Vol. 37 (2017)
No. 5

数 学 杂 志
J. of Math. (PRC)

FORCING AN ω_1-REAL WITHOUT ADDING A REAL

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Abstract: In this paper, we study strong Mathias forcing and its generalizations to uncountable cases. By applying the method of forcing, we show that strong Mathias forcing adds unbounded but not dominating reals, which is in contrast to usual Mathias forcing. We also show that the ω_1-version of strong Mathias forcing adds unbounded but not dominating ω_1-reals and meanwhile, this forcing adds no new reals, which are applied to the consistency of cardinal invariants on the real line.

Keywords: ω_1-real; Mathias forcing; dominating reals; unbounded reals; cardinal invariants

2010 MR Subject Classification: 03E17; 03E35; 03E50

Document code: A **Article ID:** 0255-7797(2017)05-0911-05

1 Introduction

Forcing is a mechanism of obtaining independent results over the commonly accepted foundation of mathematics, the Zermelo-Fraenkel axiom system with the axiom of choice. Since its formulation by Cohen [1], forcing has become a powerful tool in advanced set theory as well as other related fields and proved numerous results.

Roughly speaking, forcing usually works in the following way: to prove that a statement $\neg\phi$ is unprovable from ZFC, one builds a model of ZFC together with ϕ. One starts with a ground model $\mathbb{V}$ and force with a generic filter G to get a bigger model, the generic extension $\mathbb{V}[G]$, which witnesses the failure of $\neg\phi$. The generic filter G usually collects information which approximates the desired statement. This process is in a sense similar to a field extension. For instance, $\mathbb{C}$ would disagree with $\mathbb{R}$ on the statement $\exists x(x^2 = -1)$. One start with $\mathbb{R}$ and imposes a solution i of the polynomial $x^2 + 1 = 0$ to get $\mathbb{C} = \mathbb{R}[i]$.

The beginner is referred to [2] and [5] for an elementary introduction to forcing. More details can be found in the textbooks [3] and [6]. Forcing is also related to boolean valued models, see [10] for instance.

The Cohen forcing adds reals to the universe. In set theory, reals are identified with a countable binary sequence or countable sequence on ω. Cohen forcing approaches the generic Cohen real by finite strings. There are other forcings which adjoin different types of reals. For instance, Mathias forcing [7] is defined in Section 2.

* **Received date:** 2016-03-30 **Accepted date:** 2016-06-08

Foundation item: Supported by National Natural Science Foundation of China (11401567).

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The rest of this article is organised as follows: Section 2 introduces the Mathias forcing and its properties; Section 3 deals with a variation of Mathias forcing and focuses on the difference with the usual Mathias forcing; Section 4 generalizes to uncountable space and proves the main theorem.

2 Mathias Forcing

Definition 1 A pair (s, S) is called a Mathias pair, if

1. s is an strictly increasing function from some finite number n to ω.
2. S is an infinite subset of ω.
3. $\text{range}(s) \subseteq \min(S)$.

Definition 2 Mathias forcing consists of all Mathias pairs with the partial order defined as: $(s, S) \leq (t, T)$ if s end extends t, $\text{range}(s) \setminus \text{range}(t) \subseteq T$ and $S \subseteq T$.

For a generic filter G, the Mathias real is defined as $g = \bigcup\{t : \exists T (t, T) \in G\}$, which is an increasing function from ω to ω in the generic extension.

Definition 3 A real g dominates a real f, if $\exists m \in \omega\ \forall n > m\ (f(n) < g(n))$.

Definition 4 $g \in \omega^\omega \cap \mathbb{V}[G]$ is dominating over $\mathbb{V}$ if $\forall f \in \omega^\omega \cap \mathbb{V}$, g dominates f.

Definition 5 $g \in \omega^\omega \cap \mathbb{V}[G]$ is unbounded by $\mathbb{V}$ if $\forall f \in \omega^\omega \cap \mathbb{V}$, g is not dominated by f.

It is clear that a dominating real is unbounded. It can be proved that a Mathias real is dominating over the ground model, while a Cohen real is unbounded but not dominating.

3 Strengthen the Mathias Forcing

In this section, we consider a stronger version of the Mathias forcing.

Consider the strong Mathias forcing $\mathbb{P}_0$ which consists of all Mathias pairs. $(s, S) \leq (t, T)$ if s end extends t, $\text{range}(s) \setminus \text{range}(t) \subseteq T$, $S \subseteq T$ and $T \setminus S$ is finite.

Namely, we enhance the usual Mathias forcing with the last requirement. As with the Mathias forcing, we can define the typical generic real $g = \bigcup\{t : \exists T (t, T) \in G\}$.

In contrast to Mathias forcing, the typical generic real of $\mathbb{P}_0$ is not a dominating real.

Theorem 6 g is not dominating over the ground model.

Proof A key observation is that, below any condition, there are exactly $\aleph_0$ many conditions. So this forcing is in a sense a disjoint union of $\mathfrak{c}$ many Cohen forcings.

Suppose g is dominating, let $r \Vdash \dot{g}$ is dominating. List the conditions below r as $\{p_n : n \in \omega\}$. Let $q_n \leq p_n$ decide $g(n)$. Namely, let $k_n \in \omega$ be such that $q_n \Vdash \dot{g}(n) = k_n$. Define in the ground model that $f : \omega \to \omega$, $f(n) = k_n$. Since $r \Vdash \exists m \forall n > m (\dot{g}(n) > f(n))$. There is some $r' \leq r$ and m such that $\forall n > m\ r' \Vdash \dot{g}(n) > f(n)$. By the observation above, there is some $n > m$ such that $p_n \leq r'$. Then $q_n \leq r'$. Therefore,

$$q_n \Vdash \dot{g}(n) = k_n \wedge \dot{g}(n) > f(n).$$

A contradiction. However, we still have the following.

Theorem 7 g is unbounded over the ground model.

Proof Suppose otherwise. Let $(t,T) \Vdash f$ dominate $\dot{g}$. Moreover, by passing to a stronger condition, we can assume without loss of generality that

$$\exists m \forall n > m\ (t,T) \Vdash f(n) > \dot{g}(n).$$

Now we let $n = \max\{m+1, |t|\}$ and extend (t,T) to (s,S) such that $|s| > n$ and $s(n) \geq f(n)$. Since $(s,S) \Vdash \dot{g}(n) = s(n)$, we get a contradiction.

4 Generalization to Uncountable Space

In this section, we generalize the results to uncountable case. We will focus on the first uncountable cardinal ω_1.

An ω_1-real is a function from ω_1 to ω_1.

Definition 8 A pair (s,S) is called a ω_1-Mathias pair, if

1. s is an strictly increasing function from some countable ordinal α to ω_1.
2. S is an uncountable subset of ω_1.
3. range$(s) \subseteq \min(S)$.

Definition 9 Consider the strong ω_1-Mathias forcing $\mathbb{P}_1$ which consists of all ω_1-Mathias pairs. $(s,S) \leq (t,T)$ if s end extends t, range$(s) \setminus$ range$(t) \subseteq T$, $S \subseteq T$ and $T \setminus S$ is countable.

Remark 10 We might study the ω_1-Mathias forcing. The generic ω_1-real added by ω_1-Mathias forcing is dominating. However, the question that whether reals were added is more complicated and we will not explain in this article.

Now we begin to study the properties of $\mathbb{P}_1$.

The main theorem we wish to prove is the following.

Theorem 11 The forcing $\mathbb{P}_1$ adds no reals; the generic ω_1-real is unbounded. If CH holds, then the generic ω_1-real is not dominating.

Chain condition is usually a key to study a forcing. Set theoretic topologist usually prefer the forcing with countable chain condition (c.c.c.), as Martin's Axiom can be applied there. However, we have the following lemma.

Lemma 12 $\mathbb{P}_1$ does not satisfy the countable chain condition.

Proof Fix a partition $\omega_1 = \bigcup\{I_\alpha : \alpha < \omega_1\}$, with each I_α uncountable. Then the $(\emptyset, I_\alpha)$'s are mutually incompatible forcing notions.

The chain conditions plays an important role in preserving cardinal. Since the failure of c.c.c., one would ask if ω_1 is preserved.

Fortunately, a closure property can be expected for $\mathbb{P}_1$.

Theorem 13 $\mathbb{P}_1$ is ω-closed.

Proof Let $(s_0,S_0) \geq (s_1,S_1) \geq \cdots \geq (s_n,S_n) \geq \cdots$ be a decreasing sequence of conditions. Define $s = \bigcup\{s_n : n < \omega\}$. Then s is an increasing function into ω_1. Let domain$(s_n) = \alpha_n$, domain$(s) = \alpha$, then $\alpha = \sup\{\alpha_n : n \in \omega\}$ is a countable ordinal.

Let $S = \bigcap\{S_n : n < \omega\}$. Then $S = S_0 \setminus \bigcup\{S_n \setminus S_{n+1} : n \in \omega\}$. So S is uncountable.

Since for each n, $\mathrm{range}(s_n) \subseteq \min(S_n) \leq \min(S)$, $\mathrm{range}(s) \subseteq \min(S)$.

Therefore, (s, S) is a condition. Since for each n, $S_n \setminus S = \bigcup\{S_k \setminus S_{k+1} : k \geq n\}$ is a countable set, (s, S) is a lower bound for the given decreasing sequence.

It is well known that ω-closed forcing does not add countable sequence of ordinals. In other words, it is ω-distributive. See [3], for instance. So ω_1 is preserved. Also, we have

Corollary 14 $\mathbb{P}_1$ does not add a real.

ω-closed forcing is proper, see [8]. The class of forcing notions that are both ω-distributive and proper was studied in [9], which focus on the forcing axiom and preservation of a couple of combinatorial properties. An immediate consequence is the following.

Corollary 15 Assume CH in the ground model, then in a generic extension by $\mathbb{P}_1$, or a countable support iteration of $\mathbb{P}_1$, all the following cardinal invariants are equal to $\aleph_1$

$$\begin{aligned}&\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{m}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u},\\ &\mathrm{add}(\mathcal{B}), \mathrm{cov}(\mathcal{B}), \mathrm{non}(\mathcal{B}), \mathrm{cof}(\mathcal{B}), \mathrm{add}(\mathcal{L}), \mathrm{cov}(\mathcal{L}), \mathrm{non}(\mathcal{L}), \mathrm{cof}(\mathcal{L}).\end{aligned}$$

Remark 16 This result is in contrast to the case of usual Mathias forcing. With a countable support iteration of the usual Mathias forcing, $\mathfrak{e}, \mathfrak{m}, \mathfrak{p}, \mathfrak{t}, \mathrm{add}(\mathcal{B}), \mathrm{cov}(\mathcal{B}), \mathrm{add}(\mathcal{L})$, $\mathrm{cov}(\mathcal{L})$ remain $\aleph_1$ while the rest become $\mathfrak{c}$.

As before, if G is a generic filter of $\mathbb{P}$, then let $g = \bigcup\{t : \exists T(t, T) \in G\}$. g is a generic ω_1-real.

Unboundedness and dominating properties can be defined for a generic ω_1-real, similar to Definitions 3 to 5.

Theorem 17 g is an unbounded ω_1-real over the ground model.

Proof The proof is similar to the proof of Theorem 7.

Theorem 18 Assume CH holds in the ground model, then g is not dominating over the ground model.

In fact, we can prove a stronger theorem.

Theorem 19 Assume for any condition r in a forcing $\mathbb{Q}$, $\{q \in \mathbb{Q} : q \leq r\}$ has cardinality $\aleph_1$. Then for any ω_1-real h in the generic extension, there is in the ground model an ω_1-real f, which is not dominated by h.

Proof The proof is similar to that of Theorem 6. Suppose the conclusion fails. Then there is some $r \in \mathbb{Q}$ such that $r \Vdash \dot{h}$ is a dominating ω_1-real.

List $\{r' \in \mathbb{Q} : r' \leq r\} = \{p_\alpha : \alpha < \omega_1\}$. For each α, let $q_\alpha \leq p_\alpha$, $\beta_\alpha < \omega_1$ be such that $q_\alpha \Vdash \dot{h}(\alpha) = \beta_\alpha$. Now define in the ground model an ω_1-real f with $f(\alpha) = \beta_\alpha$.

Since $r \Vdash f$ is dominated by h, there is some $p \leq r$ and $\gamma < \omega_1$ such that $p \Vdash \forall\alpha > \gamma\ (f(\alpha) < \dot{h}(\alpha))$. Since there are uncountably many successor of p, find some $\alpha > \gamma$ such that $p_\alpha \leq p$. Then $q_\alpha \leq p$ and $q_\alpha \Vdash \dot{h}(\alpha) = f(\alpha)$. A contradiction.

Proof of Theorem 18 Assume CH holds in the ground model. Obviously, for any given condition (t, T) of $\mathbb{P}_1$, the cardinality of the set $\{(t', T') \in \mathbb{P}_1 : (t', T') \leq (t, T)\}$ is

$$|\omega_1^{<\omega_1} \times \omega_1^{<\omega_1}| = |\omega_1^{<\omega_1}| = \aleph_1 \cdot |\omega_1^{\omega}| = \aleph_1 \cdot \aleph_1 \cdot \aleph_0^{\aleph_0} = \aleph_1.$$

The second equality is due to the Hausdorff's formula; the last equality is CH.

Thus Theorem 19 applies and we conclude that the forcing $\mathbb{P}_1$ adds an unbounded ω_1-real but adds no dominating ω_1-real.

Remark The results in this section can be generalize to any regular cardinal κ with $\kappa^{<\kappa} = \kappa$.

References

[1] Cohen P J. The independence of the continuum hypothesis[J]. Proc. Nat. Acad. Sci., 1963, 50: 1143–1148,

[2] Chow Timothy Y. A beginner's guide to forcing[A]. Communicating Mathematics, Vol. 479 of Contemp. Math.[C]. Providence RI: Amer. Math. Soc., 2009: 25–40.

[3] Jech T. Set theory (3rd ed)[M]. Amsterdamlin: Springer-Verlag, 2003.

[4] Jech T. Multiple forcing[M]. Vol. 88 of Cambridge Tracts Math., Cambridge: Cambridge Univ. Press, 1986.

[5] Jech T. What is $\cdots$ forcing?[J]. Notices Amer. Math. Soc., 2008, 55(6): 692–693.

[6] Kunen K. Set theory: an introduction to independence proofs[M]. Vol. 102 of Studies in Logic and the Foundations of Mathematics, Amsterdam: North-Holland Publishing Co., 1980.

[7] Mathias A. Happy families[J]. Ann. Math. Logic., 1977, 12(1): 59–111.

[8] Shelah S. Proper forcing[M]. Number 940 in Lecture Notes in Mathematics, Berlin: Springer-Verlag, 1982.

[9] Zhu Huiling. Distributive proper forcing axiom and cardinal invariants[J]. Arch. Math. Logic, 2013, 52(5-6):497–506,

[10] Luan Jingwen. Boolean valued model of polymerized axiom system[J]. J. Math., 1997, 17(2): 64–70.

添加超实数而不加实数的力迫

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摘要: 本文研究了加强型Mathias力迫及其在不可数情形下的推广. 通过力迫法, 证明了Mathias力迫添加支配性实数, 而加强型Mathias力迫添加的是无界、非支配性的实数. 还证明了ω_1上的Mathias 型力迫添加的是无界、非支配性的ω_1 类实数且不添加新的实数. 这些结论可应用于对实数上的基数不变量的研究.

关键词: ω_1-超实数; Mathias力迫; 支配性实数; 无界实数; 基数不变量

MR(2010)**主题分类号**: 03E17; 03E35; 03E50 **中图分类号**: O144.3