing p-form by some authors such as Bochner and Yano, Gallot and Meyer, Tachibana and Yamaguchi, Liu Jizhi and Cai Kairen (see [8]). In 1999, Gursky [5] proved a vanishing theorem for conformal vector fields on four-manifolds of negative scalar curvature, whose assumptions were conformally invariant, and in the case of locally conformally flat manifolds reduced to a sign condition on the Euler characteristic. The proof due to Gursky is actually a refinement of the Bochner method which had been used to prove classical vanishing theorems. Recently, inspired by Gursky's two papers [4, 5], Hu and Li [6] proved Theorem A as follows.

Theorem A Let (M, g) be an n-dimensional compact oriented Riemannian manifold with scalar curvature $R < 0$. If there exists a non-trivial Killing vector field on (M, g), then we have

$$\int_M \frac{1}{R}(|E|^2 - \frac{R^2}{n(n-1)}) \leq 0, \tag{1.3}$$

where E denotes the trace-free Ricci tensor. Moreover, equality is attained in (1.3) if and only if R is constant and the Riemannian universal cover of (M, g) is isometric to a Riemannian product $\mathbb{R} \times N^{n-1}$ for some Einstein manifold N^{n-1} with constant scalar curvature R.

We follow their methods [3, 5, 6] to improve Theorem A and obtain the following results.

Theorem 1.1 Let (M, g) be an n-dimensional compact oriented Riemannian manifold with scalar curvature R. If there exists a non-trivial Killing vector field on (M, g), then we have

$$\int_M |E| \geq -\int_M \frac{R}{\sqrt{n(n-1)}}, \tag{1.4}$$

where E denotes the trace-free Ricci tensor. Moreover, equality holds in (1.4) if and only if R is nonpositive constant and the Riemannian universal cover of (M, g) is isometric to a Riemannian product $\mathbb{R} \times N^{n-1}$ for some Einstein manifold N^{n-1} with constant scalar curvature R.

Corollary 1.2 Let (M, g) be an n-dimensional compact oriented Riemannian manifold with scalar curvature $R < 0$. If there exists a non-trivial Killing vector field on (M, g), then we have

$$\int_M |E| \geq -\int_M \frac{R}{\sqrt{n(n-1)}},$$

where E denotes the trace-free Ricci tensor. Moreover, equality holds in the above if and only if R is constant and the Riemannian universal cover of (M, g) is isometric to a Riemannian product $\mathbb{R} \times N^{n-1}$ for some Einstein manifold N^{n-1} with constant scalar curvature R.

2 Proof of Theorem 1.1

Let (M, g) be an n-dimensional Riemannian manifold. Let $\{e_1, \cdots, e_n\}$ with respect to the Riemannian metric g be a local orthonormal basis of TM, and $\{\theta_1, \cdots, \theta_n\}$ be its dual

basis. Let $\{\theta_{ij}\}$ be the connection forms of (M,g), one has the structure equations

$$d\theta_i = \sum_j \theta_{ij} \wedge \theta_j, \quad \theta_{ij} + \theta_{ji} = 0, \tag{2.1}$$

$$d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \theta_k \wedge \theta_l, \tag{2.2}$$

where R_{ijkl} are the components of the Riemannian curvature tensor of (M,g). The Ricci curvature tensor R_{ij} and the scalar curvature R of (M,g) are defined by

$$R_{ij} = \sum_k R_{ikjk} \text{ and } R = \sum_i R_{ii}, \tag{2.3}$$

respectively. For a vector field $V = \sum_i V_i e_i$ on (M,g), we define the covariant derivative $V_{i,j}$ and the second covariant derivative $V_{i,jk}$, respectively, by $\nabla V = \sum_{i,j} V_{i,j} \theta_j \otimes e_i$, i.e.,

$$\sum_j V_{i,j} \theta_j = dV_i + \sum_j V_j \theta_{ji} \tag{2.4}$$

and

$$\sum_k V_{i,jk} \theta_k = dV_{i,j} + \sum_k V_{k,j} \theta_{ki} + \sum_k V_{i,k} \theta_{kj}. \tag{2.5}$$

Using exterior derivation of (2.4), one gets the Ricci identity

$$V_{i,kl} = V_{i,lk} + \sum_j V_j R_{jikl}. \tag{2.6}$$

Now we assume that $V = \sum_i V_i e_i$ is a Killing vector. Note that from (1.2), $V = \sum_i V_i e_i$ is a Killing vector field is equivalent to

$$V_{i,j} + V_{j,i} = 0 \text{ for all } i, j. \tag{2.7}$$

From (2.5) and (2.7), we have

$$\begin{aligned} \sum_{j,k} V_{j,jk} \theta_k &= \sum_j (dV_{j,j} + \sum_k V_{k,j} \theta_{kj} + \sum_k V_{j,k} \theta_{kj}) \\ &= d(\sum_j V_{j,j}) + \sum_{j,k} (V_{k,j} + V_{j,k}) \theta_{kj} \\ &= 0, \end{aligned}$$

and thus for any k,

$$\sum_j V_{j,jk} = 0. \tag{2.8}$$

Combing (2.6), (2.7) with (2.8), we get the following Weitzenböck formula (see [1])

$$\begin{aligned}\frac{1}{2}\triangle|V|^2 &= \sum_{i,j} V_{i,j}^2 + \sum_{i,j} V_i V_{i,jj} = |\nabla V|^2 - \sum_{i,j} V_i V_{j,ij} \\ &= |\nabla V|^2 - \sum_{i,j} V_i (V_{j,ji} + \sum_k V_k R_{kjij}) \\ &= |\nabla V|^2 - \sum_{i,j} V_i V_j R_{ij}. \end{aligned} \tag{2.9}$$

Lemma 2.1 Let $V = \sum_i V_i e_i$ be a Killing vector field on the n-dimensional Riemannian manifold (M, g). Then we have

$$\sum_i V_i^2 \sum_{i,j} V_{i,j}^2 \geq 2 \sum_j (\sum_i V_i V_{i,j})^2. \tag{2.10}$$

Remark 2.2 In [5], Gursky observed that (2.10) still holds for every conformal vector field V. In [6], Hu and Li proved Lemmas 2.1 and 2.3. For completeness, we write the proofs of Hu and Li.

Proof It suffices to prove (2.10) for any fixed point $p \in \Omega_0 := \{x \in M | V(x) \neq 0\}$. Note that on Ω_0, (2.10) is equivalent to

$$|\nabla V|^2 \geq 2|\nabla |V||^2. \tag{2.11}$$

Around p, we choose $\{e_i\}$ such that $V(p) = V_1(p)e_1(p)$; that is, $V_2 = \cdots = V_n = 0$ at p. From (2.7), we have

$$V_{1,1} = \cdots = V_{n,n} = 0; \;\; V_{1,j} = -V_{j,1}, \;\; 2 \leq j \leq n.$$

Then at p, we have

$$2\sum_j (\sum_i V_i V_{i,j})^2 = 2V_1^2 \sum_j V_{1,j}^2 = 2V_1^2 \sum_{j=2}^n V_{1,j}^2 \leq V_1^2 \sum_{i,j} V_{i,j}^2 = \sum_i V_i^2 \sum_{i,j} V_{i,j}^2.$$

This proves (2.10), or equivalently proves (2.11) on Ω_0.

Combing (2.6) and (2.7) with (2.8), we have

$$\triangle V_i = \sum_j V_{i,jj} = -\sum_j V_j R_{ij}, \quad 1 \leq i \leq n. \tag{2.12}$$

Form (2.12) and the unique continuation result of Kazdan [7], we know that, for a non-trivial Killing vector field V on the compact Riemannian manifold (M, g), the set $M \backslash \Omega_0$ is of measure zero. Combining this fact with (2.9) and (2.11), one has the following lemma.

Lemma 2.3 Let V be a non-trivial Killing vector field on a compact Riemannian manifold (M, g). Then

$$\frac{1}{2}\triangle|V|^2 \geq 2|\nabla|V||^2 - \sum_{i,j} V_i V_j R_{ij} \tag{2.13}$$

holds on M in the sense of distributions.

In order to prove Theorem 1.1, we need the following algebraic lemma, which can be proved by the standard method of Lagrange multipliers, and was observed by Hu and Li [6].

Lemma 2.4 Let $A = (a_{ij})_{n\times n}$ be a real symmetric matrix with $\sum\limits_i a_{ii} = 0$ and $x_1, \cdots, x_n \in \mathbb{R}$. Then

$$-\sqrt{\frac{n-1}{n}\sum_{i,j} a_{ij}^2}(\sum_i x_i^2) \le \sum_{i,j} a_{ij}x_ix_j \le \sqrt{\frac{n-1}{n}\sum_{i,j} a_{ij}^2}(\sum_i x_i^2). \tag{2.14}$$

Moreover, when $\sum\limits_i x_i^2 \neq 0$, one of the equalities in (2.14) is attained if and only if there exists an orthogonal $n \times n$ matrix T such that

$$TAT^{-1} = \begin{pmatrix} (n-1)\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda \end{pmatrix} \tag{2.15}$$

and $(x_1, \cdots, x_n)$ correspondingly takes the value $((n-1)\lambda, 0, \cdots, 0)$, where $\lambda > 0$ holds if there is equality in the right-hand side of (2.14), and $\lambda < 0$ holds if there is equality in the left-hand side of (2.14).

Proof of Theorem 1.1 Let $V = \sum\limits_i V_ie_i$ be a non-trivial Killing vector field on (M, g). Denote by E the trace-free part of the Ricci tensor Ric, i.e., $E_{ij} = R_{ij} - (R/n)\delta_{ij}$. Then, applying Lemma 2.4, we get that

$$-\sum_{i,j} V_iV_jR_{ij} = -\sum_{i,j} V_iV_jE_{ij} - \frac{R}{n}|V|^2 \ge -\sqrt{\frac{n-1}{n}}|E||V|^2 - \frac{R}{n}|V|^2 \tag{2.16}$$

and the second equality holds at a point $p \in M$ with $V(p) \neq 0$ if and only if by choosing suitable $\{e_i\}$, E can be diagonalized as

$$E = \begin{pmatrix} (n-1)\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda \end{pmatrix}, \tag{2.17}$$

correspondingly, $V_1 = (n-1)\lambda$ and $V_2 = \ldots = V_n = 0$ for some $\lambda > 0$.

Combining (2.13) with (2.16), we get

$$\frac{1}{2}\triangle|V|^2 \ge 2|\nabla|V||^2 - \sqrt{\frac{n-1}{n}}|E||V|^2 - \frac{R}{n}|V|^2. \tag{2.18}$$

For $\varepsilon > 0$, we define a function $f_\varepsilon = \sqrt{|V|^2 + \varepsilon^2}$. Thus we

$$|\nabla f_\varepsilon|^2 = \frac{|V|^2|\nabla|V||^2}{|V|^2 + \varepsilon^2} \le |\nabla|V||^2. \tag{2.19}$$

From (2.18) and (2.19), we directly compute

$$\begin{aligned} f_\varepsilon \triangle f_\varepsilon + |\nabla f_\varepsilon|^2 &= \frac{1}{2}\triangle f_\varepsilon^2 = \frac{1}{2}\triangle |V|^2 \\ &\geq 2|\nabla |V||^2 - \sqrt{\frac{n-1}{n}}|E||V|^2 - \frac{R}{n}|V|^2 \\ &\geq 2|\nabla f_\varepsilon|^2 - \sqrt{\frac{n-1}{n}}|E| f_\varepsilon^2 - \frac{R}{n} f_\varepsilon^2 \end{aligned} \tag{2.20}$$

and thus

$$f_\varepsilon \triangle f_\varepsilon \geq |\nabla f_\varepsilon|^2 - \sqrt{\frac{n-1}{n}}|E| f_\varepsilon^2 - \frac{R}{n} f_\varepsilon^2 \tag{2.21}$$

and thus

$$\triangle \log f_\varepsilon = f_\varepsilon^{-2}(f_\varepsilon \triangle f_\varepsilon - |\nabla f_\varepsilon|^2) \geq -\sqrt{\frac{n-1}{n}}|E| - \frac{R}{n}. \tag{2.22}$$

Integrating (2.22) on M, we obtain

$$0 = \int_M \Delta \log f_\varepsilon \geq -\int_M (\sqrt{\frac{n-1}{n}}|E| + \frac{R}{n}),$$

i.e.,

$$\int_M |E| \geq -\frac{1}{\sqrt{(n-1)n}} \int_M R. \tag{2.23}$$

If the equality holds in (2.23), then equality (2.16) must hold at each point of M. Thus at each point of M, E can be diagonalized as in (2.17). So it satisfies

$$|E| \equiv -\frac{R}{\sqrt{n(n-1)}} \quad \text{on} \quad M. \tag{2.24}$$

Furthermore, combining (2.17) with (2.24) gives $\lambda = -R/n(n-1)$. Then we must have $V_1 = -R/n$, $V_2 = \ldots = V_n = 0$ and

$$\mathrm{Ric} = (R_{ij}) = \begin{pmatrix} 0 & & & \\ & \frac{R}{n-1} & & \\ & & \ddots & \\ & & & \frac{R}{n-1} \end{pmatrix}. \tag{2.25}$$

This implies that $\sum\limits_{ij} V_i V_j R_{ij} = 0$, and thus from (2.9), by the maximum principle we get that $|V| = -R/n$ is constant, then $\nabla V = 0$, i.e., V is a parallel vector field on (M, g). By the de Rham decomposition theorem and (2.25), we see that the Riemannian universal cover $\widetilde{M}$ of M is the product of $\mathbb{R}$ and an Einstein manifold $\widetilde{N}$ with nonpositive constant scalar curvature R.

Remark 2.5 We see that if $R<0$, one has

$$\begin{aligned}-(\sqrt{\frac{n-1}{n}}|E|+\frac{R}{n})|V|^2 &= -\frac{n-1}{2R}[(|E|+\frac{R}{\sqrt{n(n-1)}})^2-(|E|^2-\frac{R^2}{n(n-1)})]|V|^2\\ &\geq \frac{n-1}{2R}(|E|^2-\frac{R^2}{n(n-1)})|V|^2.\end{aligned}$$

Then

$$-(\sqrt{\frac{n-1}{n}}|E|+\frac{R}{n})\geq\frac{n-1}{R}(|E|^2-\frac{R^2}{n(n-1)})$$

holds on M in the sense of distributions. Hence Corollary 1.2 can be considered as generalization of Theorem 1 in [6], i.e., Theorem A.

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具有负数量曲率的紧致黎曼流形的Killing向量场

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摘要: 本文研究了具有负数量曲率的紧致黎曼流形上的Killing向量场. 利用Bochner方法, 得到在此类流形上非平凡的Killing向量场的存在的必要条件. 这个结果拓广了文献[6]中的定理1.

关键词: Killing 向量场; 负数量曲率; 无迹Ricci 曲率张量

MR(2010)**主题分类号**: 53C20; 53C24 **中图分类号**: O186.12