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LIOUVILLE TYPE THEOREMS FOR A NONLINEAR ELLIPTIC EQUATION

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Abstract: Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold. In this paper, we consider the Liouville type theorems for positive solutions to the following nonlinear elliptic equation: $\Delta_f u + au\log u = 0$, where a is a nonzero constant. By applying Bochner formula and the maximum principle, we obtain local gradient estimates of the Li-Yau type for positive solutions of the above equation on Riemannian manifolds with Bakry-Emery Ricci curvature bounded from below and some relevant Liouville type theorems, which improve some results of [7].

Keywords: gradient estimate; nonlinear elliptic equation; Liouville-type theorem; maximum principle

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1 Introduction

Let (M^n, g) be an n-dimensional complete Riemannian manifold. The drifting Laplacian is defined by $\Delta_f = \Delta - \nabla f \nabla$, where f is a smooth function on M. The N-Bakry-Emery Ricci tensor is defined by

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \bigtriangledown^2 f - \frac{1}{N} df \otimes df$$

for $0 \le N < \infty$ and $N = 0$ if and only if $f = 0$, where f is some smooth function on M, ∇^2 is the Hessian and Ric is the Ricci tensor. The ∞-Bakry-Emery Ricci tensor is defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \bigtriangledown^2 f.$$

In particular, $\mathrm{Ric}_f = \lambda g$ is called a gradient Ricci soliton which is extensively studied in Ricci flow.

In this paper, we want to study positive solutions of the nonlinear elliptic equation with the drifting Laplacian

$$\Delta_f u + au\log u = 0 \tag{1.1}$$

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on an n-dimensional complete Riemannian manifold (M^n, g), where a is a nonzero constant. When f = constant, the above equation (1.1) reduces to

$$\Delta u + au \log u = 0. \tag{1.2}$$

Equation (1.2) is closely related to Ricci soliton [9] and the famous Gross Logarithmic Sobolev inequality [6]. Ma [9] first studied the positive solutions of equation (1.2) and derived a local gradient estimate for the case $a < 0$. Then the gradient estimate for the case $a > 0$ is obtained in [4] and [15] by studying the related heat equation of (1.2). More progress of this and related equations can be found in [2, 8, 10, 13, 14] and the references therein. Recently, inspired by the method used by Brighton in [1], Huang and Ma [7] derived local gradient estimates of the Li-Yau type for positive solutions of equations (1.2). These estimates are different from those in [4, 9, 15]. Using these estimate, they can easily get some Liouville type theorems. We want to generalize their results to equation (1.1) and we obtain the following results

Theorem 1.1 Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_f^N(B_P(2R)) \geq -K$, where K is a nonnegative constant. Assuming that u is a positive solution of the nonlinear elliptic eq. (1.1). Then on $B_p(R)$, we have the following inequalities

(1) If $a > 0$, then

$$|\nabla u| \leq M\sqrt{\frac{(n+N+3)^2}{2(n+N)}(a+K) + \frac{(n+N+2)c_1^2 + (n-1+\sqrt{nK}R)c_1 + c_2}{R^2}}; \tag{1.3}$$

(2) If $a < 0$, then

$$|\nabla u| \leq M\sqrt{\frac{(5n+5N+6)^2}{36(n+N)}\max\{0, a+K\} + \frac{1}{R^2}\left[(\frac{75n+75N+12}{6})c_1^2 + (n-1+\sqrt{nK}R)c_1 + c_2\right]}, \tag{1.4}$$

where $M = \sup\limits_{x\in B_p(2R)} u(x)$, the c_1 and c_2 is a positive constants.

Let $R \to \infty$, we have the following gradient estimates on complete noncompact Riemannian manifolds.

Corollary 1.2 Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $\mathrm{Ric}_f^N \geq -K$, where K is a nonnegative constant. Assuming that u is a positive solution of the nonlinear elliptic eq. (1.1). Then the following inequalities hold

(1) if $a > 0$, then

$$|\nabla u| \leq \frac{(n+N+3)M}{\sqrt{2(n+N)}}\sqrt{a+K}; \tag{1.5}$$

(2) if $a < 0$, then

$$|\nabla u| \leq \frac{(5n+5N+6)M}{6\sqrt{n+N}}\sqrt{\max\{0, a+K\}}, \tag{1.6}$$

where $M = \sup\limits_{x\in M^n} u(x)$.

In particular, for $a < 0$, if $a \leq -K$, then $\max\{0, a+K\} = 0$. Thus, (1.5) implies $|\nabla u| \leq 0$ whenever u is a bounded positive solution of the nonlinear elliptic (1.1). Hence $u \equiv 1$. Therefore the following Liouville-type result follows.

Corollary 1.3 Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $\mathrm{Ric}_f^N \geq -K$, where K is a nonnegative constant. Assuming that u is a bounded positive solution of (1.1) with $a < 0$. If $a \leq -K$, then $u \equiv 1$.

In particular, we have the following conclusion

Corollary 1.4 Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $\mathrm{Ric}_f^N \geq 0$. Assuming that u is a bounded positive solution defined of (1.1) with $a < 0$, then $u \equiv 1$.

The above results are obtained under the assumption that Ric_f^N is bounded by below. We can also obtain similar results under the assumption that Ric_f is bounded by below.

Theorem 1.5 Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_f(B_P(2R)) \geq -(n-1)H$, and $|\nabla f| \leq K$, where K and H is a nonnegative constant. Assuming that u is a positive solution of the nonlinear elliptic eq. (1.1) on $B_p(2R)$. Then on $B_p(R)$, the following inequalities hold

(1) if $a > 0$, then

$$|\nabla u| \leq M\sqrt{\frac{(n+3)^2}{2n}\left[a+(n-1)H\right] + \frac{(n+2)c_1^2 + [(n-1)(1+\sqrt{HR}) + KR)]c_1 + c_2}{R^2}}; \tag{1.7}$$

(2) if $a < 0$, then

$$|\nabla u| \leq M\sqrt{\frac{(5n+6)^2}{36n}\max\{0, a+(n-1)H\} + \frac{1}{R^2}\left\{(\frac{75n+12}{6})c_1^2 + \left[(n-1)(1+\sqrt{HR}) + KR\right]c_1 + c_2\right\}}, \tag{1.8}$$

where $M = \sup\limits_{x \in B_p(2R)} u(x)$, the c_1 and c_2 is a positive constants.

Corollary 1.6 Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $\mathrm{Ric}_f \geq -(n-1)H$, and $|\nabla f| \leq K$, where K and H is a nonnegative constant. Assuming that u is a positive solution of the nonlinear elliptic eq. (1.1) on the following inequalities hold

(1) if $a > 0$, then

$$|\nabla u| \leq \frac{(n+3)M}{\sqrt{2n}}\sqrt{a+(n-1)H}; \tag{1.9}$$

(2) if $a < 0$, then

$$|\nabla u| \leq \frac{(5n+6)M}{6\sqrt{n}}\sqrt{\max\{0, a+(n-1)H\}}, \tag{1.10}$$

where $M = \sup\limits_{x \in M^n} u(x)$.

In particular, for $a < 0$, if $a \le -(n-1)H$, then $\max\{0, a + (n-1)H\} = 0$. Thus, (1.10) implies $|\nabla u| \le 0$ whenever u is a bounded positive solution to (1.1). Hence, that $u \equiv 1$. Therefore, the following Liouville-type result follows

Corollary 1.7 Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $\mathrm{Ric}_f \ge -(n-1)H$, and $|\nabla f| \le K$, where K and H is a nonnegative constant. Assuming that u is a bounded positive solution of (1.1) with $a < 0$. If $a \le -(n-1)H$, then $u \equiv 1$.

In particular, we have the following conclusion.

Corollary 1.8 Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with $\mathrm{Ric}_f \ge 0$. Assuming that u is a bounded positive solution of (1.1) with $a < 0$, then $u \equiv 1$.

2 The Proof of Theorems

Now we are in the position to give the proof of Theorem 1.1. First we recall the following key lemma.

Lemma 2.1 Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_f^N(B_P(2R)) \ge -K$, where K is a nonnegative constant. Assuming that u is a positive solution to nonlinear elliptic eq. (1.1) on $B_p(2R)$. Then on $B_p(R)$, the following inequalities hold

(1) If $a > 0$, then

$$\frac{1}{2}\Delta_f|\nabla h|^2 \ge \frac{5(n+N)}{18}\frac{|\nabla h|^4}{h^2} - \frac{(n+N)}{3}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - (a+K)|\nabla h|^2, \tag{2.1}$$

where $h = u^{\frac{3}{n+N+3}}$.

(2) If $a < 0$, then

$$\frac{1}{2}\Delta_f|\nabla h|^2 \ge \frac{37(n+N)}{36}\frac{|\nabla h|^4}{h^2} - \frac{5(n+N)}{6}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - (a+K)|\nabla h|^2, \tag{2.2}$$

where $h = u^{\frac{6}{5(n+N)+6}}$.

Proof of Lemma 2.1 Let $h = u^\epsilon$, where $\epsilon \ne 0$ is a constant to determined. Then we have

$$\log h = \epsilon \log u.$$

A simple calculation implies

$$\begin{aligned} \Delta_f h = \Delta_f(u^\epsilon) &= \epsilon(\epsilon-1)u^{\epsilon-2}|\nabla u|^2 + \epsilon u^{\epsilon-1}\Delta_f u \\ &= \epsilon(\epsilon-1)u^{\epsilon-2}|\nabla u|^2 - a\epsilon u^\epsilon \log u \\ &= \tfrac{\epsilon-1}{\epsilon}\tfrac{|\nabla h|^2}{h} - ah\log h. \end{aligned} \tag{2.3}$$

Therefore we get

$$\begin{aligned}\nabla h\nabla\Delta_f h &= \nabla h\nabla\left(\frac{\epsilon-1}{\epsilon}\frac{|\nabla h|^2}{h}-ah\log h\right)\\ &= \frac{\epsilon-1}{\epsilon}\nabla h\left(\frac{\nabla(|\nabla h|^2)h-|\nabla h|^2\nabla h}{h^2}\right)-a\nabla h\left(\nabla h\log h+h\nabla\log h\right)\\ &= \frac{\epsilon-1}{\epsilon h}\nabla h\nabla(|\nabla h|^2)-\frac{\epsilon-1}{\epsilon}\frac{|\nabla h|^4}{h^2}-ah\log h\frac{|\nabla h|^2}{h}-a|\nabla h|^2.\end{aligned}\tag{2.4}$$

Applying (2.3) and (2.4) into the famous Bochner formula to h, we have

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 &\geq \frac{1}{n+N}(\Delta_f h)^2+\nabla h\nabla\Delta_f h+Ric_f^N(\nabla h,\nabla h)\\ &\geq \frac{1}{n+N}\left(\frac{\epsilon-1}{\epsilon}\frac{|\nabla h|^2}{h}-ah\log h\right)^2+\nabla h\nabla\Delta_f h-K|\nabla h|^2\\ &= \left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}\right]\frac{|\nabla h|^4}{h^2}-a\left[\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right]h\log h\frac{|\nabla h|^2}{h}\\ &\quad+\frac{a^2}{n+N}(h\log h)^2+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+K)|\nabla h|^2.\end{aligned}\tag{2.5}$$

Now we let

$$a\left[\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right]\geq 0.\tag{2.6}$$

Then for a fixed point p, if there exist a positive constant δ such that $h\log h\leq\delta\frac{|\nabla h|^2}{h}$, then (2.5) becomes

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 &\geq \left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}-a\delta\left(\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right)\right]\frac{|\nabla h|^4}{h^2}\\ &+ \frac{a^2}{n+N}(h\log h)^2+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+K)|\nabla h|^2\\ &\geq \left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}-a\delta\left(\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right)\right]\frac{|\nabla h|^4}{h^2}\\ &\quad+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+K)||\nabla h|^2.\end{aligned}$$

On the contrary, at the point p, if $h\log h\geq\delta\frac{|\nabla h|^2}{h}$, then (2.5) becomes

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 &\geq \left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}\right]\frac{|\nabla h|^4}{h^2}+\left[\frac{a^2}{n+N}-\frac{a}{\delta}\left(\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right)\right](h\log h)^2\\ &\quad+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+K)|\nabla h|^2\\ &\geq \left\{\left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}\right]+\delta^2\left[\frac{a^2}{n+N}-\frac{a}{\delta}\left(\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right)\right]\right\}\frac{|\nabla h|^4}{h^2}\\ &\quad+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+K)|\nabla h|^2\\ &\geq \left\{\left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}\right]-a\delta\left[\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right]\right\}\frac{|\nabla h|^4}{h^2}\\ &\quad+\frac{\epsilon-1}{\epsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+K)|\nabla h|^2\end{aligned}\tag{2.7}$$

as long as

$$\frac{a^2}{n+N}-\frac{a}{\delta}\left[\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right]>0.\tag{2.8}$$

In order to obtain the bound of $|\nabla h|$ by applying the maximum principle to (2.7), it is sufficient to choose the coefficient of $\frac{|\nabla h|^4}{h^2}$ in (2.7) is positive, that is

$$\left[\frac{(\epsilon-1)^2}{(n+N)\epsilon^2}-\frac{\epsilon-1}{\epsilon}\right]-a\delta\left[\frac{2(\epsilon-1)}{(n+N)\epsilon}+1\right]>0. \tag{2.9}$$

Then we divide it into two cases.

Case 1 $a>0$. In this case, provided $\epsilon\in\left(\frac{2}{n+N+2},\frac{6}{(5-\sqrt{13})(n+N)+6}\right)$, there will exist an δ satisfying (2.6), (2.8) and (2.9). In particular, we choose

$$\epsilon=\frac{3}{n+N+3}$$

and

$$\delta=\frac{n+N}{2a}.$$

Then (2.7) becomes

$$\frac{1}{2}\Delta_f|\nabla h|^2\geq\frac{5(n+N)}{18}\frac{|\nabla h|^4}{h^2}-\frac{(n+N)}{3}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+K)|\nabla h|^2. \tag{2.10}$$

Case 2 $a<0$. In this case, provided $\epsilon\in\left(\frac{6}{(5+\sqrt{13})(n+N)+6},\frac{2}{n+N+2}\right)$, there will exist an δ satisfying (2.6), (2.8) and (2.9). In particular , we choose

$$\epsilon=\frac{6}{5(n+N)+6}$$

and

$$\delta=\frac{-3(n+N)}{4a}.$$

Then (2.7) becomes

$$\frac{1}{2}\Delta_f|\nabla h|^2\geq\frac{37(n+N)}{36}\frac{|\nabla h|^4}{h^2}-\frac{5(n+N)}{6}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+K)|\nabla h|^2. \tag{2.11}$$

Now we begin to prove Theorem 1.1 which will follow by applying comparison theorems and Bochner formula to an appropriate function h.

Proof of Theorem 1.1 We first prove the case of $a>0$. Let m be a cut-off function such that $m(r)=1$ for $r\leq 1$, $m(r)=0$ for $r\geq 2$, $0\leq m(r)\leq 1$, and

$$0\geq m^{-\frac{1}{2}}(r)m^{'}(r)\geq -c_1,\ \ m^{''}(r)\geq -c_2$$

for positive constants c_1 and c_2. Denote by $\rho(x)=d(x,p)$ the distance between x and p in (M^n,g). Let

$$\phi(x)=m\left(\frac{\rho(x)}{R}\right).$$

Making use of an argument of Calabi [3] (see also Cheng and Yau [5]), we can assume without loss of generality that the function ϕ is smooth in $B_p(2R)$. Then we have

$$\frac{|\nabla\phi|^2}{\phi} \le \frac{c_1{}^2}{R^2}. \tag{2.12}$$

It was shown by Qian [11] that

$$\Delta_f(\rho^2) \le n\left\{1+\sqrt{1+\frac{4K\rho^2}{n}}\right\}.$$

Hence we have

$$\begin{aligned}\Delta_f\rho &= \tfrac{1}{2\rho}\left(\Delta_f(\rho^2)-2|\nabla\rho|^2\right) \le \tfrac{n-2}{2\rho}+\tfrac{n}{2\rho}\left(1+\sqrt{1+\tfrac{4K\rho^2}{n}}\right)\\ &= \tfrac{n-1}{\rho}+\sqrt{nK}.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta_f\phi &= \tfrac{m''(r)|\nabla\rho|^2}{R^2}+\tfrac{m'(r)\Delta_f\rho}{R}\\ &\ge -\tfrac{(n-1+\sqrt{nK}R)c_1+c_2}{R^2}.\end{aligned} \tag{2.13}$$

Define $G=\phi|\nabla h|^2$, we will use the maximum principle for G on $B_p(2R)$. Assume G achieves its maximum at the point $x_0\in B_p(2R)$ and assume $G(x_0)>0$ (otherwise this is obvious). Then at the point x_0, it holds that

$$\Delta_f G\le 0, \qquad \nabla(|\nabla h|^2) = -\frac{|\nabla h|^2}{\phi}\nabla\phi.$$

Using (2.1) in Lemma 2.1, we obtain

$$\begin{aligned}0 &\ge \Delta_f G\\ &= \phi\Delta_f(|\nabla h|^2)+|\nabla h|^2\Delta_f\phi+2\nabla\phi\nabla(|\nabla h|^2)\\ &= \phi\Delta_f(|\nabla h|^2)+\tfrac{\Delta_f\phi}{\phi}G-2\tfrac{|\nabla\phi|^2}{\phi^2}\nabla\phi\\ &\ge 2\phi\left[\tfrac{5(n+N)}{18}\tfrac{|\nabla h|^4}{h^2}-\tfrac{n+N}{3}\tfrac{\nabla h}{h}\nabla(|\nabla h|^2)-(a+k)|\nabla h|^2\right]+\tfrac{\Delta_f\phi}{\phi}G-2\tfrac{|\nabla\phi|^2}{\phi^2}\nabla\phi\\ &= \tfrac{5(n+N)}{9}\tfrac{G^2}{\phi h^2}+\tfrac{2(n+N)G}{3\phi}\nabla\phi\tfrac{\nabla h}{h}-2(a+K)G+\tfrac{\Delta_f\phi}{\phi}G-2\tfrac{|\nabla\phi|^2}{\phi^2}G,\end{aligned} \tag{2.14}$$

where the second inequality used (2.10). Multiplying both sides of (2.14) by $\frac{\phi}{G}$, we obtain

$$\frac{5(n+N)}{9}\frac{G^2}{h} \le -\frac{2(n+N)}{3}\nabla\phi\frac{\nabla h}{h}+2\phi(a+K)-\Delta_f\phi+2\frac{|\nabla\phi|^2}{\phi}. \tag{2.15}$$

Then using the Cauchy inequality, we have

$$\begin{aligned}-\tfrac{2(n+N)}{3}\nabla\phi\tfrac{\nabla h}{h} &\le \tfrac{2(n+N)}{3}|\nabla\phi|\tfrac{|\nabla h|}{h}\le\tfrac{(n+N)}{3\varepsilon}\tfrac{|\nabla\phi|^2}{\phi}+\tfrac{(n+N)\varepsilon}{3h^2}\phi|\nabla h|^2\\ &= \tfrac{(n+N)}{3\varepsilon}\tfrac{|\nabla\phi|^2}{\phi}+\tfrac{(n+N)\varepsilon}{3h^2}G,\end{aligned}$$

where $\varepsilon \in (0, \frac{5}{3})$ is a positive constant. Taking the above inequality into (2.15), we have

$$\begin{aligned} \tfrac{(5-3\varepsilon)(n+N)}{9}\tfrac{G}{h^2} &\leq 2\phi(a+K) - \Delta_f\phi + (2+\tfrac{n+N}{3\varepsilon})\tfrac{|\nabla\phi|^2}{\phi} \\ &\leq 2(a+K) - \Delta_f\phi + (2+\tfrac{n+N}{3\varepsilon})\tfrac{|\nabla\phi|^2}{\phi}. \end{aligned} \tag{2.16}$$

In particular, choosing $\varepsilon = \frac{1}{3}$ in (2.16) and using (2.12) and (2.13), we have

$$\begin{aligned} \tfrac{4(n+N)G}{9h^2} &\leq 2(a+K) - \Delta_f\phi + (n+N+2)\tfrac{|\nabla\phi|^2}{\phi} \\ &\leq 2(a+K) + \tfrac{(n+N+2)c_1^2+(n-1+\sqrt{nK}R)c_1+c_2}{R^2}. \end{aligned}$$

So for $x_0 \in B_p(R)$, we have

$$\begin{aligned} \tfrac{4(n+N)}{9}G(x) &\leq \tfrac{4(n+N)}{9}G(x_0) \\ &\leq h^2(x_0)\left[\tfrac{(n+N+3)^2}{2(n+N)}(a+K) + \tfrac{(n+N+2)c_1^2+(n-1+\sqrt{nK}R)c_1+c_2}{R^2}\right]. \end{aligned}$$

This shows

$$|\nabla u|^2(x) \leq M^2\left[\frac{(n+N+3)^2}{2(n+N)}(a+K) + \frac{(n+N+2)c_1^2+(n-1+\sqrt{nK}R)c_1+c_2}{R^2}\right]$$

and

$$|\nabla u| \leq M\sqrt{\frac{(n+N+3)^2}{2(n+N)}(a+K) + \frac{(n+N+2)c_1^2+(n-1+\sqrt{nK}R)c_1+c_2}{R^2}},$$

where $M = \sup\limits_{x\in B_p(2R)} u(x)$. This yields the desired inequality (1.3) of Theorem 1.1.

Next, we prove the case $a < 0$. Define $\overline{G} = \phi|\nabla h|^2$, we will use the maximum principle for $\overline{G}$ on $B_p(2R)$. Assume $\overline{G}$ achieves its maximum at the point $\overline{x_0} \in B_p(2R)$ and assume $\overline{G}(\overline{x_0}) > 0$ (otherwise this is obvious). Then at the point $\overline{x_0}$, it holds that

$$\Delta_f\overline{G} \leq 0, \quad \nabla(|\nabla h|^2) = -\frac{|\nabla h|^2}{\phi}\nabla\phi.$$

In a similar way as the case $a > 0$, we have

$$\begin{aligned} 0 &\geq \Delta_f\overline{G} \\ &= \phi\Delta_f(|\nabla h|^2) + \tfrac{\Delta_f\phi}{\phi}\overline{G} - 2\tfrac{|\nabla\phi|^2}{\phi^2}\overline{G} \\ &\geq 2\phi\left[\tfrac{37(n+N)}{36}\tfrac{|\nabla h|^4}{h^2} - \tfrac{5(n+N)}{6}\tfrac{\nabla h}{h}\nabla(|\nabla h|^2) - (a+K)|\nabla h|^2\right] + \tfrac{\Delta_f\phi}{\phi}\overline{G} - 2\tfrac{|\nabla\phi|^2}{\phi^2}\overline{G} \\ &= \tfrac{37(n+N)}{18}\tfrac{\overline{G}^2}{\phi h^2} + \tfrac{5(n+N)\overline{G}}{3\phi}\nabla\phi\tfrac{\nabla h}{h} - 2(a+K)\overline{G} + \tfrac{\Delta_f\phi}{\phi}\overline{G} - 2\tfrac{|\nabla\phi|^2}{\phi^2}\overline{G}, \end{aligned} \tag{2.17}$$

where the second inequality used (2.11). Multiplying both sides of (2.17) by $\frac{\phi}{\overline{G}}$, we obtain

$$\frac{37(n+N)}{18}\frac{\overline{G}^2}{h^2} \leq -\frac{5(n+N)}{3}\nabla\phi\frac{\nabla h}{h} + 2\phi(a+K) - \Delta_f\phi + 2\frac{|\nabla\phi|^2}{\phi}. \tag{2.18}$$

Using Cauchy inequality, we can get

$$-\tfrac{5(n+N)}{3}\nabla\phi\tfrac{\nabla h}{h} \leq \tfrac{5(n+N)}{3}|\nabla\phi|\tfrac{|\nabla h|}{h} \leq \tfrac{5(n+N)}{6\varepsilon}\tfrac{|\nabla\phi|^2}{\phi} + \tfrac{5(n+N)\varepsilon}{6h^2}\overline{G},$$

where $\varepsilon \in (0, \frac{37}{15})$ is a positive constant. Taking the above inequality into (2.18) gives

$$\begin{aligned}\tfrac{(37-15\varepsilon)(n+N)}{18}\tfrac{\overline{G}}{h^2} &\leq 2\phi(a+K) - \Delta_f\phi + \left(2+\tfrac{5(n+N)}{6\varepsilon}\right)\tfrac{|\nabla\phi|^2}{\phi}\\ &\leq 2\max\{0, a+K\} - \Delta_f\phi + \left(2+\tfrac{5(n+N)}{6\varepsilon}\right)\tfrac{|\nabla\phi|^2}{\phi}.\end{aligned}$$

Hence, choosing $\varepsilon = \frac{1}{15}$ in (2.16) and using (2.12) and (2.13), we obtain

$$\frac{2(n+N)\overline{G}^2}{h^2} \leq 2\max\{0, a+K\} + \frac{1}{R^2}\left[(\frac{75n+75N+12}{6})c_1^2 + (n-1+\sqrt{nK}R)c_1 + c_2\right].$$

Therefore, it holds on $B_p(R)$,

$$|\nabla u| \leq M\sqrt{\frac{(5n+5N+6)^2}{36(n+N)}\max\{0, a+K\} + \frac{1}{R^2}\left[(\frac{75n+75N+12}{6})c_1^2 + (n-1+\sqrt{nK}R)c_1 + c_2\right]}.$$

This concludes the proof of inequality (1.4) of Theorm 1.1.

Now we are in the position to give a brief proof of Theorem 1.5.

Skept of the Proof of Theorem 1.5 Noticing that we have the following Bochner formula to h with Ric_f,

$$\frac{1}{2}\Delta_f|\nabla h|^2 = |\nabla^2 h|^2 + \nabla h\nabla\Delta_f h + Ric_f(\nabla h, \nabla h),$$

then (2.5) becomes

$$\begin{aligned}\tfrac{1}{2}\Delta_f|\nabla h|^2 &= |\nabla^2 h|^2 + \nabla h\nabla\Delta_f h + Ric_f(\nabla h, \nabla h)\\ &\geq \tfrac{1}{n}\left(\tfrac{\epsilon-1}{\epsilon}\tfrac{|\nabla h|^2}{h} - ah\log h\right)^2 + \nabla h\nabla\Delta_f h - (n-1)H|\nabla h|^2\\ &= \left[\tfrac{(\epsilon-1)^2}{n\epsilon^2} - \tfrac{\epsilon-1}{\epsilon}\right]\tfrac{|\nabla h|^4}{h^2} - a\left[\tfrac{2(\epsilon-1)}{n\epsilon}+1\right]h\log h\tfrac{|\nabla h|^2}{h}\\ &\quad + \tfrac{a^2}{n}(h\log h)^2 + \tfrac{\epsilon-1}{\epsilon}\tfrac{\nabla h}{h}\nabla(|\nabla h|^2) - \Big[a+(n-1)H\Big]|\nabla h|^2.\end{aligned}$$

Moreover, the comparison theorem holds true in the following form (see Theorem 1.1 in [12]): if $\mathrm{Ric}_f \geq -K$ and $|\nabla f| \leq K$, we have

$$\Delta_f\rho \leq (m-1)\sqrt{H}\coth(\sqrt{H}\rho) + K.$$

Hence $\Delta_f\rho \leq (m-1)(\frac{1}{\rho}+\sqrt{H}) + K$. So (2.12) and (2.13) also hold true in almost the same forms

$$\frac{|\nabla\phi|^2}{\phi} \leq \frac{{c_1}^2}{R^2}$$

and

$$-\Delta_f\phi \leq \frac{[(n-1)(1+\sqrt{HR}) + KR]c_1 + c_2}{R^2}.$$

Noticing the above facts, the proof of Theorem 1.5 is the same to that of Theorem 1.1, so we omit it here.

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一类非线性椭圆方程的刘维尔型定理

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摘要: 设(M^n, g) 是一个n维非紧的完备黎曼流行. 本文考虑有正解的非线性椭圆方程$\Delta_f u + au\log u = 0$的刘维尔型定理, 其中$a$ 是一个非零常数. 利用Bochner 公式和极大值原理, 获得了以上方程在Bakry-Emery里奇曲率有下界时正解的Li-Yau 型梯度估计和某些有关的刘维尔理论, 推广了文献[7]的结果.

关键词: 梯度估计; 非线性椭圆方程; 刘维尔型定理; 极大值原理

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