THE GROWTH ON ENTIRE SOLUTIONS OF FERMAT TYPE \( q \)-DIFFERENCE DIFFERENTIAL EQUATIONS

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Abstract: This paper is devoted to consider the entire solutions on Fermat type \( q \)-difference differential equations. Using the classical and difference Nevanlinna theory and functional equations theory, we obtain some results on the growth of the Fermat type \( q \)-difference differential equations.

Keywords: \( q \)-difference differential equations; entire solutions; finite order

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1 Introduction

Let \( f(z) \) be a meromorphic function in the complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [5, 16]. As we all know that Nevanlinna theory was extensively applied to considering the growth, value distribution, and solvability of meromorphic solutions of differential equations [6]. Recently, difference analogues of Nevanlinna theory were established, which also be used to consider the corresponding properties of meromorphic solutions on difference equations or \( q \)-difference equations, such as [2, 4, 7–12, 14, 17].

Let us recall the classical Fermat type equation

\[
 f(z)^2 + g(z)^2 = 1. \tag{1.1}
\]

Equation (1.1) has the entire solutions \( f(z) = \sin(h(z)) \) and \( g(z) = \cos(h(z)) \), where \( h(z) \) is any entire function, no other solutions exist. However, the above result fails to give more precise informations when \( g(z) \) has a special relationship with \( f(z) \). Yang and Li [15] first considered the entire solutions of the Fermat type differential equation

\[
 f(z)^2 + f'(z)^2 = 1, \tag{1.2}
\]

and they proved the following result.

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Theorem A [15, Theorem 1] The transcendental meromorphic solutions of (1.2) must satisfy
\[ f(z) = \frac{1}{2} \left( Pe^{-iz} + \frac{1}{P} e^{iz} \right) = \sin(z + B), \]
where \( P \) is a non-zero constant and \( e^{iB} = \frac{i}{P} \).

Tang and Liao [13] further investigated the entire solutions of a generalization of (1.2) as follows
\[ f(z)^2 + P(z)^2 f^{(k)}(z)^2 = Q(z), \tag{1.3} \]
where \( P(z), Q(z) \) are non-zero polynomials and obtained the next result.

Theorem B [13, Theorem 1] If the differential equation (1.3) has a transcendental meromorphic solution \( f \), then
\[ P(z) \equiv A, \quad Q(z) \equiv B, \quad k \text{ is odd and} \quad f(z) = b \sin(az + d), \]
where \( a, b, d \) are constants such that \( Aa^k = \pm 1, b^2 = B \).

Recently, the difference analogues of Nevanlinna theory were used to consider the solutions properties of Fermat type difference equations. Liu, Cao and Cao [8] investigated the finite order entire solutions of the difference equation
\[ f(z)^2 + f(z + c)^2 = 1, \tag{1.4} \]
here and in the following, \( c \) is a non-zero constant and \( P(z), Q(z) \) are non-zero polynomial, unless otherwise specified. The result can be stated as follows.

Theorem C [8, Theorem 1.1] The transcendental entire solutions with finite order of (1.4) must satisfy \( f(z) = \sin(Az + B) \), where \( B \) is a constant and \( A = \frac{(4k+1)\pi}{2c} \), \( k \) is an integer.

Furthermore, Liu and Yang [10] considered a generalization of (1.4) as follows
\[ f(z)^2 + P(z)^2 f(z + c)^2 = Q(z), \tag{1.5} \]
and obtained the following result.

Theorem D Let \( P(z), Q(z) \) be non-zero polynomials. If the difference equation (1.5) admits a transcendental entire solution of finite order, then \( P(z) \equiv \pm 1 \) and \( Q(z) \) reduces to a constant \( q \).

If an equation includes the \( q \)-difference \( f(qz) \) and the derivatives of \( f(z) \) or \( f(z + c) \), then this equation can be called \( q \)-difference differential equation. Liu and Cao [11] considered the entire solutions on Fermat type \( q \)-difference differential equation
\[ f'(z)^2 + f(qz)^2 = 1, \tag{1.6} \]
and obtained the following result.

Theorem E [11, Theorem 3.1] The transcendental entire solutions with finite order of (1.6) must satisfy \( f(z) = \sin(z + B) \) when \( q = 1 \), and \( f(z) = \sin(z + k\pi) \) or \( f(z) = -\sin(z + k\pi + \frac{\pi}{2}) \) when \( q = -1 \). There are no transcendental entire solutions with finite order when \( q \neq \pm 1 \).

By comparing with the above five theorems, we state the following questions which will be considered in this paper.
**Question 1** From Theorem A to Theorem E, we remark that the order of all transcendental entire solutions with finite order of different equations are equal to one. Hence, considering a generalization of equation (1.6), such as

\[ f'(z)^2 + P(z)^2 f(qz)^2 = Q(z), \quad (1.7) \]

it is natural to ask if the finite order of the entire solutions of (1.7) is equal to one or not?

**Question 2** From Theorem B to Theorem E, the existence of finite order entire solutions of (1.3) and (1.5) forces the polynomial \( P(z) \) reduce to a constant. Is it also remain valid for equation (1.7)?

However, Examples 1 and 2 below show that Questions 1 and 2 are false in generally.

**Example 1** Entire function \( f(z) = \sin z^n \) solves

\[ f'(z)^2 + n^2 z^{2(n-1)} f(qz)^2 = n^2 z^{2(n-1)}, \]

where \( q \) satisfies \( q^n = 1 \). It implies that the solutions order of (1.7) may take arbitrary numbers and \( P(z)^2 = n^2 z^{2(n-1)} \) is not a constant.

**Example 2** We can construct a general solution from Example 1. Entire function \( f(z) = \sin(h(z)) \) solves

\[ f'(z)^2 + [h'(z)]^2 f(qz)^2 = [h'(z)]^2, \]

where \( q \) satisfies \( q^n = 1 \) and \( h(z) \) is a non-constant polynomial.

**Example 3** Function \( f(z) = \sinh z \) is also an entire solution of \( f'(z)^2 - f(qz)^2 = 1 \) and \( f(z) = \cosh z \) is an entire solution of \( f'(z)^2 - f(qz)^2 = -1 \), where \( q = -1 \).

From Example 1 to Example 3, we also remark that if \( P(z)^2 = \pm 1 \), the transcendental entire solutions \( f(z) \) are of order one, if \( P(z) = nz^{(n-1)} \), the transcendental entire solutions \( f(z) \) are of order \( n \). Hence, it is reasonable to conjecture that the order of entire solutions of (1.7) is equal to \( \rho(f) = 1 + \deg P(z) \). In this paper, we will answer the above conjecture and obtain the following result.

**Theorem 1.1** If \( |q| > 1 \), then the entire solution of (1.7) should be a polynomial. If there exists a finite order transcendental entire solution \( f \) of (1.7), then \( \rho(f) = 1 + \deg P(z) \) and \( |q| = 1 \).

In the following, we will consider another \( q \)-difference differential equation

\[ f'(z + c)^2 + P(z)^2 f(qz)^2 = Q(z), \quad (1.8) \]

and obtain the following result.

**Theorem 1.2** If \( |q| > 1 \), then the entire solution of (1.8) should be a polynomial. If there exist a finite order transcendental entire solution \( f \) of (1.8), then \( \rho(f) = 1 + \deg P(z) \) and \( |q| = 1 \).

**Example 4** Function \( f(z) = \sin z \) is an entire solution of \( f'(z + c)^2 + f(qz)^2 = 1 \), where \( c = \pi \) and \( q = -1 \).
Finally, we consider other $q$-difference equation
\[ f(z + c)^2 + P(z)^2 f(qz)^2 = Q(z). \] (1.9)

**Theorem 1.3** If $|q| > 1$, then the entire solution $f(z)$ of (1.9) should be a polynomial. If $P(z)^2 = 1$ in (1.9), the following example shows that we can not give the precise expression of finite order entire solution and the order of $f(z)$ does not satisfy $\rho(f) = 1 + \deg P(z)$ and $|q| = 1$.

**Example 5** [11] If $q = -1$, $c = \frac{\pi}{2}$, thus $f(z) = \sin z$ satisfies $f(z + \frac{\pi}{2})^2 + f(-z)^2 = 1$. If $q = \frac{1+i\sqrt{3}}{2}$, $c = \frac{1-i\sqrt{3}}{2}$, and $p(z) = \frac{1}{3}z^3 + z^2 + z + \frac{4}{3}i\pi + \frac{1}{3} + k\pi$, thus
\[ p(z + c) + p(qz) = \frac{3i\pi}{2} + 2ki\pi \]
and $k$ is an integer. Thus
\[ f(z) = \frac{e^{p(z - \frac{1-i\sqrt{3}}{2})} - e^{-p(z - \frac{1+i\sqrt{3}}{2})}}{2} \]
satisfies
\[ f(z + \frac{1-i\sqrt{3}}{2})^2 + f(\frac{1+i\sqrt{3}}{2}z)^2 = 1. \]

**Remark 1** The proofs of Theorem 1.2 and Theorem 1.3 are similar as the proof of Theorem 1.1. Hence we will not give the details here.

### 2 Some Lemmas

For the proofs of Theorems 1.1, 1.2 and 1.3, we need the following results.

**Lemma 2.1** [3, Lemma 3.1] Let $\Phi : (1, \infty) \to (0, \infty)$ be a monotone increasing function, and let $f$ be a nonconstant meromorphic function. If for some real constant $\alpha \in (0, 1)$, there exist real constants $K_1 > 0$ and $K_2 \geq 1$ such that
\[ T(r, f) \leq K_1 \Phi(\alpha r) + K_2 T(\alpha r, f) + S(\alpha r, f), \]
then
\[ \rho(f) \leq \frac{\log K_2}{\log \alpha} + \limsup_{r \to \infty} \frac{\log \Phi(r)}{\log r}. \]

**Lemma 2.2** [11, Lemma 2.15] Let $p(z)$ be a non-zero polynomial with degree $n$. If $p(qz) - p(z)$ is a constant, then $q^n = 1$ and $p(qz) = p(z)$. If $p(qz) + p(z)$ is a constant, then $q^n = -1$ and $p(qz) + p(z) = \pm 2a_0$, where $a_0$ is the constant term of $p(z)$.

**Lemma 2.3** [2, Theorem 2.1] Let $f(z)$ be transcendental meromorphic function of finite order $\rho$. Then for any $\varepsilon > 0$, we have
\[ T(r, f(z + c)) = T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r) = T(r, f) + S(r, f). \] (2.1)
Lemma 2.4 [16, Theorem 1.62] Let \( f_j(z) \) be meromorphic functions, \( f_k(z) \) \((k = 1, 2, \cdots, n - 1)\) be not constants, satisfying \( \sum_{j=1}^{n} f_j = 1 \) and \( n \geq 3 \). If \( f_n(z) \not\equiv 0 \) and

\[
\sum_{j=1}^{n} N(r, \frac{1}{f_j}) + (n-1)\sum_{j=1}^{n} N(r, f_j) < (\lambda + o(1))T(r, f_k),
\]

where \( \lambda < 1 \) and \( k = 1, 2, \cdots, n - 1 \), then \( f_n(z) \equiv 1 \).

3 Proof of Theorem 1.1

If \(|q| > 1\) and \( f(z) \) is an entire solution of (1.7), we use the observation (see [1]) that

\[
T(r, f(qz)) = T(|q|r, f(z)) + O(1)
\]

holds for any meromorphic function \( f \) and any constant \( q \). If \( f(z) \) is a transcendental entire function, then from (1.7) and Valiron-Mohon’ko theorem, we have

\[
T(|q|r, f(z)) + O(1) \leq T(r, f(qz)) + S(r, f) \leq T(r, f(z)) + S(r, f).
\]

Let \( \alpha = \frac{1}{|q|} \) and \(|q| > 1\). Then we have

\[
T(|q|\alpha r, f(z)) \leq T(\alpha r, f(z)) + S(\alpha r, f(z)).
\]

Hence, we have \( T(r, f(z)) \leq T(\alpha r, f(z)) + S(\alpha r, f(z)) \). From Lemma 2.1, we have \( \rho(f) = 0 \). Combining Hadamard factorization theorem, we have \( f'(z) + iP(z)f(qz) = Q_1(z) \) and \( f'(z) - iP(z)f(qz) = Q_2(z) \), thus \( f'(z) = \frac{Q_1(z) + Q_2(z)}{2} \) is a polynomial, which is a contradiction with \( f(z) \) is a transcendental entire function. Thus \( f(z) \) should be a polynomial.

Assume that \( f(z) \) is a transcendental entire solution of (1.7) with finite order, then

\[
[f'(z) + iP(z)f(qz)][f'(z) - iP(z)f(qz)] = Q(z). \tag{3.1}
\]

Thus both \( f'(z) + iP(z)f(qz) \) and \( f'(z) - iP(z)f(qz) \) have finitely many zeros. Combining (3.1) with the Hadamard factorization theorem, we assume that

\[
f'(z) + iP(z)f(qz) = Q_1(z)e^{h(z)}
\]

and

\[
f'(z) - iP(z)f(qz) = Q_2(z)e^{-h(z)},
\]

where \( h(z) \) is a non-constant polynomial provided that \( f(z) \) is of finite order transcendental and \( Q_1(z)Q_2(z) = Q(z) \), where \( Q_1(z), Q_2(z) \) are non-zero polynomials. Thus we have

\[
f'(z) = \frac{Q_1(z)e^{h(z)} + Q_2(z)e^{-h(z)}}{2} \tag{3.2}
\]

and

\[
f(qz) = \frac{Q_1(z)e^{h(z)} - Q_2(z)e^{-h(z)}}{2iP(z)}. \tag{3.3}
\]
From (3.2), we have
\[ f'(qz) = \frac{Q_1(qz)e^{h(qz)} + Q_2(qz)e^{-h(qz)}}{2}. \] (3.4)

Taking first derivative of (3.3), we have
\[ f'(qz) = \frac{A(z)e^{h(z)} - B(z)e^{-h(z)}}{2iqP(z)^2}, \] (3.5)

where
\[ A(z) = P(z)Q_1'(z) + Q_1(z)[P(z)h'(z) - P'(z)] \] (3.6)

and
\[ B(z) = P(z)Q_2'(z) - Q_2(z)[P(z)h'(z) + P'(z)]. \] (3.7)

From (3.4) and (3.5), we have
\[ \frac{A(z)e^{h(qz)+h(z)}}{iqP(z)^2Q_2(qz)} - \frac{B(z)e^{h(qz)-h(z)}}{iqP(z)^2Q_2(qz)} - \frac{Q_1(qz)}{Q_2(qz)}e^{2h(qz)} \equiv 1. \] (3.8)

Obviously, if \( h(qz) \) is a constant, then \( h(z) \) is a constant, thus \( f(z) \) should be a polynomial. If \( h(qz) \) is a non-constant entire function, then \( h(qz) - h(z) \) and \( h(qz) + h(z) \) are not constants simultaneously. The following, we will discuss two cases.

**Case 1** If \( h(qz) - h(z) \) is not a constant, from Lemma 2.4, we know that
\[ \frac{A(z)e^{h(qz)+h(z)}}{iqP(z)^2Q_2(qz)} \equiv 1. \] (3.9)

Since \( f(z) \) is a finite order entire solution, then \( h(z) \) should satisfies \( h(z) = a_nz^n + \cdots + a_0 \) is a non-constant polynomial, thus \( |q| = 1 \) follows for avoiding a contradiction. From Lemma 2.2, we have \( h(qz) + h(z) = 2a_0 \). Hence, we have
\[ A(z) = iqP(z)^2Q_2(qz)e^{-2a_0}. \] (3.10)

In addition, from (3.8), we also get
\[ \frac{B(z)e^{h(qz)-h(z)}}{iqP(z)^2Q_2(qz)} + \frac{Q_1(qz)}{Q_2(qz)}e^{2h(qz)} \equiv 0, \] (3.11)

which implies that
\[ B(z) = -iqQ_1(qz)P(z)^2e^{2a_0}. \] (3.12)

Thus
\[ A(z)B(z) = q^2P(z)^4Q(qz). \] (3.13)

Substitute (3.6) and (3.7) into (3.13), we have
\[ \{P(z)Q_1'(z) + Q_1(z)[P(z)h'(z) - P'(z)]\} \{P(z)Q_2'(z) - Q_2(z)[P(z)h'(z) + P'(z)]\} = q^2P(z)^4Q(qz). \] (3.14)
Since $f(z)$ is a finite order entire solution, by comparing with the degree of both hand side of (3.14), we have

$$\deg(h(z)) = 1 + \deg P(z).$$

It implies that $\rho(f) = 1 + \deg P(z)$.

**Case 2** If $h(qz) + h(z)$ is not a constant, from Lemma 2.4, we know that

$$-B(z)e^{h(qz)-h(z)} \equiv 1.$$ 

Hence $|q| = 1$ follows for avoiding a contradiction. Assume that $h(z) = a_n z^n + \cdots + a_0$, thus $h(qz) = h(z)$. Hence we have

$$-B(z) = iqP(z)^2Q_2(qz).$$

In addition, from (3.8), we also get

$$A(z) = iqQ_1(qz)P(z)^2.$$  \hspace{1cm} (3.16)

Thus, similar as the above, we also get $\rho(f) = 1 + \deg P(z)$.

**References**


费马 $q$-差分微分方程整函数解的增长性研究

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摘要: 本文研究了费马 $q$-差分微分方程的整函数解的相关问题, 利用经典和差分的Nevanlinna理论和函数方程理论的研究方法, 获得了 $q$-差分微分方程整函数解增长性的几个结果。

关键词: $q$-差分微分方程, 整函数解, 有穷级

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