ON A NEW NONTRIVIAL ELEMENT INVOLVING THE THIRD PERIODICITY $\gamma$-FAMILY IN $\pi_*S$

WANG Yu-yu¹, WANG Jian-bo²

(1. School of Mathematics and Science, Tianjin Normal University, Tianjin 300387, China)
(2. Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, China)

Abstract: In this paper, we discuss stable homotopy groups of spheres. By making a non-trivial secondary differential as geometric input in the Adams spectral sequence, the convergence of $h_0g_n$ ($n > 3$) in $\pi_*S$ is given. Furthermore, by the knowledge of Yoneda products, a new nontrivial element in $\pi_*S$ is detected. The scale of the nontrivial elements is expanded by our results.

Keywords: stable homotopy groups of spheres; Toda-Smith spectrum; Adams spectral sequence; May spectral sequence; Adams differential

2010 MR Subject Classification: 55Q45

1 Introduction

Let $S$ denote the sphere spectrum localized at $p$ and $p$ denote an odd prime. From [14], the homotopy group of $n$-dimensional sphere $\pi_{n+r}S^n$ ($r > 0$) is a finite group. So the determination of $\pi_{n+r}S^n$ has become one of the central problems in algebraic topology.

Ever since the introduction of the Adams spectral sequence (ASS) in the late 1950’s (see [1]), the study of the homotopy groups of spheres $\pi_*S$ was split into algebraic and geometric problems, including the computation of $\text{Ext}^*_{A}(\mathbb{Z}_p, \mathbb{Z}_p)$ and the detection which element of $\text{Ext}^*_{A}(\mathbb{Z}_p, \mathbb{Z}_p)$ can survive to $E_2^*, \pi_*S$. Here $A$ is the mod $p$ Steenrod algebra, $\text{Ext}^*_{A}(\mathbb{Z}_p, \mathbb{Z}_p)$ is the $E_2$-term of the ASS. By $[2]$,

$$E_2^{s,t} \cong \text{Ext}^*_{A}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s}S,$$

and the Adams differential is $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$.

In addition, we also have the Adams-Novikov spectral sequence (ANSS) [12, 13] based on the Brown-Peterson spectrum BP in the determination of $\pi_*S$.

Many wonderful results were obtained, however, it is still far from the total determination of $\pi_*S$. After the detection of $\eta_j \in \pi_{p^j q + pq - 2}S$ for $p = 2$, $j \neq 2$, by Mahowald in [11],
which was represented by $h_1 h_j \in \text{Ext}_A^{2p^2 + pq}(\mathbb{Z}_p, \mathbb{Z}_p)$, many nontrivial elements in $\pi_9 S$ were found. Please see references [5–9] for details. In recent years, the first author established several convergence of elements by an arithmetic method, see [16–18, 21].

In [5], Cohen made the nontrivial secondary Adams differential $d_2(h_i) = a_0 b_{i-1}$ $(p > 2, i > 0)$ as geometric input, then, a nontrivial element $\xi_i \in \pi_{(p+1)^2} S$ $(i \geq 0)$ is detected.

In this paper, we also detect a new family in $\pi_9 S$ by geometric method, the only geometric input used in the proof is the secondary nontrivial differential given in [20].

The main result is obtained as follows.

**Theorem 1.1** Let $3 \leq s < p - 1$, $n > 3$, $p \geq 7$, then

$$0 \neq \gamma_s h_0 g_n \in \text{Ext}_A^{s+3, p^2 + 2pq + sp^2 q + p + (s-1)q + s-3} (\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element of order $p$ in $\pi_{p^2 + 2pq + sp^2 q + p + (s-1)q + s-3} S$.

The paper is organized as follows. After giving some necessary preliminaries and useful knowledge about the MSS in Section 2. The proof of Theorem 1.1 and some results on Ext groups will be given in Section 3.

2 Related Spectrum and the May Spectral Sequence

For the convenience of the reader, let us briefly indicate the necessary preliminaries in the proof of the propositions and theorems.

Let $M$ be the Moore spectrum modulo an odd prime $p$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$

Let $\alpha$: $\sum^q M \rightarrow M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\sum^q M \xrightarrow{\alpha} M \xrightarrow{j'} V(1) \xrightarrow{j''} \Sigma^{q+1} M.$$

Let $\beta$: $\sum^{(p+1)q} V(1) \rightarrow V(1)$ be the $\nu_2$-mapping and $V(2)$ be the cofibre of $\beta$ sitting in the cofibration

$$\sum^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i''} V(2) \xrightarrow{i'''} \Sigma^{(p+1)q+1} V(1).$$

Furthermore, $\gamma$: $\sum^{(p^2 + p+1)q} V(2) \rightarrow V(2)$ is the $\nu_3$-mapping and the $\gamma$-element $\gamma_s = jj''''$ is a nontrivial element in $\pi_{sp^2 q + (s-1)pq + (s-2)q-3} S$, where $p \geq 7$ (see [15]).

From [19], we know that the third periodicity family $\gamma_s$ is represented by the third Greek letter family element

$$\tilde{\gamma}_s \in \text{Ext}_A^{s, sp^2 q + (s-1)pq + (s-2)q + s-3} (\mathbb{Z}_p, \mathbb{Z}_p)$$

in the ASS, which is represented by the element

$$s(s - 1)(s - 2)a_3^{-3} h_{3,0} h_{2,1} h_{1,2}.$$
in the May spectral sequence (MSS).

Let $L$ be the cofibre of $\alpha_1 = j \omega i: \Sigma^{q-1} S \to S$ given by the cofibration

$$\Sigma^{q-1} S \xrightarrow{\alpha_1} S \xrightarrow{j} L \xrightarrow{f} \Sigma^q S.$$  

From [10], we can see that $\text{Ext}_{A}^{1+\ast}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-bases

$$a_0 \in \text{Ext}_{A}^{1}(\mathbb{Z}_p, \mathbb{Z}_p), \quad h_i \in \text{Ext}_{A}^{1}(\mathbb{Z}_p, \mathbb{Z}_p)(i \geq 0).$$

$\text{Ext}_{A}^{2}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-bases

$$\alpha_2, \quad a_0 h_i \quad (i > 0), \quad g_i \quad (i \geq 0), \quad k_i \quad (i \geq 0), \quad b_i \quad (i \geq 0), \quad \text{and} \quad h_i h_j \quad (j \geq i + 2, i \geq 0),$$

whose internal degrees are

$$2q + 1, 2, p' q + 1, 2p' q + i + 1 q, 2p' i + 1 + p' q, p' i + 1 q \text{ and } p' q + p' q,$$

respectively. Aikawa computed $\text{Ext}_{A}^{1}(\mathbb{Z}_p, \mathbb{Z}_p)$ by $\lambda$-algebra in [3].

In the following, recall the Adams resolution of some spectra related to $S$ from [4]. Let

$$\cdots \xrightarrow{\alpha_2} \Sigma^{-2} E_2 \xrightarrow{\alpha_1} \Sigma^{-1} E_1 \xrightarrow{\alpha_0} E_0 = S$$

be the minimal Adams resolution of the sphere spectrum $S$ which satisfies

(A) $\quad E_s \xrightarrow{\sigma_s} K G_s \xrightarrow{\sigma_{s+1}} \Sigma E_s \quad$ are cofibrations for all $s \geq 0$, which induce short exact sequences in $\mathbb{Z}_p$-cohomology

$$0 \longrightarrow H^s E_{s+1} \xrightarrow{\sigma_s} H^s K G_s \xrightarrow{\sigma_{s+1}} H^s E_s \longrightarrow 0.$$ 

(B) $K G_s$ are the graded wedge sums of Eilenberg-Maclane spectrum $K \mathbb{Z}_p$ of type $\mathbb{Z}_p$.

(C) $\pi_t K G_s$ are the $E_1^{0, \ast}$-terms of the ASS,

$$(\bar{b}_p, \bar{c}_s) : \pi_t K G_{s-1} \longrightarrow \pi_t K G_s$$

are the $d_{s-1}^{0, \ast}$-differentials of the ASS, and $\pi_t K G_s \cong \text{Ext}_{A}^{s+\ast}(\mathbb{Z}_p, \mathbb{Z}_p)$. Then, an Adams resolution of an arbitrary spectrum $V$ can be obtained by smashing $V$ to (2.1).

Remark 2.1 In the ANSS, $h_0$ is a permanent cycle and converges to the corresponding homotopy element $i' i \omega i (\alpha_1 = j \omega i \in \pi_{q-1} K)$. Furthermore, if some suppositions on Ext groups are given, then there exists $\bar{V} \in \pi_{q+2} \mathbb{Z}_p \mathbb{Z}_p$ such that $i' i \xi = \alpha' \cdot \bar{V}$ (mod $F^i \pi_t K$) and $\bar{V}$ is represented by $(i' i)_s(g_n) \in \text{Ext}_{A}^{2}(p^{2q} + 2 p^{2q-2} p^q K^H (\mathbb{Z}_p, \mathbb{Z}_p)$ in the ASS, where $\xi \in \pi_{q+2} \mathbb{Z}_p \mathbb{Z}_p$ is the homotopy element which is represented by $h_0 g_n \in \text{Ext}_{A}^{2}(p^{2q} + 2 p^{2q-2} p^q K^H (\mathbb{Z}_p, \mathbb{Z}_p)$ in the ASS and $F^i \pi_t K$ denotes the group consisting of all elements in $\pi_t K$ with filtration no less than 4.
To detect $\pi_s S$ with the ASS, we must compute the $E_2$-term of the ASS, $\text{Ext}^*_A(Z_p, Z_p)$. The most successful method for computing it is the MSS.

From [13], there is a MSS $\{E^{*,t,*}_r, d_r\}$, which converges to $\text{Ext}^*_A(Z_p, Z_p)$ with $E_1$-term

$$E^{*,t,*}_1 = E(h_{i,j} \mid i > 0, j \geqslant 0) \otimes P(b_{i,j} \mid i > 0, j \geqslant 0) \otimes P(a_i \mid i \geqslant 0),$$  \hspace{1cm} (2.2)

where $E(\ )$ denotes the exterior algebra, $P(\ )$ denotes the polynomial algebra, and

$$h_{i,j} \in E_1^{1,2(p^i-1)p^{i+1}-1}, b_{i,j} \in E_1^{2,2(p^i-1)p^{i+1}-1}, a_i \in E_1^{1,2p^{i+1}-1}. \hspace{1cm} (2.3)$$

One has $d_r: E^{*,t,M}_r \rightarrow E^{*,t+1,M-r}_r$ ($r \geqslant 1$). If $x \in E^{*,t,*}_r$ and $y \in E^{*,t',*}_r$, then

$$d_r(x \cdot y) = d_r(x)y + (-1)^s x d_r(y). \hspace{1cm} (2.4)$$

Furthermore, the May $E_1$-term is graded commutative in the sense that

$$a_m h_{n,j} = h_{n,j} a_m, \hspace{0.5cm} h_{m,k} h_{n,j} = -h_{n,j} h_{m,k},$$

$$a_m b_{n,j} = b_{n,j} a_m, \hspace{0.5cm} h_{m,k} b_{n,j} = b_{n,j} h_{m,k},$$

$$a_m a_n = a_n a_m, \hspace{0.5cm} b_{m,n} b_{i,j} = b_{i,j} b_{m,n}. \hspace{1cm} (2.5)$$

The first May differential $d_1$ is given by

$$d_1(h_{i,j}) = - \sum_{0<k<i} h_{i-k,k+j} h_{k, j}, \hspace{1cm} d_1(a_i) = - \sum_{0<k<i} h_{i-k,k} a_k, \hspace{1cm} d_1(b_{i,j}) = 0. \hspace{1cm} (2.6)$$

For each element $x \in E^{*,t,*}_1$, if we denote $\dim x = s$, $\deg x = t$, we have

$$\left\{ \begin{align*}
\dim h_{i,j} &= \dim a_i = 1, \hspace{0.5cm} \dim b_{i,j} = 2, \\
\deg h_{i,j} &= 2(p^i-1)p^j = (p^{i+j}-1 + \cdots + p^j)q, \\
\deg b_{i,j} &= 2(p^i-1)p^{i+1} = (p^{i+j} + \cdots + p^{i+1})q, \\
\deg a_i &= 2p^i - 1 = (p^{i-1} + \cdots + 1)q + 1, \\
\deg a_0 &= 1.
\end{align*} \right. \hspace{1cm} (2.7)$$

Remark 2.2 Any positive integer $t$ can be expressed uniquely as $t = q(c_0 + c_{n-1}p^{n-1} + \cdots + c_{i}p + c_0) + c$, where $0 \leqslant c_i < p$ ($0 \leqslant i < n)$, $0 < c_n < p$, $0 \leqslant c < q$. Then, it is easy to get the following result from [16].

**Proposition 2.3** In the MSS, we have $E^{*,t,*}_1 = 0$ for some $j$ ($0 \leqslant j \leqslant n$), $s < c_j$, where $s$ is also a positive integer with $0 < s < p$.

### 3 Some Adams $E_2$-Terms

In this section, we mainly give some important results about Adams $E_2$-terms. At the end, the proof of Theorem 1.1 will be given.
Proposition 3.1 Let $3 \leq s < p - 1$, $n > 3$, $p \geq 7$, then

$$0 \neq \gamma_3 h_0 g_n \in \text{Ext}_A^{s+3,p^{n+1}q+2p^nq+s^2q+(s-1)pq+(s-1)q+s-3}(\mathbb{Z}_p, \mathbb{Z}_p).$$

Proof Consider the structure of $E_1^{s+2,t,*}$ in the MSS, where $t = p^{n+1}q + 2p^nq + sp^2q + (s-1)pq + (s-1)q + s - 3$. Due to $3 \leq s < p - 1$, then $5 \leq s + 2 < p + 1$.

Case 1 $5 \leq s + 2 < p$. Let $h = x_1 x_2 \cdots x_n$ be the generator of $E_1^{s+2,t,*}$, where $x_i$ is one of $a_k$, $h_{i,j}$ or $b_{u,z}$, $0 \leq k \leq n + 2$, $0 < i + j \leq n + 2$, $0 < u + z \leq n + 1$, $i > 0$, $j > 0$, $u > 0$, $z \geq 0$.

Assume that $\deg x_i = q(c_{i,n+1}p^{n+1} + \cdots + c_{i,1}p + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or $1$, $e_i = 1$ if $x_i = a_k$ or $e_i = 0$, then

$$\deg h = \sum_{i=1}^{m} \deg x_i = q((\sum_{i=1}^{m} c_{i,n+1})p^{n+1} + (\sum_{i=1}^{m} c_{i,n})p^n + \cdots + (\sum_{i=1}^{m} c_{i,0})) + (\sum_{i=1}^{m} e_i) = q(p^{n+1} + 2p^n + sp^2 + (s-1)p + (s-1)) + s - 3,$$

$$\dim h = \sum_{i=1}^{m} \dim x_i = s + 2.$$ 

Note that $\dim x_i = 1$ or $2$, we can see that $m \leq s + 2 < p$ from $\sum_{i=1}^{m} \dim x_i = s + 2$. By the fact that $c_{i,j} = 0$ or $1$, $e_i = 0$ or $1$, $m \leq s + 2 < p$, we have

$$\sum_{i=1}^{m} e_i = s - 3, \sum_{i=1}^{m} c_{i,0} = s - 1, \sum_{i=1}^{m} c_{i,1} = s - 1, \sum_{i=1}^{m} c_{i,2} = s,$$

$$\sum_{i=1}^{m} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0, \sum_{i=1}^{m} c_{i,n} = 2, \sum_{i=1}^{m} c_{i,n+1} = 1.$$ 

From the above results, we can see that $b_{1,n} b_{1,n-1} h_{1,n}$, $h_{2,n} h_{1,n}$, $h_{2,n} b_{1,n-1}$, $b_{2,n} h_{1,n}$, $b_{1,n-1} b_{2,n-1}$, $b_{1,n} b_{2,n-1}$, $h_{1,n+1} b_{2,n-1}$ and $h_{1,n+1} b_{1,n-1} h_{1,n}$ are contained in the $x_i$. By the commutativity of $E_1^{*,*,*}$, we can denote

$$h_1 = x_1 x_2 \cdots x_{m-3} b_{1,n} h_{1,n} b_{1,n-1}, \quad h_1' = x_1 x_2 \cdots x_{m-3} \in E_1^{s-3,t',*},$$

$$h_2 = x_1 x_2 \cdots x_{m-3} b_{1,n} b_{1,n-1}, \quad h_2' = x_1 x_2 \cdots x_{m-3} \in E_1^{s-4,t',*},$$

$$h_3 = x_1 x_2 \cdots x_{m-3} h_{1,n+1} b_{2,n-1}, \quad h_3' = x_1 x_2 \cdots x_{m-3} \in E_1^{s-3,t',*},$$

$$h_4 = x_1 x_2 \cdots x_{m-3} h_{2,n} h_{1,n}, \quad h_4' = x_1 x_2 \cdots x_{m-2} \in E_1^{s,t',*},$$

$$h_5 = x_1 x_2 \cdots x_{m-3} h_{2,n} b_{1,n-1}, \quad h_5' = x_1 x_2 \cdots x_{m-2} \in E_1^{s-1,t',*},$$

$$h_6 = x_1 x_2 \cdots x_{m-3} b_{2,n-1} h_{1,n}, \quad h_6' = x_1 x_2 \cdots x_{m-2} \in E_1^{s-1,t',*},$$

$$h_7 = x_1 x_2 \cdots x_{m-3} b_{2,n-1} b_{1,n-1}, \quad h_7' = x_1 x_2 \cdots x_{m-2} \in E_1^{s-2,t',*},$$

$$h_8 = x_1 x_2 \cdots x_{m-3} h_{1,n+1} b_{1,n-1}, \quad h_8' = x_1 x_2 \cdots x_{m-3} \in E_1^{s-2,t',*},$$

where $t' = sp^2q + (s-1)pq + (s-1)q + s - 3$.

We list all the possibilities of $h_1'$ in the following table ($i = 1, 2, \cdots, 8$), thus $h$ doesn’t exist in this case.
For the same reason, the following: if this contradicts to c

Table 1: the possibilities of \( h_i' \)

<table>
<thead>
<tr>
<th>The possibility</th>
<th>Analysis</th>
<th>The existence of ( h_i' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_1' )</td>
<td>( s - 3 &lt; \sum_{i=1}^{m-3} c_{i,2} = s )</td>
<td>Nonexistent</td>
</tr>
<tr>
<td>( h_2' )</td>
<td>( s - 4 &lt; \sum_{i=1}^{m-3} c_{i,2} = s )</td>
<td>Nonexistent</td>
</tr>
<tr>
<td>( h_3' )</td>
<td>( s - 3 &lt; \sum_{i=1}^{m-3} c_{i,2} = s )</td>
<td>Nonexistent</td>
</tr>
<tr>
<td>( h_4' )</td>
<td>( h_4' = a_{s-3}^3 h_{2,0}^2 h_{1,0} = 0 )</td>
<td>Nonexistent</td>
</tr>
<tr>
<td>( h_5' )</td>
<td>( s - 1 &lt; \sum_{i=1}^{m-2} c_{i,2} = s )</td>
<td>Nonexistent</td>
</tr>
<tr>
<td>( h_6' )</td>
<td>( s - 1 &lt; \sum_{i=1}^{m-2} c_{i,2} = s )</td>
<td>Nonexistent</td>
</tr>
<tr>
<td>( h_7' )</td>
<td>( s - 2 &lt; \sum_{i=1}^{m-3} c_{i,2} = s )</td>
<td>Nonexistent</td>
</tr>
<tr>
<td>( h_8' )</td>
<td>( s - 2 &lt; \sum_{i=1}^{m-3} c_{i,2} = s )</td>
<td>Nonexistent</td>
</tr>
</tbody>
</table>

**Case 2** If \( s + 2 = \pi \), then \( E_1^{s+2,t'',*} = E_1^{p,t'',*} \), where \( t'' = p^{n+1}q + 2p^nq + (p - 2)p^2q + (p - 3)pq + (p - 3)q + p - 5 \). Let \( h = x_1 x_2 \cdots x_r \) be the generator of \( E_1^{p,t'',*} \), and assume that

\[
\deg x_i = q(c_{i,n+1} p^{n+1} + c_{i,n} p^n + \cdots + c_{i,1} p + c_{i,0}) + e_i,
\]

where \( c_{i,j} = 0 \) or 1, \( e_i = 1 \) if \( x_i = a_{k_i} \) or \( e_i = 0 \), then

\[
\deg h = \sum_{i=1}^{r} \deg x_i = q((\sum_{i=1}^{r} c_{i,n+1}) p^{n+1} + (\sum_{i=1}^{r} c_{i,n}) p^n + \cdots + (\sum_{i=1}^{r} c_{i,0})) + (\sum_{i=1}^{r} e_i)
\]

\[
= q(p^{n+1} + 2p^n + (p - 2)p^2 + (p - 3)p + (p - 3)) + p - 5,
\]

\[
\dim h = \sum_{i=1}^{r} \dim x_i = p.
\]

We claim that \( \sum_{i=1}^{r} c_{i,0} \), \( \sum_{i=1}^{r} c_{i,1} \) and \( \sum_{i=1}^{r} c_{i,2} \) are impossible to constitute \( p \). The reason is the following: if \( \sum_{i=1}^{r} c_{i,0} = p \), because of \( \sum_{i=1}^{r} e_i = p - 5 \), then

\[
q((\sum_{i=1}^{r} c_{i,n+1}) p^{n+1} + (\sum_{i=1}^{r} c_{i,n}) p^n + \cdots + (\sum_{i=1}^{r} c_{i,0})) + (\sum_{i=1}^{r} e_i) = \sum_{i=1}^{r} e_i \pmod{p},
\]

this contradicts to \( q(p^{n+1} + 2p^n + (p-2)p^2 + (p-3)p + (p-3)) + p - 5 = (p-3)q + p - 5 \pmod{p} \).

For the same reason, \( \sum_{i=1}^{r} c_{i,1} \) and \( \sum_{i=1}^{r} c_{i,2} \) are impossible to constitute \( p \).

From \( \dim x_i = 1 \) or 2 and \( \sum_{i=1}^{r} \dim x_i = p \), we can see that \( r \leq p \). By Remark 2.2 and
When \( r \leq p, c_{i,j} = 0 \) or 1, \( e_i = 0 \) or 1, we have
\[
\sum_{i=1}^{r} c_i = p - 5, \quad \sum_{i=1}^{r} c_{i,0} = p - 3, \quad \sum_{i=1}^{r} c_{i,1} = p - 3, \quad \sum_{i=1}^{r} c_{i,2} = p - 2,
\]
so
\[
(r \sum_{i=1}^{r} c_{i,3}) p^3 + \cdots + r \sum_{i=1}^{r} c_{i,n} p^{n-3} + (\sum_{i=1}^{r} c_{i,n+1}) p^{n-1} = p^{n+1} + 2p^n, \quad (3.1)
\]

Thus \( p \mid \sum_{i=1}^{r} c_{i,3} \). Note that \( c_{i,3} = 0 \) or 1, \( r \leq p \), it is known that \( \sum_{i=1}^{r} c_{i,3} = 0 \) or \( p \).

**Case 2.1** When \( \sum_{i=1}^{r} c_{i,3} = 0 \), we have
\[
(\sum_{i=1}^{r} c_{i,4}) p + \cdots + (\sum_{i=1}^{r} c_{i,n} p^{n-3} + (\sum_{i=1}^{r} c_{i,n+1}) p^{n-2} = p^{n-2} + 2p^{n-3}.
\]

**Case 2.1.1** When \( n > 4 \), we claim that \( \sum_{i=1}^{r} c_{i,4} = 0 \). Otherwise, if \( \sum_{i=1}^{r} c_{i,4} = p \), then
\( r = p \). So \( \dim x_i = 1(1 \leq i \leq p) \) and \( \deg x_i = \) (higher terms) + \( p^kq \) + (lower terms).
Because of \( \sum_{i=1}^{r} c_i = p - 5 \), \( \deg a_k \equiv 1(\mod q) \), \( \dim h_{i,j} \equiv 0(\mod q) \) and \( \dim b_{u,z} \equiv 0(\mod q) \), there exist factors \( a_{i_1}a_{i_2}\cdots a_{i_{p-5}} \) among the generators \( x_i (j_i \geq 5, 1 \leq i \leq p - 5) \). Thus, \( \sum_{i=1}^{r} c_{i,3} \geq p - 5 \), which contradicts to \( \sum_{i=1}^{r} c_{i,3} = 0 \), so \( \sum_{i=1}^{r} c_{i,4} = 0 \). By induction on \( j \), we can get \( \sum_{i=1}^{r} c_{i,j} = 0 \) (5 \( \leq j \leq n - 1 \)), \( \sum_{i=1}^{r} c_{i,n} = 2 \), \( \sum_{i=1}^{r} c_{i,n+1} = 1 \).

**Case 2.1.2** When \( n = 4 \), it is easy to get \( \sum_{i=1}^{r} c_{i,4} = 2 \) and \( \sum_{i=1}^{r} c_{i,5} = 1 \).

From the above discussion of Case 2.1.1 and Case 2.1.2, similarly to Case 1, we can see that \( b_{1,n}b_{1,n-1}h_{1,n}, b_{2,n}h_{1,n}, b_{2,n-1}h_{1,n}, b_{1,n-1}b_{2,n-1}, b_{1,n}b_{2,n-1}, b_{1,n-1}h_{1,n+1} \) and \( h_{1,n+1}b_{1,n-1} \) are contained in the \( x_i \), so \( h \) is impossible to exist.

**Case 2.2** When \( \sum_{i=1}^{r} c_{i,3} = p \), then \( r = p \). We get \( \dim x_i = 1 \) from \( \dim h = p \), then \( h = x_1x_2\cdots x_p, x_i \in E(h_{i,j} | i > 0, j \geq 0) \otimes P(a_k | k \geq 0) \).

**Case 2.2.1** When \( n > 4 \), we get
\[
p \cdot p^3 + (\sum_{i=1}^{r} c_{i,4}) p^4 + \cdots + (\sum_{i=1}^{r} c_{i,n}) p^n + (\sum_{i=1}^{r} c_{i,n+1}) p^{n+1} = p^{n+1} + 2p^n,
\]
that is \( (1 + \sum_{i=1}^{r} c_{i,4}) + (\sum_{i=1}^{r} c_{i,5}) p + \cdots + (\sum_{i=1}^{r} c_{i,n+1}) p^{n-3} = p^{n-3} + 2p^{n-4} \), thus \( p \mid (1 + \sum_{i=1}^{r} c_{i,4}) \),
so \( \sum_{i=1}^{r} c_{i,4} = p - 1 \) from \( c_{i,4} = 0 \) or 1 and \( r = p \). By induction on \( j \), we can get
\[
\sum_{i=1}^{r} c_{i,j} = p - 1(4 \leq j \leq n - 1), \quad \sum_{i=1}^{r} c_{i,n} = 1, \quad \sum_{i=1}^{r} c_{i,n+1} = 1.
\]
By the reason of degree and the Proposition 2.3, $h$ is impossible to exist.

**Case 2.2.2** When $n = 4$, we know that $\sum_{i=1}^{c_{i,4}} = 1$, $r \sum_{i=1}^{c_{i,5}} = 1$ from (3.2), then

$$\deg h = q(p^5 + 2p^4 + (p - 2)p^2 + (p - 3)p + p - 3) + (p - 5).$$

By the reason of degree and the Proposition 2.3, $h$ is impossible to exist.

From the above discussion, for $5 \leq s + 2 < p + 1$, $E_{r}^{s+2,t,*} = 0$, so $E_{r}^{s+2,t,*} = 0$ ($r \geq 2$). It is known that $h_{a,n}h_{b,1,n}, h_{a,n}, a_{a}^{s-3}h_{b,1,n}h_{1,2} \in E_{r}^{s,t,*}$ are permanent cycles in the MSS and converge nontrivially to $g_{n}, h_{n}, \bar{\gamma}_{s} \in \text{Ext}_{A}^{s,t,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ ($n \geq 0$), respectively, so $a_{a}^{s-3}h_{b,1,n}h_{1,2}h_{1,n}h_{1,0} \in E_{r}^{s+2,t,*}$ is a permanent cycle in the MSS and converges nontrivially to $\bar{\gamma}_{s}h_{0}g_{n} \in \text{Ext}_{A}^{s+3,t}$. Note that $E_{r}^{s+2,t,*} = 0$ ($r \geq 1$), thus the permanent cycle is not $d_{i}$-boundary and converges nontrivially to $\bar{\gamma}_{s}h_{0}g_{n} \in \text{Ext}_{A}^{s+3,t}$. That is, when $5 \leq s + 2 < p + 1$, $h_{a,n}h_{b,1,n}h_{1,2}h_{1,n}h_{1,0} \in \text{Ext}_{A}^{s+3,t,*}(\mathbb{Z}_p, \mathbb{Z}_p)$.

**Proposition 3.2** Let $3 \leq s < p - 1$, $n > 3$, $p \geq 7$, $2 \leq r < s + 3$, then

$$\text{Ext}_{A}^{s+3-r,t,*} = 0$$

**Proof** We only need to prove that $E_{r}^{s+3-r,t,*} = 0$ in the MSS; where $t = p^{n+1}q + 2p^nq + sp^2q + (s - 1)pq + s - r - 2$. Let $h = \chi_{i}x_{1}x_{2} \cdots x_{m}$ be the generator of $E_{r}^{s+3-r,t,*}$, where $x_{i}$ is $a_{k}, h_{i,j}$ or $b_{u,i}$, $0 \leq k \leq n + 2$, $0 < i + j \leq n + 2$, $0 < u + z \leq n + 1$, $i > 0, j > 0$, $u > 0, z \geq 0$.

Assume that $\deg x_{i} = q(c_{i,n}p^{n+1} + c_{i,n}p^n + \cdots + c_{i,0}) + e_{i}$, where $c_{i,j} = 0$ or $1$, $e_{i} = 1$ if $x_{i} = a_{k}$ or $e_{i} = 0$, then

$$\deg h = \sum_{i=1}^{m} \deg x_{i} = q(\sum_{i=1}^{m} c_{i,n+1})p^{n+1} + \left( \sum_{i=1}^{m} c_{i,n}p^n + \cdots + \left( \sum_{i=1}^{m} c_{i,0} \right) \right) + \left( \sum_{i=1}^{m} e_{i} \right) = q(p^{n+1} + 2p^n + sp^2q + (s - 1)p + (s - 1)) + s - r - 2,$$

$$\dim h = \sum_{i=1}^{m} \dim x_{i} = s + 3 - r.$$
where \( t(r) = sp^2q + (s - 1)pq + (s - 1)q + s - 2 - r. \)

For \( h', s - 2 - r < \sum_{i=1}^{m-3} c_i, = s, \) by Proposition 2.3, we get that \( h'_i \) is impossible to exist.

For the same reason, \( h'_i (i = 2, 3, \cdots, 8) \) are impossible to exist. So we have \( E_{1}^{s+3-r,t} = 0, \) that is \( \text{Ext}_{A}^{s+3-r,t}(Z_{p}, Z_{p}) = 0. \)

**Proposition 3.3** Let \( p \geq 7, \) \( tq = p^{n+1}q + 2p^s q, n > 3, \) then

\[
\begin{align*}
\text{(1)} & \quad \text{Ext}_{A}^{1,tq+r+q}(Z_{p}, Z_{p}) = 0 \quad (r = 2, 3, 4, \quad u = -1, 0 \quad \text{or} \quad r = 3, 4, \quad u = 1); \\
& \quad \text{Ext}_{A}^{1,tq}(Z_{p}, Z_{p}) \cong Z_{p}\{h_{0}l_{n}\}; \quad \text{Ext}_{A}^{4,tq}(Z_{p}, Z_{p}) = 0; \\
& \quad \text{Ext}_{A}^{4,tq+2+1}(Z_{p}, Z_{p}) \cong Z_{p}\{a_{0}g_{n}\}, a_{0}g_{n} \neq 0.
\end{align*}
\]

\[
\begin{align*}
\text{(2)} & \quad \text{Ext}_{A}^{5,tq+r+q+1}(Z_{p}, Z_{p}) = 0 \quad (r = 1, 3, 4); \quad \text{Ext}_{A}^{5,tq+r}(Z_{p}, Z_{p}) = 0 \quad (r = 2, 3); \\
& \quad \text{Ext}_{A}^{5,tq+2+1}(Z_{p}, Z_{p}) \cong Z_{p}\{h_{2}\}; \quad \text{Ext}_{A}^{5,tq+2}(Z_{p}, Z_{p}) \cong Z_{p}\{a_{0}g_{n}\}; \\
& \quad \text{Ext}_{A}^{5,tq+1}(Z_{p}, Z_{p}) = 0.
\end{align*}
\]

**Proof** (1) Consider the second degrees (mod \( p^{n+1}q \)) of the generators in the \( E_1 \)-terms of the MSS, where \( 0 \leq j \leq n + 1, \)

\[
\begin{align*}
\text{deg } h_{s,j} & = (p^{s+j-1} + \cdots + p^j)q \pmod{p^{n+1}q}, \quad 0 \leq j < s + j - 1 < n + 1, \\
& = (p^n + \cdots + p^j)q \pmod{p^{n+1}q}, \quad 0 \leq j < s + j - 1 < n + 1; \\
\text{deg } b_{s,j-1} & = (p^{s+j-1} + \cdots + p^j)q \pmod{p^{n+1}q}, \quad 1 \leq j < s + j - 1 < n + 1, \\
& = (p^n + \cdots + p^j)q \pmod{p^{n+1}q}, \quad 1 \leq j < s + j - 1 < n + 1; \\
\text{deg } a_{j+1} & = (p^{s+j} + \cdots + 1)q + 1 \pmod{p^{n+1}q}, \quad 0 \leq j < n + 1, \\
& = (p^n + \cdots + 1)q + 1 \pmod{p^{n+1}q}, \quad j = n + 1.
\end{align*}
\]

For the second degree \( k = tq+r+u \) \((0 \leq r \leq 4, -1 \leq u \leq 2) = 2p^s q + r + u \pmod{p^{n+1}q}, \) and excluding the factor which has second degree \( \geq tq + p_q, \) we can get that the possibility of the factor of the generators in \( E_1^{w,tq+r+q+u} \) \((4 \leq w \leq 5)\) are \( a_0, a_1, h_{1,0}, h_{1,n+1}, h_{1,n}, h_{2,n}, \)
Thus from the degree we know that

\[ E_1^{4, tq + ru + u, *} = 0 \quad (r = 3, 4, u = 1); \]
\[ E_1^{4, tq + ru + u, *} = 0 \quad (r = 2, 3, 4, u = -1, 0); \]
\[ E_1^{4, tq, *} \cong Z_p\{b_{1, n-1} b_{2, n-1}\}; \]
\[ E_1^{4, tq + ru, *} \cong Z_p\{h_{1, 0} h_{1, n} b_{2, n-1}, h_{2, n} b_{1, n-1} h_{1, 0}\}; \]
\[ E_1^{4, tq + 2q + 1, *} \cong Z_p\{a_1 h_{1, 0} h_{1, n} b_{2, n}\}. \]

In the MSS, note that \( d_r(xy) = d_r(x)y + (-1)^s xd_r(y) (x \in E_1^{s, t, *}, y \in E_1^{s', t', *}) \). Since \( d_r(b_{1, n-1} h_{2, n} h_{1, 0}) \neq 0 \), then \( E_1^{2, tq} = Z_p\{b_{2, n-1} h_{1, n} h_{1, 0}\} \) \((r \geq 2)\). Moreover, \( h_{2, n} h_{1, n} h_{1, 0} \) is permanent cycle in the MSS which converges to \( h_{0} g_a \in \text{Ext}^{5, *}_A(Z_p, Z_p) \), then \( d_r(E_1^{5, tq + ru, *}) = 0 \) for \( r \geq 1 \), so that \( b_{2, n-1} h_{1, n} h_{1, 0} \) is not \( d_r \)-boundary and it converges nontrivially to \( h_{0} g_a \).

In addition, we say that \( \text{Ext}^{5, tq}_A(Z_p, Z_p) = 0 \), since \( E_1^{5, tq, *} \cong Z_p\{b_{1, n-1} b_{2, n-1}\} \), where \( b_{1, n-1} \) converges to \( b_{n-1} \), while in the \( \text{Ext}^{2, *}_A(Z_p, Z_p) \), there is no element in relation to \( b_{2, n-1} \in E_1^{2, pq + ru, q, *}. \)

(2) Similarly, due to the degree of the reason, we can get the following results

\[ E_1^{5, tq + q + 1, *} \cong Z_p\{a_1 b_{1, n-1} b_{2, n-1}, a_0 h_{1, 0} h_{2, n} b_{1, n-1}, a_0 h_{1, 0} b_{2, n-1} h_{1, n}\}; \]
\[ E_1^{5, tq + ru + q + 1, *} = 0 \quad (r = 3, 4); \]
\[ E_1^{5, tq + ru, *} = 0 \quad (r = 2, 3); \]
\[ E_1^{5, tq + 2q + 1, *} \cong Z_p\{a_1 h_{1, 0} b_{2, n-1} h_{1, n}, h_{2, n} b_{1, n-1} h_{1, 0} a_1\}; \]
\[ E_1^{5, tq + 2q + 2, *} \cong Z_p\{a_0 h_{1, 0} h_{2, n-1}, a_0^2 h_{2, n} b_{1, n-1}\}; \]
\[ E_1^{5, tq + 1, *} \cong Z_p\{a_1 b_{1, n-1} b_{2, n-1}, a_0 h_{1, 0} h_{1, n} b_{1, n+1}\}. \]

The generators of \( E_1^{5, tq + q + 1, *} \) in the MSS all die, this is because that

\[ d_1(a_1 b_{1, n-1} b_{2, n-1}) = -a_0 h_{1, 0} b_{1, n-1} b_{2, n-1} \neq 0; \]
\[ a_0 h_{1, 0} b_{2, n-1} b_{1, n-1} = -d_1(a_1 h_{2, n} b_{1, n-1}) \]

and

\[ a_0 h_{1, 0} b_{2, n-1} h_{1, n} = -d_1(a_1 h_{2, n-1} h_{1, n}). \]

So we have \( \text{Ext}^{5, tq + q + 1}_A(Z_p, Z_p) = 0 \). In addition, with a similar proof of (1), we know that \( d_r(E_1^{5, tq + 2q + 1, *}) = 0 \). So the generator of \( E_1^{5, tq + 2q + 1, *} \) in the MSS converges to \( a_0 h_{1, 0} h_{1, n+1} \).

Since \( d_1(h_{2, n-1} b_{1, n-1} h_{1, 0}) \neq 0 \), then \( E_1^{5, tq + 2q + 1} = Z_p\{b_{2, n-1} h_{1, n} h_{1, 0} a_1\} \) for \( r \geq 2 \). Moreover, \( h_{2, n} h_{1, n} h_{1, 0} a_1 \) is a permanent cycle in the MSS which converges to \( \tilde{\alpha}_2 \in \text{Ext}^{*, *}_A(Z_p, Z_p) \), then \( d_r(E_1^{5, tq + 2q + 1}) = 0 \) for \( r \geq 1 \). Thus \( b_{2, n-1} h_{1, n} h_{1, 0} a_1 \) is not \( d_r \)-boundary and converges nontrivially to \( \tilde{\alpha}_2 a_1 \).

Since \( a_0, b_{n-1}, h_{n} \) and \( h_{n+1} \) are all permanent cycles in the MSS and converge to \( a_0, b_{n-1}, h_{n} \) and \( h_{n+1} \), respectively, it is easy to get that \( a_0 b_{1, n-1} h_{1, n} h_{1, n+1} \) is a permanent cycle in the MSS and converges to \( a_0 b_{n-1} h_{n} h_{n+1} \) which equals \( 0 \in \text{Ext}^{5, tq + 1}_A(Z_p, Z_p) \) by
in the ASS, the elements
$k \in \phi$ because $n > \delta$ for $\tilde{L}$ is a permanent cycle in the ASS, and converges to a nontrivial element in $\pi_{p^{n+1}}(\mathbb{Z}_p, \mathbb{Z}_p)$.

**Theorem 3.4** Let $p \geq 7$, $n > 3$, then

$$h_0g_n \in \text{Ext}_A^{3, p^{n+1} + 2p^n q + 2q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS, and converges to a nontrivial element in $\pi_{p^{n+1} + 2p^n q + q - 3S}$.

**Proof** From [20, Theorem 1.1], there is a nontrivial differential $d_2(g_n) = a_0l_n(n \geq 1)$ in the ASS, the elements $g_n$ and $l_n$ are called a pair of $a_0$-related elements. The condition of Theorem A in [7] can be established by the $\mathbb{Z}_p$-bases of $\text{Ext}_A^{s}(\mathbb{Z}_p, \mathbb{Z}_p)$ ($s \leq 3$) in [10] and Proposition 3.3 in the above. Furthermore, we have $\kappa \cdot (\alpha_1)_L = (1_{E_4} \land p)f$ with $f \in [\sum_{i=0}^{n-1} L, E]$ (see [7], 9.2.34), then $(1_{E_4} \land i)\kappa \cdot (\alpha_1)_L = 0$. Thus

$$(1_{E_4} \land L \land i)(\kappa \land L)\phi = (1_{E_4} \land L \land i)(\kappa \land L)((\gamma_1)L \land L)\tilde{\gamma}_\omega = 0,$$

where $\tilde{\gamma}_\omega \in \pi_q L \land L$ such that $((\alpha_1)_L \land 1_{L})\tilde{\gamma}_\omega = \phi$. It can be easily proved that $(\kappa \land 1_{L})\phi = (\tau_2 \land 1_{L})\sigma \phi$, where $\sigma \phi \in \pi_{q+2p}(KG_3 \land L)$ is a $d_1$-cycle which represents $(\phi, (\sigma) \in \text{Ext}_A^{3, q+2q}(H^* L, \mathbb{Z}_p)$. Thus

$$(\tau_2 \land 1_{L} \land M)(1_{KG_3} \land i)\sigma \phi = 0.$$ 

So we can get that $(1_{L} \land i)_s \phi_s (g_n) \in \text{Ext}_A^{3, p^{n+1} q + 2p^n q + q}(H^* L \land M, \mathbb{Z}_p)$ is a permanent cycle in the ASS. Then Theorem 3.4 will be concluded by Theorem C in [7], here $\phi \in [\sum_{i=0}^{q-1} S, L]$, $\kappa \in \pi_{q+1} E_4$.

**The Proof of Theorem 1.1** From Theorem 3.4, $h_0g_n \in \text{Ext}_A^{3, p^{n+1} q + 2p^n q + q}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\varphi \in \pi_{p^{n+1} q + 2p^n q + q - 3S}$ for $n > 3$.

Consider the following composition of mappings

$$\tilde{f} : \Sigma^{p^{n+1} + 2p^n q + q - 3S} \xrightarrow{\text{Ext}_A^{0, 0}(\mathbb{Z}_p, \mathbb{Z}_p)} V(2) \xrightarrow{\gamma_\omega} \Sigma^{-s(p^2 + p + 1)q + (p+1)q} \mathbb{Z}_p,$$

because $\varphi$ is represented by $h_0g_n$ in the ASS, then the above $\tilde{f}$ is represented by

$$\tilde{g} = (jj''i'')(s)(\gamma^s)(S, S, (i''i''')(h_0g_n) = (jj''i''s) \gamma_s (i''i''')(h_0g_n)$$

in the ASS. Furthermore, we know that $\gamma_s = jj''i''s \gamma_s (i''i'') \in \pi_s S$ is represented by $\tilde{\gamma}_s$ in the ASS. By using the Yoneda products, we know that the composition

$$\text{Ext}_A^{0, 0}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{(i''i'')^s} \text{Ext}_A^{0, 0}(H^* V(2), \mathbb{Z}_p) \xrightarrow{(jj''i'')(\gamma_\omega)^s} \text{Ext}_A^{s(p^2 + s-1)q + (s+1)q + q - 3}(\mathbb{Z}_p, \mathbb{Z}_p)$$
is a multiplication (up to nonzero scalar) by
\[ \tilde{\gamma}_s \in \text{Ext}_A^{s,p^2q+(s-1)pq+(s-2)q+s-3}(\mathbb{Z}_p, \mathbb{Z}_p). \]

Hence, the composite map \( \tilde{f} \) is represented (up to nonzero scalar) by
\[ \tilde{\gamma}_s h_0 g_n \in \text{Ext}_A^{s+3,p^{s+1}q+2p^s q+p^s q+(s-1)pq+(s-1)q+s-3}(\mathbb{Z}_p, \mathbb{Z}_p) \]
in the ASS.

From Proposition 3.1, we see that \( \tilde{\gamma}_s h_0 g_n \neq 0 \). Moreover, from Proposition 3.2, it follows that \( \tilde{\gamma}_s h_0 g_n \) cannot be hit by any differential in the ASS. Thus \( \tilde{\gamma}_s h_0 g_n \) survives nontrivially to a homotopy element in \( \pi_\ast S \).

References


球面稳定同伦群中第三周期$\gamma$类非平凡新元素

王玉玉$^1$, 王健波$^2$

(1.天津师范大学数学科学学院, 天津 300387)
(2.天津大学理学院数学系, 天津 300072)

摘要: 本文研究了球面稳定同伦群的问题. 以Adams谱序列中的第二非平凡微分为几何输入, 给出了
球面稳定同伦群中$hag_n(n > 3)$的收敛性. 同时, 由Yoneda乘积的知识, 发掘了球面稳定同伦群中的一个非
平凡新元素. 非平凡元素的范围将被我们的结果进一步扩大.

关键词: 球面稳定同伦群; Toda-Smith谱; Adams谱序列; May谱序列; Adams微分