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ON THE SUM OF k-POWER OF ALL DISTANCES IN BIPARTITE GRAPHS

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Abstract: Denote the sum of k-power of all distances between all pairs of vertices in G by $S_k(G)$. In this paper, by applying the vertex partition method, sharp bound of all connected n-vertex bipartite graphs of diameter d on the $S_k(G)$ is obtained, and the extremal graphs with the minimal $S_k(G)$ are also characterized.

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1 Introduction

In this paper, we only consider connected, simple and undirected graphs and assume that all graphs are connected, and refer to Bondy and Murty [2] for notation and terminologies used but not defined here.

Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G. $G - v, G - uv$ denote the graph obtained from G by deleting vertex $v \in V_G$ or edge $uv \in E_G$, respectively (this notation is naturally extended if more than one vertex or edge is deleted). Similarly, $G + uv$ is obtained from G by adding an edge $uv \notin E_G$. For $v \in V_G$, let $N_G(v)(N(v)$ for short) denote the set of all the adjacent vertices of v in G and $d(v) = |N_G(v)|$, the degree of v in G.

A bipartite graph G is a simple graph, whose vertex set V_G can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. A bipartite graph in which every two vertices from different partition classes are adjacent is called complete, which is denoted by $K_{m,n}$, where $m = |V_1|, n = |V_2|$.

The distance $d(u, v)$ between vertices u and v in G is defined as the length of a shortest path between them. The diameter of G is the maximal distance between any two vertices of G. Let B_n^d be the class of all bipartite graphs of order n with diameter d.

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Let $S_k = S_k(G)$ be the sum of k-power of distances between all pairs of vertices of G, which is denoted by

$$S_k = S_k(G) = \sum_{u,v \in V_G} d_G^k(u,v) = \frac{1}{2} \sum_{v \in V_G} H_G(v),$$

where $H_G(v)$ is the sum of k-power of all diatances from v in G.

When $k = 1$, S_k is Wiener index. The Wiener index is popular in chemical literatures. This quantity was introduced by Mustapha Aouchich and Pierre Hansen in [1] and was extensively studied in the monograph. Recently, $S_2(G)$ is applied to the research of distance spectral radius. Zhou and Trinajstić [19] proved an upper bound using the order n in addition to the sum of the squares of the distances $S_2(G)$, see [18, 20]. They also proved a lower bound on the distance spectral radius of a graph using only $S_2(G)$. As a continuance of it, in this paper, we determine the extremal graphs with the minimal $S_k(G)$ for the class of all connected n-vertex bipartite graphs of diameter d. For surveys and some up-to-date papers related to Wiener index of trees and line graphs, see [7, 9, 11–15, 17] and [3, 6, 8, 10, 16], respectively.

In this paper we study the quantity S_k in the case of n-vertex bipartite graphs, which is an important class of graphs in graph theory. Based on the structure of bipartite graphs, sharp bounds on S_k among B_n^d are determined. The corresponding extremal graph is also identified.

Further on we need the following lemma, which is the direct consequence of the definition of S_k.

Lemma 1.1 Let G be a connected graph of order n and not isomorphic to K_n. Then for each edge $e \in \overline{G}, S_k(G) > S_k(G+e)$.

2 The Graph with Minimum S_k among B_n^d

Let G be a graph in B_n^d. Clearly there exists a partition $V_0, V_1, \cdots, V_d$ of V_G such that $|V_0| = 1$ and $d(u,v) = i$ for each vertex $v \in V_i$ and $u \in V_0$ $(i = 0, 1, \cdots, d)$. We call V_i a block of V_G. Two blocks V_i, V_j of V_G are adjacent if $|i-j| = 1$. For convenience, let $|V_i| = l_i$ throughout this section.

Lemma 2.1 [15] For any graph $G \in B_n^d$ with the above partition of V_G, $G[V_i]$ induces an empty graph (i.e., containing no edge) for each $i \in (i = 0, 1, \cdots, d)$.

Given a complete bipartite graph $K_{\lfloor \frac{n-d+3}{2} \rfloor, \lceil \frac{n-d+3}{2} \rceil}$ with bipartition (X, Y) satisfying $|Y| = \lceil \frac{n-d+3}{2} \rceil$ and $|X| = \lfloor \frac{n-d+3}{2} \rfloor \geq 2$, choose a vertex x (resp. y) in X (resp. Y) and let $G' = K_{\lfloor \frac{n-d+3}{2} \rfloor, \lceil \frac{n-d+3}{2} \rceil} - xy$, where G' is depicted in Figure 1. Let G^* be the graph obtained from G' by attaching paths $P_{\frac{d-3}{2}}$ and $P_{\frac{d-3}{2}}$ at x and y, respectively. It is routine to check that $\hat{G}[p,q] \in B_n^d$ for odd d.

Given a complete bipartite graph $K_{p,q}$ with bipartition (X, Y) satisfying $|X| = p \geq 3, |Y| = q \geq 2$, and $p + q = n - d + 4$, choose two different vertices, say x, y in X and let

$$G'' = K_{p,q} - \{xw : w \in V' \subsetneq Y\} - \{yw' : w' \in Y \backslash V'\},$$

where G'' is depicted in Figure 1. Let $\hat{G}[p,q]$ be the graph obtained from G'' by attaching paths $P_{\frac{d-4}{2}}$ and $P_{\frac{d-4}{2}}$ at x and y, respectively. It is routine to check that $\hat{G}[p,q] \in B_n^d$ for even d. Set $\mathbb{B} = \{\hat{G}[p,q] : p+q = n-d+4, |(p-2)-q| \le 1\}$.

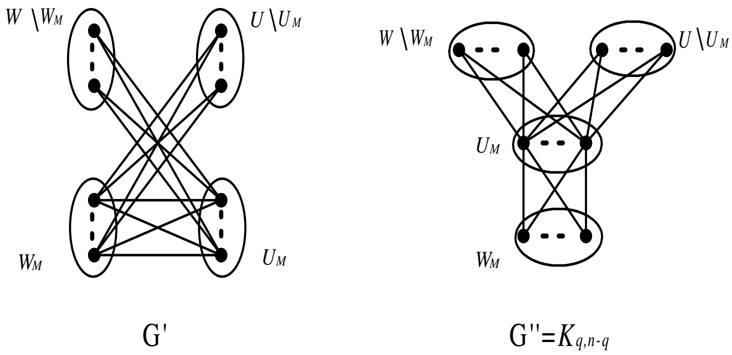

Figure 1: Graphs G' and G''

Theorem 2.2 Let G be in B_n^d with the minimum $S_k(G)$.

(i) If $d = 2$, then $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

(ii) If $d \ge 3$, then $G \cong G^*$ for odd d and $G \in \mathbb{B}$ for even d, where G^* and $\mathbb{B}$ are defined as above.

Proof Choose $G \in B_n^d$ with bipartition (X, Y) such that $S_k(G)$ is as small as possible.

(i) If $d = 2$, then by Lemma 1.1, $G \cong K_{n-t,t}$, where $t \ge 2$ or $n - t \ge 2$. Let $|X| = n - t, |Y| = t$. Then it is routine to check that, for all x (resp. y) in X (resp. Y), one has

$$H_G(x) = 2^k n - (2^k - 1)t - 2^k, H_G(y) = (2^k - 1)t + n - 2^k,$$

which gives

$$\begin{aligned} S_k(K_{n-t,t}) &= \frac{1}{2}\left(\sum_{x \in X} H_G(x) + \sum_{y \in Y} H_G(y)\right) \\ &= \frac{1}{2}(n-t)(2^k n - (2^k-1)t - 2^k) + \frac{1}{2}t((2^k-1)t + n - 2^k) \\ &= \frac{1}{2}\left(2^k n^2 + 2(2^k-1)t^2 - 2(2^k-1)nt - 2^k n\right). \\ &= 2^{k-1}n^2 + (2^k-1)t^2 - (2^k-1)nt - 2^{k-1}n. \end{aligned}$$

If n is odd, then $S_k(K_{n-t,t}) \ge \frac{2^k+1}{4}n^2 - 2^{k-1}n + \frac{2^k-1}{4}$ with equality if and only if $t = \frac{n-1}{2}$, or $t = \frac{n+1}{2}$, i.e. $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$; And if n is even, then $S_k(K_{n-t,t}) \ge \frac{2^k+1}{4}n^2 - 2^{k-1}n$ with equality if and only if $t = \frac{n}{2}$, i.e., $G \cong K_{\frac{n}{2},\frac{n}{2}}$, as desired.

(ii) First we show the following facts.

Fact 1 $G[V_{i-1}, V_i]$ induces a complete bipartite subgraph for each $i \in (0, 1, \cdots d)$, and $|V_d| = 1$ for $d \ge 3$.

Proof of Fact 1 The first part follows directly from Lemmas 1.1 and 2.1. So in what follows, we prove the second part.

Let $d \geq 3$, $z \in V_d$ and $w \in V_{d-3}$. If $|V_d| > 2$, then $G+zw \in B_n^d$ and $V_0 \cup V_1 \cup \big(V_{d-3} \backslash \{w\}\big) \cup V_{d-2} \cup \big(V_{d-1} \cup \{w\}\big) \cup V_d$ is a partition of V_{G+zw}. By Lemma 1.1 $S_k(G+zw) < S_k(G)$, a contradiction.

This completes the proof of Fact 1.

Fact 2 Consider the vertex partition $V_G = V_0 \cup V_1 \cup \cdots \cup V_d$ of G.

(i) If d is odd, then

$$|V_0| = |V_1| = \cdots = \left|V_{\frac{d-1}{2}-1}\right| = \left|V_{\frac{d-1}{2}+2}\right| = \cdots = |V_{d-1}| = |V_d| = 1, \left||V_{\frac{d-1}{2}}| - |V_{\frac{d-1}{2}+1}|\right| \leq 1.$$

(ii) If d is even, then

$$|V_0| = |V_1| = \cdots = \left|V_{\frac{d}{2}-2}\right| = \left|V_{\frac{d}{2}+2}\right| = \cdots = |V_{d-1}| = |V_d| = 1, \left||V_{\frac{d}{2}}| - (|V_{\frac{d}{2}-1}| + |V_{\frac{d}{2}+1}|)\right| \leq 1.$$

Proof of Fact 2 (i) Note that $|V_0| = |V_d| = 1$, here we only show that $|V_1| = 1$ holds. Similarly, we can also show $|V_2| = \cdots = \left|V_{\frac{d-1}{2}-1}\right| = \left|V_{\frac{d-1}{2}+2}\right| = \cdots = |V_{d-1}| = 1$, we omit the procedure here.

In fact, if $d = 3$, our result is trivial. So we consider that $d \geq 5$. If $|V_1| \geq 2$, then choose $u \in V_1$ and let $G' = G - v_0u + \{ux : x \in V_4\}$. In fact, the vertex partition of G' is $V_0 \cup \big(V_1 \backslash \{u\}\big) \cup V_2 \cup \big(V_3 \cup \{u\}\big) \cup V_4 \cup \cdots \cup V_d$, in view of Fact 1 and the choice of G, any two of adjacent blocks of $V_{G'}$ induce a complete bipartite subgraph and $|V_d| = 1$ for $d \geq 5$.

Note that $\sum\limits_{i=4}^{d} \big((i-1)^k - (i-3)^k\big) l_i \geq \sum\limits_{i=4}^{d} \big((i-1)^k - (i-3)^k\big) > (3^k - 1)$,

$$\begin{aligned}
H_G(u) &= H_{G'}(u) + \sum_{i=4}^{d} \big((i-1)^k - (i-3)^k)\big) l_i - (3^k - 1); \\
H_G(v) &= H_{G'}(v) - (3^k - 1), \ \ \forall v \in V_0; \\
H_G(v) &= H_{G'}(v), \ \ \forall v \in \big(V_1 \backslash \{u\}\big) \cup V_2 \cup V_3; \\
H_G(v) &= H_{G'}(v) + (i-1)^k - (i-3)^k, \ \ \forall v \in V_4 \cup V_5 \cup \cdots \cup V_d.
\end{aligned}$$

This gives

$$\begin{aligned}
& S_k(G) - S_k(G') = \frac{1}{2}\left(\sum_{v \in V_G} H_G(v) - \sum_{v \in V_{G'}} H_{G'}(v)\right) \\
= \ & \frac{1}{2}\left(\sum_{v \in V_0} (H_G(v) - H_{G'}(v)) + H_G(u) - H_{G'}(u)\right) + \sum_{i=4}^{d} \sum_{v \in V_i} (H_G(v) - H_{G'}(v)) \\
= \ & \frac{1}{2}\Big[-2(3^k - 1) + 2\sum_{i=4}^{d} \big((i-1)^k - (i-3)^k\big) l_i\Big] \\
= \ & \Big[-(3^k - 1) + \sum_{i=4}^{d} \big((i-1)^k - (i-3)^k\big) l_i\Big] > 0,
\end{aligned}$$

i.e., $S_k(G) > S_k(G')$, a contradiction to the choice of G. Hence, $|V_1| = 1$.

Next we show that if d is odd, then $\left||V_{\frac{d-1}{2}}|-|V_{\frac{d-1}{2}+1}|\right|\leq 1$. Without loss of generality, we assume that $\left|V_{\frac{d-1}{2}}\right|\geq\left|V_{\frac{d-1}{2}+1}\right|$. Then it suffices to show that $\left|V_{\frac{d-1}{2}}\right|-\left|V_{\frac{d-1}{2}+1}\right|\leq 1$. If this is not true, then $|V_{\frac{d-1}{2}}|-|V_{\frac{d-1}{2}+1}|\geq 2$. Choose $w\in V_{\frac{d-1}{2}}$, let

$$G'=G-\left\{ux:x\in V_{\frac{d-3}{2}}\cup V_{\frac{d+1}{2}}\right\}+\left\{wy:y\in V_{\frac{d-1}{2}}\cup V_{\frac{d+3}{2}}\right\},$$

then the vertex partition of G' is

$$V_0\cup V_1\cup\cdots\cup V_{\frac{d-3}{2}}\cup\left(V_{\frac{d-1}{2}}\backslash\{w\}\right)\cup\left(V_{\frac{d+1}{2}}\cup\{w\}\right)\cup V_{\frac{d+3}{2}}\cup\cdots\cup V_d$$

and each of the two adjacent blocks of $V_{G'}$ induces a complete bipartite graph. By direct calculation, we have

$$\begin{aligned}S_k(G')-S_k(G) &= \left[(|V_{\frac{d-1}{2}}|-1)+2^k|V_{\frac{d+1}{2}}|\right]-\left[2^k(|V_{\frac{d-1}{2}}|-1)+|V_{\frac{d+1}{2}}|\right]\\ &= -(2^k-1)\left(|V_{\frac{d-1}{2}}|-|V_{\frac{d+1}{2}}|-1\right)\leq-(2^k-1)<0,\end{aligned}$$

a contradiction to the choice of G.

(ii) By the same discussion as the proof of the first part of (i) as above, we can show that $|V_0|=|V_1|=\cdots=\left|V_{\frac{d}{2}-2}\right|=\left|V_{\frac{d}{2}+2}\right|=\cdots=|V_{d-1}|=|V_d|=1$, we omit the procedure here.

Now, we show that if d is even, then $\left||V_{\frac{d}{2}}|-(|V_{\frac{d}{2}-1}|+|V_{\frac{d}{2}+1}|)\right|\leq 1$. Without loss of generality, we assume that $\left|V_{\frac{d}{2}}\right|<\left|V_{\frac{d}{2}-1}\right|+\left|V_{\frac{d}{2}+1}\right|$. Then it suffices to show that

$$\left|V_{\frac{d}{2}-1}\right|+\left|V_{\frac{d}{2}+1}\right|-\left|V_{\frac{d}{2}}\right|\leq 1.$$

If this is not true, then $\left|V_{\frac{d}{2}-1}\right|+\left|V_{\frac{d}{2}+1}\right|-\left|V_{\frac{d}{2}}\right|\geq 2$. It is routine to check that at least one of $V_{\frac{d}{2}-1}$ and $V_{\frac{d}{2}+1}$ contains at least two vertices. Hence, we assume without loss of generality that $\left|V_{\frac{d}{2}-1}\right|\geq 2$. Choose $w\in V_{\frac{d}{2}-1}$ and let

$$G^*=G-\left\{wx:x\in V_{\frac{d}{2}-2}\cup V_{\frac{d}{2}}\right\}+\left\{wy:y\in V_{\frac{d}{2}-1}\cup V_{\frac{d}{2}+1}\right\},$$

then the vertex partition of G^* is

$$V_0\cup V_1\cup\cdots\cup V_{\frac{d-3}{2}}\cup\left(V_{\frac{d}{2}-1}\backslash\{w\}\right)\cup\left(V_{\frac{d}{2}}\cup\{w\}\right)\cup V_{\frac{d}{2}+1}\cup\cdots\cup V_d$$

and each of the two adjacent blocks of V_{G^*} induces a complete bipartite graph. By direct calculation, we have

$$\begin{aligned}S_k(G^*)-S_k(G) &= \left[(|V_{\frac{d}{2}-1}|+|V_{\frac{d}{2}+1}|-1)+2^k|V_{\frac{d}{2}}|\right]-\left[2^k(|V_{\frac{d}{2}-1}|+|V_{\frac{d}{2}+1}|-1)+|V_{\frac{d}{2}}|\right]\\ &= -(2^k-1)\left[|V_{\frac{d}{2}-1}|+|V_{\frac{d}{2}+1}|-(|V_{\frac{d}{2}}|+1)\right]\leq-(2^k-1)<0,\end{aligned}$$

a contradiction to the choice of G.

This completes the proof of Fact 2.

Now we come back to show the second part of Theorem 2.2. In view of Fact 2(i), if d is odd, note that $|V_{\frac{d-1}{2}}| + |V_{\frac{d-1}{2}+1}| = n - d + 1$, together with $\left||V_{\frac{d-1}{2}}| - |V_{\frac{d-1}{2}+1}|\right| \le 1$, we obtain that $G \cong G^*$, as desired.

In view of Fact 2(ii), if d is even, note that $|V_{\frac{d}{2}}| + |V_{\frac{d}{2}-1}| + |V_{\frac{d}{2}+1}| = n - d + 2$, together with $\left||V_{\frac{d}{2}}| - (|V_{\frac{d}{2}-1}| - |V_{\frac{d}{2}+1}|)\right| \le 1$, we obtain that $G \in \mathbb{B}$. Furthermore,

$$\mathbb{B} = \left\{\widehat{G}[p,q] : p + q = n - d + 4, p = \frac{n-d+6}{2}\right\}$$

for even n and $\mathbb{B} = \left\{\widehat{G}[p,q] : p + q = n - d + 4, p = \frac{n-d+7}{2}\right\}$ for odd n.

This completes the proof.

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二部图的距离k次方和问题

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摘要: 本文定义 $S_k(G)$ 为 G 中所有点对之间距离的 k 次方之和. 利用顶点划分的方法得到了直径为 d 的 n 顶点连通二部图 $S_k(G)$ 的下界, 并确定了达到下界所对应的的极图.

关键词: 二部图; 直径; 极图

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