

ON PROJECTIVE RICCI FLAT KROPINA METRICS

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Abstract: In this paper, we study and characterize projective Ricci flat Kropina metrics. By using the formulas of S -curvature and Ricci curvature for Kropina metrics, we obtain the formula of the projective Ricci curvature for Kropina metrics. Based on this, we obtain the necessary and sufficient conditions for Kropina metrics to be projective Ricci flat metrics. Further, as a natural application, we study and characterize projective Ricci flat Kropina metrics defined by a Riemannian metric and a Killing 1-form of constant length. We also characterize projective Ricci flat Kropina metrics with isotropic S -curvature. In this case, the Kropina metrics are Ricci flat metrics.

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1 Introduction

(α, β) -metrics form a special and very important class of Finsler metrics which can be expressed in the form $F = \alpha\phi(\frac{\beta}{\alpha})$, where α is a Riemannian metric and β is a 1-form and $\phi = \phi(s)$ is a C^∞ positive function on an open interval. In particular, when $\phi = 1 + s$, the Finsler metric $F = \alpha + \beta$ is called Randers metric. When $\phi = \frac{1}{s}$, the Finsler metric $F = \frac{\alpha^2}{\beta}$ is called Kropina metric. Randers metrics and Kropina metrics are both C -reducible. However, Randers metrics are regular Finsler metrics and Kropina metrics are Finsler metrics with singularity. Kropina metrics were first introduced by Berwald when he studied the two dimensional Finsler spaces with rectilinear extremal and were investigated by Kropina (see [4, 5]). Kropina metrics seem to be among the simplest nontrivial Finsler metrics with many interesting application in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics (see [4, 5, 7]). Recently, some geometers found some interesting and important geometric properties of Kropina metrics (see [10–12]).

The Ricci curvature in Finsler geometry is the natural extension of the Ricci curvature in Riemannian geometry and plays an important role in Finsler geometry. A Finsler metric

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Biography: Cheng Xinyue (1958–), male, born at Chongqing, professor, major in differential geometry and its applications.

$F = F(x, y)$ on an n -dimensional manifold M is called an Einstein metric if it satisfies the following equation on the Ricci curvature **Ric**

$$\mathbf{Ric}(x, y) = (n - 1)\sigma F^2(x, y). \quad (1.1)$$

where $\sigma = \sigma(x)$ is a scalar function on M . In particular, a Finsler metric F is called a Ricci flat metric if F satisfies (1.1) with $\sigma = 0$, that is, **Ric** = 0.

The S -curvature $\mathbf{S} = \mathbf{S}(x, y)$ is an important non-Riemannian quantity in Finsler geometry which was first introduced by Shen when he studied volume comparison in Riemann-Finsler geometry (see [8]). Shen proved that the S -curvature and the Ricci curvature determine the local behavior of the Busemann-Hausdorff measure of small metric balls around a point (see [9]). He also established a volume comparison theorem for the volume of metric balls under a lower Ricci curvature bound and a lower S -curvature bound and generalized Bishop-Gromov volume comparison theorem in the Riemannian case (see [9]). Recent study shows that the S -curvature plays a very important role in Finsler geometry (see [2, 9]). The Finsler metric F is said to be of isotropic S -curvature if $\mathbf{S}(x, y) = (n + 1)cF(x, y)$, where $c = c(x)$ is a scalar function on M . Further, if $c(x) = \text{constant}$, then F is said to be of constant S -curvature.

It is natural to consider the geometric quantities defined by Ricci curvature and S -curvature. Recently, Shen defined the concept of projective Ricci curvature in Finsler geometry. Concretely, for a Finsler metric F on an n -dimensional manifold M , the projective Ricci curvature **PRic** is defined by

$$\mathbf{PRic} = \mathbf{Ric} + (n - 1) \{ \bar{\mathbf{S}}_{|m} y^m + \bar{\mathbf{S}}^2 \}, \quad (1.2)$$

where $\bar{\mathbf{S}} := \frac{1}{n+1}\mathbf{S}$ and “ $|$ ” denotes the horizontal covariant derivative with respect to Berwald connection or Chern connection of F . We can easily rewrite the projective Ricci curvature as follows

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1} \mathbf{S}_{|m} y^m + \frac{n-1}{(n+1)^2} \mathbf{S}^2. \quad (1.3)$$

It is easy to show that, if two Finsler metrics are pointwise projectively related Finsler metrics on a manifold with a fixed volume form, then their projective Ricci curvature are equal. In other words, the projective Ricci curvature is projective invariant with respect to a fixed volume form.

On the other hand, the projective Ricci curvature is actually a kind of weighted Ricci curvatures. See [6] and the definition of S -curvature in Section 2. We call a Finsler metric F the projective Ricci flat metric if F satisfies **PRic** = 0. In [1], the authors characterized projective Ricci flat Randers metrics.

To state our main results, let us introduce some common notations for Kropina metrics. Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n -dimensional manifold M . Put

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where “;” denotes the covariant derivative with respect to the Levi-Civita connection of α . Further, put

$$\begin{aligned} r^i_j &:= a^{im}r_{mj}, & s^i_j &:= a^{im}s_{mj}, & r_j &:= b^mr_{mj}, & s_j &:= b^ms_{mj}, \\ q_{ij} &:= r_{im}s^m_j, & t_{ij} &:= s_{im}s^m_j, & q_j &:= b^iq_{ij} = r_ms^m_j, & t_j &:= b^it_{ij} = s_ms^m_j, \end{aligned}$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. We will denote $r_{i0} := r_{ij}y^j$, $s_{i0} := s_{ij}y^j$ and $r_{00} := r_{ij}y^iy^j$, $r_0 := r_iy^i$, $s_0 := s_iy^i$, etc..

In this paper, by using Busemann-Hausdorff volume form, we will derive firstly the formula for the projective Ricci curvature of a Kropina metric in Section 3. Based on this, we can prove the following main theorem.

Theorem 1.1 Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n -dimensional manifold M . Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations

$${}^\alpha\mathbf{Ric} = b^{-4}\lambda(x)\alpha^2 - (n-2)b^{-4}\left[-(r_0+s_0)^2 + b^2s_{0;0} + b^2r_{0;0}\right] \quad (1.4)$$

$$\begin{aligned} &-2b^{-2}q_{00} + nb^{-4}rr_{00} - b^{-2}b^mr_{00;m} - b^{-2}r_{00}r^m_m, \\ (n-1)b^2t_0 + (n-3)b^2q_0 - nrs_0 + b^2b^ms_{0;m} + b^2s_0r^m_m - b^4s^m_{0;m} + \lambda(x)\beta &= 0, \end{aligned} \quad (1.5)$$

where $b := \|\beta_x\|_\alpha$ denotes the length of β with respect to α and $\lambda(x) = \frac{1}{2}(n-1)b^2t^m_m - (n-3)r_ms^m + b^2s^m_{;m}$.

By the definition, the 1-form β is said to be a Killing form on Riemannian manifold (M, α) if $r_{ij} = 0$. The 1-form β is said to be a constant length Killing 1-form if it is a Killing form and has constant length with respect to α , equivalently $r_{ij} = 0$ and $s_i = 0$.

For a Kropina metric $F = \alpha^2/\beta$, if β is a constant length Killing 1-form with respect to α , we have the following theorem.

Theorem 1.2 Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric with constant length Killing form β on an n -dimensional manifold M . Then F is a projective Ricci flat metric if and only if there exists a function $\lambda = \lambda(x)$ such that α is an Einstein metric ${}^\alpha\mathbf{Ric} = \lambda\alpha^2$ and β satisfies the following equations

$$\lambda\beta - s^m_{0;m} = 0, \quad (1.6)$$

$$s^j_ms^m_j = 0. \quad (1.7)$$

For 1-form $\beta = b_i(x)y^i$ on M , we say that β is a conformal form with respect to α if it satisfies $b_{i;j} + b_{j;i} = \rho a_{ij}$, where $\rho = \rho(x)$ is a function on M and “;” is the horizontal covariant derivative with respect to α . If $\rho = 0$, β is just a Killing form with respect to α .

In fact, for a Kropina metric F , the following four conditions are equivalent (see [10]):

- (a) F has an isotropic S -curvature, $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a function on M ;
- (b) $r_{00} = k(x)\alpha^2$, where $k = k(x)$ is a function on M ;
- (c) $\mathbf{S} = 0$;
- (d) β is a conformal form with respect to α .

So we can get the following conclusion.

Corollary 1.3 Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n -dimensional manifold M . Assume that F is of isotropic S -curvature, i.e., $\mathbf{S} = (n+1)cF$. Then F is a projective Ricci flat metric if and only if F is Ricci flat metric.

2 Preliminaries

Let F be a Finsler metric on an n -dimensional manifold M and G^i be the geodesic coefficients of F , which are defined by

$$G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}. \quad (2.1)$$

For any $x \in M$ and $y \in T_x M \setminus \{0\}$, the Riemann curvature $\mathbf{R}_y := R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m} \frac{\partial G^m}{\partial y^k}. \quad (2.2)$$

The Ricci curvature is the trace of the Riemann curvature, which is defined by $\mathbf{Ric} = R^m_m$.

For a Finsler metric $F = F(x, y)$ on an n -dimensional manifold M , define the Busemann-Hausdorff volume form of F by $dV_F = \sigma_F(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$, where

$$\sigma_F(x) = \frac{\text{Vol}(\mathbf{B}^n(1))}{\text{Vol}\{y \in \mathbf{R}^n \mid F(x, y) < 1\}}$$

and Vol denotes the Euclidean volume and $\mathbf{B}^n(1)$ denotes the unit ball in \mathbf{R}^n . Then the S -curvature \mathbf{S} of F is given by

$$\mathbf{S}(x, y) = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial (\log \sigma_F)}{\partial x^m}. \quad (2.3)$$

The S -curvature \mathbf{S} measures the average rate of change of $(T_x M, F_x)$ in the direction $y \in T_x M$. It is known that $\mathbf{S} = 0$ for Berwald metrics.

(α, β) -metrics form an important class of Finsler metrics which can be expressed in the form $F = \alpha \phi\left(\frac{\beta}{\alpha}\right)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_\alpha < b_0$ on a manifold. It was proved that $F = \alpha \phi(\beta/\alpha)$ is a positive definite Finsler metric if and only if $\phi = \phi(s)$ is a positive C^∞ positive function on $(-b_0, b_0)$ satisfying the following condition (see [2])

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0. \quad (2.4)$$

Randers metric $F = \alpha + \beta$ is just the (α, β) -metric with $\phi = 1 + s$. When $\phi = 1/s$, the metric $F = \frac{\alpha^2}{\beta}$ is just the Kropina metric. It is easy to see that a Kropina metric $F = \frac{\alpha^2}{\beta}$ is not a regular Finsler metric for $|s| < b$, but it is regular if $s > 0$. In this paper, we study regular Kropina metrics. Hence, we will always restrict our consideration to the domain where $\beta = b_i(x)y^i > 0$ so that $s > 0$.

Let $G^i(x, y)$ and ${}^\alpha G^i(x, y)$ denote the geodesic coefficients of an (α, β) -metric $F = \alpha \phi(\beta/\alpha)$ and α , respectively. We can express the geodesic coefficients G^i as follows (see [2]).

$$G^i = {}^\alpha G^i + \alpha Q s^i_0 + \Theta(-2\alpha Q s_0 + r_{00}) \frac{y^i}{\alpha} + \Psi(-2\alpha Q s_0 + r_{00}) b^i, \quad (2.5)$$

where

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &:= \frac{\phi'(\phi - s\phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']} - s\Psi, \\ \Psi &:= \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']}. \end{aligned}$$

In particular, for a Kropina metric $F = \frac{\alpha^2}{\beta}$, i.e., $\phi(s) = \frac{1}{s}$, it follows from (2.5) that

$$G^i = {}^\alpha G^i - \frac{F}{2} s^i{}_0 - \frac{1}{b^2} \left(s_0 + \frac{1}{F} r_{00} \right) y^i + \frac{1}{2b^2} (F s_0 + r_{00}) b^i. \quad (2.6)$$

Further, the Ricci curvature of $F = \frac{\alpha^2}{\beta}$ is given by (see [12])

$$\begin{aligned} \mathbf{Ric} &= {}^\alpha \mathbf{Ric} + \frac{3(n-1)}{b^4 F^2} r_{00}^2 + \frac{n-1}{F b^4} (2r_{00}s_0 - 4r_{00}r_0 - 4F r_0 s_0 - F s_0^2) \\ &\quad + \frac{n-1}{b^2 F} (r_{00;0} + F s_{0;0} + F^2 s_m s_0^m) + \frac{1}{b^4} [(r_0 + s_0)^2 - r(F s_0 + r_{00})] \\ &\quad + \frac{1}{b^2} \left\{ F b^m s_{0;m} + b^m r_{00;m} - r_{0;0} - s_{0;0} + (r_{00} + F s_0) r_m^m + 2n r_{0m} s_0^m \right. \\ &\quad \left. - F r_m s_0^m - F r_{0m} s^m - \frac{F^2}{2} s^m s_m \right\} - F s_{0;m}^m - \frac{F^2}{4} s_m^j s_j^m. \end{aligned} \quad (2.7)$$

Note that $\sigma_F(x) = \left(\frac{2}{b}\right)^n \sqrt{\det(a_{ij})}$ for Kropina metric. From this and by (2.3) and (2.6), one obtains the S -curvature of the Kropina metric $F = \frac{\alpha^2}{\beta}$,

$$\mathbf{S}(x, y) = \frac{n+1}{F b^2} (F r_0 - r_{00}), \quad (2.8)$$

which is proved by Zhang and Shen (see Proposition 5.1 in [12]).

3 Projective Ricci Flat Kropina Metrics

In this section, we will first derive a formula for the projective Ricci curvature of a Kropina metric. Then we will characterize projective Ricci flat Kropina metrics. By (1.3), the projective Ricci curvature is given by

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1} \mathbf{S}_{|m} y^m + \frac{n-1}{(n+1)^2} \mathbf{S}^2. \quad (3.1)$$

By (2.6), we have

$$\begin{aligned} G_m^i &= {}^\alpha G_m^i - \frac{F y^m}{2} s^i{}_0 - \frac{F}{2} s^i{}_m - \frac{1}{b^2} \left(s_m - \frac{F y^m}{F^2} r_{00} + \frac{2r_{0m}}{F} \right) y^i - \frac{1}{b^2} \left(s_0 + \frac{1}{F} r_{00} \right) \delta_m^i \\ &\quad + \frac{b^i}{2b^2} (F y^m s_0 + F s_m + 2r_{0m}). \end{aligned}$$

Thus

$$\begin{aligned}
 \mathbf{S}_{|m}y^m &= y^m \frac{\partial \mathbf{S}}{\partial x^m} - G_m^l y^m \frac{\partial \mathbf{S}}{\partial y^l} \\
 &= \mathbf{S}_{;m}y^m - \left[-Fs_0^m - \frac{2}{b^2} \left(s_0 + \frac{r_{00}}{F} \right) y^l + \frac{b^m}{b^2} (Fs_0 + r_{00}) \right] \frac{\partial \mathbf{S}}{\partial y^m} \\
 &= \mathbf{S}_{;m}y^m + Fs_0^m \mathbf{S}_{y^m} - \frac{b^m}{b^2} (Fs_0 + r_{00}) \mathbf{S}_{y^m} + \frac{2}{b^2} \left(s_0 + \frac{r_{00}}{F} \right) \mathbf{S}. \quad (3.2)
 \end{aligned}$$

From (2.8), we obtain

$$\mathbf{S}_{;m}y^m = -\frac{2(n+1)}{b^4} \left(r_0^2 - \frac{r_0 r_{00}}{F} + s_0 r_0 - \frac{s_0 r_{00}}{F} \right) + \frac{n+1}{b^2} \left(r_{0;0} - \frac{r_{00;0}}{F} - \frac{r_{00}^2}{\beta F} \right), \quad (3.3)$$

$$\begin{aligned}
 \left[Fs_0^m - \frac{b^m}{b^2} (Fs_0 + r_{00}) \right] \mathbf{S}_{y^m} &= \frac{n+1}{b^2} \left[Fq_0 - 2q_{00} - \frac{r}{b^2} (Fs_0 + r_{00}) + \frac{2r_0}{b^2 F} (Fs_0 + r_{00}) \right. \\
 &\quad \left. - \frac{2r_{00}}{b^2 F^2} (Fs_0 + r_{00}) + \frac{r_{00}^2}{\beta F} \right], \quad (3.4)
 \end{aligned}$$

$$\frac{2}{b^2} \left(s_0 + \frac{r_{00}}{F} \right) \mathbf{S} = \frac{2(n+1)}{b^4} \left(s_0 r_0 - \frac{s_0 r_{00}}{F} + \frac{r_0 r_{00}}{F} - \frac{r_{00}^2}{F^2} \right). \quad (3.5)$$

Here, we have used $s_m s_0^m = t_0$, $r_m s_0^m = r_{0m} s_0^m = q_0$, $r_{0m} s_0^m = q_{00}$. Plugging (3.3), (3.4) and (3.5) into (3.2) yields

$$\begin{aligned}
 \frac{n-1}{n+1} \mathbf{S}_{|m}y^m &= (n-1) \left[-\frac{2r_0^2}{b^4} + \frac{6r_0 r_{00}}{b^4 F} + \frac{r_{0;0}}{b^2} - \frac{r_{00;0}}{b^2 F} + \frac{Fq_0}{b^2} - \frac{2q_{00}}{b^2} - \frac{r(Fs_0 + r_{00})}{b^4} \right. \\
 &\quad \left. + \frac{2s_0 r_0}{b^4} - \frac{2r_{00} s_0}{b^4 F} - \frac{4r_{00}^2}{b^4 F^2} \right]. \quad (3.6)
 \end{aligned}$$

Further, we have

$$\frac{n-1}{(n+1)^2} \mathbf{S}^2 = (n-1) \left(\frac{r_0^2}{b^4} + \frac{r_{00}^2}{b^4 F^2} - \frac{2r_0 r_{00}}{b^4 F} \right). \quad (3.7)$$

Substituting (2.7), (3.6) and (3.7) into (3.1), we obtain the formula for projective Ricci curvature of Kropina metric $F = \frac{\alpha^2}{\beta}$ as follows

$$\begin{aligned}
 \mathbf{PRic} &= {}^\alpha \mathbf{Ric} + (n-2) \left[-\frac{(r_0 + s_0)^2}{b^4} + \frac{s_{0;0}}{b^2} + \frac{r_{0;0}}{b^2} \right] + (n-1) \frac{Ft_0}{b^2} + (n-3) \frac{Fq_0}{b^2} \\
 &\quad + \frac{2q_{00}}{b^2} + \frac{1}{b^2} Fb^m s_{0;m} + \frac{1}{b^2} b^m r_{00;m} + \frac{1}{b^2} (r_{00} + Fs_0) r_m^m - \frac{F^2 s^m s_m}{2b^2} \\
 &\quad - Fs_{0;m}^m - \frac{F^2}{4} t_m^m. \quad (3.8)
 \end{aligned}$$

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1 The proof of the sufficient condition in Theorem 1.1 is trivial. We will mainly prove the necessary condition in Theorem 1.1.

Assume that $\mathbf{PRic} = 0$, which is equivalent to $4b^4\beta^2 \times \mathbf{PRic} = 0$. By (3.8), we obtain the following

$$\begin{aligned} 4b^4\beta^2 \times \left\{ {}^\alpha\mathbf{Ric} + (n-2) \left[-\frac{(r_0 + s_0)^2}{b^4} + \frac{s_{0;0}}{b^2} + \frac{r_{0;0}}{b^2} \right] + (n-1) \frac{Ft_0}{b^2} + (n-3) \frac{Fq_0}{b^2} \right. \\ \left. + \frac{2q_{00}}{b^2} + \frac{1}{b^2} Fb^m s_{0;m} + \frac{1}{b^2} b^m r_{00;m} + \frac{1}{b^2} (r_{00} + Fs_0) r_m^m - \frac{F^2 s^m s_m}{2b^2} - Fs_{0;m}^m \right. \\ \left. - \frac{F^2}{4} t_m^m \right\} = 0. \end{aligned} \quad (3.9)$$

The equation (3.9) is equivalent to the following equation

$$\Xi_4 \alpha^4 + \Xi_2 \alpha^2 + \Xi_0 = 0, \quad (3.10)$$

where

$$\Xi_4 = -2b^2 s^m s_m - b^4 t_m^m, \quad (3.11)$$

$$\begin{aligned} \Xi_2 = [4(n-1)b^2 t_0 + 4(n-3)b^2 q_0 - 4nr s_0 + 4b^2 b^m s_{0;m} + 4b^2 s_0 r_m^m \\ - 4b^4 s_{0;m}^m] \beta, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \Xi_0 = \left\{ b^{4\alpha} \mathbf{Ric} + (n-2) \left[-(r_0 + s_0)^2 + b^2 s_{0;0} + b^2 r_{0;0} \right] + 2b^2 q_{00} \right. \\ \left. - nrr_{00} + b^2 b^m r_{00;m} + b^2 r_{00} r_m^m \right\} 4\beta^2. \end{aligned} \quad (3.13)$$

Rewrite (3.10) as

$$(\Xi_4 \alpha^2 + \Xi_2) \alpha^2 + \Xi_0 = 0. \quad (3.14)$$

Because α^2 and β^2 are relatively prime polynomials in y , by (3.14) and the definition of Ξ_0 , we know that there exist a scalar function $\lambda(x)$ such that

$$\begin{aligned} b^{4\alpha} \mathbf{Ric} + (n-2) \left[-(r_0 + s_0)^2 + b^2 s_{0;0} + b^2 r_{0;0} \right] \\ + 2b^2 q_{00} - nrr_{00} + b^2 b^m r_{00;m} + b^2 r_{00} r_m^m = \lambda(x) \alpha^2. \end{aligned} \quad (3.15)$$

Then (3.9) can be simplified as

$$\begin{aligned} 0 = [4(n-1)b^2 t_0 + 4(n-3)b^2 q_0 - 4nr s_0 + 4b^2 b^m s_{0;m} + 4b^2 s_0 r_m^m - 4b^4 s_{0;m}^m \\ + 4\lambda(x) \beta] \beta - b^2 (2s_m s^m + b^2 t_m^m) \alpha^2. \end{aligned} \quad (3.16)$$

Since α^2 can't be divided by β , we see that (3.16) is equivalent to the following equations

$$\begin{aligned} 0 = (n-1)b^2 t_0 + (n-3)b^2 q_0 - nr s_0 + b^2 b^m s_{0;m} + b^2 s_0 r_m^m - b^4 s_{0;m}^m \\ + \lambda(x) \beta, \end{aligned} \quad (3.17)$$

$$0 = 2s_m s^m + b^2 t_m^m. \quad (3.18)$$

First, differentiating both sides of (3.17) with respect to y^i yields

$$(n-1)b^2 t_i + (n-3)b^2 q_i - nr s_i + b^2 b^m s_{i;m} + b^2 s_i r_m^m - b^4 s_{i;m}^m + \lambda(x) b_i = 0. \quad (3.19)$$

Contracting (3.19) with b^i gives

$$0 = (n-1)b^2 s^m s_m + (n-3)b^2 r_m s^m - b^4 s^m_{;m} + b^2 \lambda(x). \quad (3.20)$$

Removing the factor b^2 from (3.20), we obtain

$$\lambda(x) = -(n-1)s^m s_m - (n-3)r_m s^m + b^2 s^m_{;m}. \quad (3.21)$$

By (3.18), we obtain

$$s_m s^m = -\frac{1}{2}b^2 t^m_m, \quad (3.22)$$

then rewrite (3.21) as following

$$\lambda(x) = \frac{1}{2}(n-1)t^m_m - (n-3)r_m s^m + b^2 s^m_{;m}. \quad (3.23)$$

This completes the proof of Theorem 1.1.

4 Applications

In this section, we will firstly study projective Ricci flat Kropina metrics with constant length Killing 1-form β and prove Theorem 1.2. Let $F = \frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric with constant length Killing 1-form β on an n -dimensional manifold M , that is $r_{ij} = 0$, $s_j = 0$. In this case, equation (3.8) simply as follows

$$\mathbf{PRic} = {}^\alpha \mathbf{Ric} - F s^m_{0;m} - \frac{F^2}{4} t^m_m. \quad (4.1)$$

Assume that $\mathbf{PRic} = 0$, which is equivalent to $4\beta^2 \times \mathbf{PRic} = 0$. By (4.1), we obtain the following

$$4{}^\alpha \mathbf{Ric} \beta^2 - 4s^m_{0;m} \alpha^2 \beta - t^m_m \alpha^4 = 0. \quad (4.2)$$

Thus ${}^\alpha \mathbf{Ric}$ is divisible by α^2 , that is, there exists a function $\lambda(x)$ such that

$${}^\alpha \mathbf{Ric} = \lambda \alpha^2. \quad (4.3)$$

Plugging (4.3) into (4.2) and dividing the common factor α^2 , we conclude that

$$4(\lambda\beta - s^m_{0;m})\beta - t^m_m \alpha^2 = 0. \quad (4.4)$$

Since α^2 can not be divided by β , we see that (4.4) is equivalent to the following equations

$$\lambda\beta - s^m_{0;m} = 0, \quad t^m_m = s^j_m s^m_j = 0.$$

This completes the proof of Theorem 1.2.

Now, let us consider projective Ricci flat Kropina metrics with isotropic S -curvature. As we mentioned in Section 1, a Kropina metric F is of isotropic S -curvature, $\mathbf{S} = (n+1)cF$, if and only if $\mathbf{S} = 0$. In this case, from the equation (1.3), we know that

$$\mathbf{PRic} = \mathbf{Ric}. \quad (4.5)$$

Hence, F is projective Ricci flat metric if and only if F is Ricci flat metric.

In [12], Zhang and Shen proved that every Einstein-Kropina metric $F = \frac{\alpha^2}{\beta}$ has vanishing S -curvature. In this case, $\mathbf{PRic} = \mathbf{Ric}$. They also have obtained the necessary and sufficient conditions for Kropina metrics to be Einstein metrics in [12].

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射影Ricci 平坦的Kropina 度量

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摘要: 本文研究和刻画了射影Ricci 平坦的Kropina度量. 利用Kropina度量的 S -曲率和Ricci曲率的公式, 得到了Kropina度量的射影Ricci曲率公式. 在此基础上得到了Kropina度量是射影Ricci 平坦度量的充分必要条件. 进一步, 作为自然的应用, 本文研究和刻画了由一个黎曼度量和一个具有常数长度的Killing 1-形式定义的射影Ricci 平坦的Kropina度量, 也刻画了具有迷向 S -曲率的射影Ricci 平坦的Kropina度量. 在这种情形下, Kropina度量是Ricci 平坦度量.

关键词: 芬斯勒度量; Kropina 度量; Ricci 曲率; S -曲率; 射影Ricci 曲率

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