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LIPSCHITZ TYPE SMOOTHNESS OF MULTILINEAR FRACTIONAL INTEGRAL ON VARIABLE EXPONENTS SPACES

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Abstract: In this paper, we study the boundedness of multilinear fractional integral operators on variable exponent spaces. It is obtained that these operators are both bounded from strong and weak Lebesgue spaces with variable exponent spaces into Lipschitz type spaces with variable exponent, which gives some new results for previous published papers. A simple way is obtained that is colsely linked with a class of fractional integral operator.

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1 Introduction

Let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the m-fold product space of $\mathbb{R}^n$, the multilinear fractional integrals on $\mathbb{R}^n$ are defined by

$$I_{\beta,m}(f_1,\cdots,f_m)(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{|(x-y_1,\cdots,x-y_m)|^{mn-\beta}} dy_1\cdots dy_m,$$

where $0<\beta<mn, |(x-y_1,\cdots,x-y_m)| = \sqrt{|x-y_1|^2+\cdots+|x-y_m|^2}$.

When $m=1$, $I_{\beta,m}=I_\beta$, where $I_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}dy$. The famous Hardy-Littlewood-Sobolev theorem tells us that the fractional integral operator I_β is a bounded operator from the usual Lebesgue spaces $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$ when $1<p_1<p_2<\infty$ and $\frac{1}{p_1}-\frac{1}{p_2}=\frac{\beta}{n}$. Kenig and Stein [1] as well as Grafakos and Kalton [2] considered the boundedness of a family of related multilinear fractional integrals. Lan [3] presented the boundedness of multilinear fractional integral operators on weak type Hardy spaces. Recently, Yasuo [4] considered the boundedness of multilinear fractional integral operators on Herz spaces.

It is well known that function spaces with variable exponents were intensively studied during the past 20 years, due to their applications to PDE with non-standard growth conditions and so on, we mention e.g. [5, 6]. A great deal of work was done to extend the theory of

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fractional integral on the classical Lebesgue spaces to the variable exponent case (see [7–9]). However, these articles do not consider the behavior of I_β when $p^+ > \frac{\beta}{n}$. Recently, Ramseyer, Salinas and Viviani [10] studied that Lipschitz type smoothness of fractional integral on variable exponent spaces, when $p^+ > \frac{\beta}{n}$. Hence, when $p^+ > \frac{\beta}{n}$, it will be an interesting problem whether we can establish the boundedness of multilinear fractional integral from Lebesgue spaces $L^{p(\cdot)}$ into Lipschitz-type spaces with variable exponents. The main purpose of this paper is to answer the above problem.

To meet the requirements in the next sections, here, the basic elements of the theory of the Lebsegue spaces with variable exponent are briefly presented.

Let $p(\cdot) : \Omega \to [1, \infty)$ be a measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ f \text{ is measurable} : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty \text{ for some constant } \lambda > 0 \right\}.$$

$L^{p(\cdot)}(\Omega)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1\}.$$

We denote $p^- := \operatorname{ess}\inf\limits_{x\in\Omega} p(x)$, $p^+ := \operatorname{ess}\sup\limits_{x\in\Omega} p(x)$.

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of measurable function $p(\cdot)$ on $\mathbb{R}^n$ with value in $[1, \infty)$ such that $1 < p_-(\mathbb{R}^n) \le p(\cdot) \le p_+(\mathbb{R}^n) < \infty$.

We say a function $p(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is locally log-Hölder continuous, if there exists a constant C such that

$$|p(x) - p(y)| \le \frac{C}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. If, for some $p(\infty) \in \mathbb{R}$ and $C > 0$, there holds $|p(x) - p(\infty)| \le \frac{C}{\log(e+|x|)}$ for all $x \in \mathbb{R}^n$, then we say $p(\cdot)$ is log-Hölder continuous at infinity.

The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for all those exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and log-Hölder continuous at infinity with $p(\infty) := \lim\limits_{|x|\to\infty} p(x)$. Moreover, we can easily show that $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ implies $p'(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

For brevity, C always means a positive constant independent of the main parameters and may change from one occurrence to another. $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $B_k = B(x_0, 2^k R)$, $A_k = B_k \setminus B_{k-1}$ and $\chi_{A_k} = \chi_k$ be the characteristic function of the set A_k for $k \in \mathbb{Z}$. $|S|$ denotes the Lebesgue measure of S. $f \sim g$ means $C^{-1}g \le f \le Cg$.

Definition 1.1 [10] Given an exponent function $p(\cdot)$ we say that a measurable function f belongs to $L^{p(\cdot),\infty}$ if there exists a constant C such that $\displaystyle\int_{\mathbb{R}^n} t^{p(x)} \chi_{\{|f|>t\}}(x)dx \le C$ for every $t > 0$.

It is not difficult to see that

$$[f]_{p(\cdot),\infty} = \inf \left\{ \lambda > 0 : \sup_{t>0} \int_{\mathbb{R}^n} \left(\frac{t}{\lambda}\right)^{p(x)} \chi_{\{|f|>t\}}(x)dx \le 1 \right\}$$

is a quasi-norm in $L^{p(\cdot),\infty}$.

Definition 1.2 [10] Given $0 < \beta < n$ and an exponent function $p(\cdot)$ with $1 < p^- \le p^+ < \infty$ we say that a locally integrable function f belongs to $\mathrm{Lip}_{\beta,p(\cdot)}$ if there exists a constant C such that

$$\frac{1}{|B|^{\frac{\beta}{n}}\|\chi_B\|_{p'(\cdot)}}\int_B |f - m_B f|dx \le C \tag{1.1}$$

for every ball $B \subset \mathbb{R}^n$ with $m_B f = \frac{1}{|B|}\displaystyle\int_B f$. The least constant C in (1.1) will be denoted by $\|f\|_{\mathrm{Lip}_{\beta,p(\cdot)}}$.

Remark It is easy to see that in definition the average can be replaced by a constant in the following sense

$$\frac{1}{2}\|f\|_{\mathrm{Lip}_{\beta,p(\cdot)}} \le \sup_{B\in\mathbb{R}^n}\inf_{c\in\mathbb{R}} \frac{1}{|B|^{\frac{\beta}{n}}\|\chi_B\|_{p'(\cdot)}}\int_B |f-c|dx \le \|f\|_{\mathrm{Lip}_{\beta,p(\cdot)}}.$$

In this paper, we consider the case of bilinear fractional integral.

Definition 1.3 [4]

$$\tilde{I}_\beta(f_1, f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\frac{1}{|(x-y_1, x-y_1)|^{2n-\beta}} - \frac{\chi_{\{|(y_1,y_2)|>1\}}}{|(y_1,y_2)|^{2n-\beta}}\right) f_1(y_1)f_2(y_2)dy_1dy_2,$$

where $0 < \beta < 2n$.

Now it is in this position to state our results.

Theorem 1.1 Let $0 < \beta < 2n$, $1 < p_i^- \le p_i^+ < \infty$ and $\frac{n}{p_1^+} + \frac{n}{p_2^+} < \beta < \frac{n}{p_1^+} + \frac{n}{p_2^+} + 1$. Suppose that $p_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ for $i = 1, 2$ and $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$, then $\tilde{I}_\beta(f_1, f_2)$ is bounded from $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ into $\mathrm{Lip}_{\beta,p(\cdot)}$.

Theorem 1.2 Let $0 < \beta < 2n$, $1 < p_i^- \le p_i^+ < \infty$ and $\frac{n}{p_1^+} + \frac{n}{p_2^+} < \beta < \frac{n}{p_1^+} + \frac{n}{p_2^+} + 1$. Suppose that $p_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ for $i = 1, 2$, there exists a positive $r_0 > 1$ such that $p_i(x) \le p_i(\infty)$ for $|x| > r_0$ and $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$. Then $\tilde{I}_\beta(f_1, f_2)$ is bounded from $L^{p_1(\cdot),\infty}(\mathbb{R}^n) \times L^{p_2(\cdot),\infty}(\mathbb{R}^n)$ into $\mathrm{Lip}_{\beta,p(\cdot)}$.

2 Lemmas

Lemma 2.1 [11] If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le r_p\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

where $r_p := 1 + 1/p^- - 1/p^+$.

Lemma 2.2 [10] Let $p(\cdot)$ be an exponent function in $\mathcal{P}^{\log}(\mathbb{R}^n)$ such that $1 < p^- \le p^+ < \infty$ and $p(x) \le p(\infty)$ for $|x| > r_0$ with $r_0 > 1$. Then there exists a positive constant C depending on r_0 and the constants associated $\mathcal{P}^{\log}(\mathbb{R}^n)$ such that $\displaystyle\int_B |f(x)|dx \le C[f]_{p(\cdot),\infty}\|\chi_B\|_{p'(\cdot)}$ for every ball B and $f \in L^{p(\cdot),\infty}$.

The following lemma see Corollary 4.5.9 in [12].

Lemma 2.3 Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then for every ball $B \subset \mathbb{R}^n$, we have

$$\|\chi_B\|_{p(\cdot)} \sim |B|^{\frac{1}{p(x)}}, \quad \text{if} \quad |B| \leq 2^n, \ x \in B$$

and

$$\|\chi_B\|_{p(\cdot)} \sim |B|^{\frac{1}{p(\infty)}}, \quad \text{if} \quad |B| \geq 1.$$

We remark that Lemma 2.4 were showed in [13] and we will give the proof of it.

Lemma 2.4 Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $x_2 \in 2B(x_1, r)$, then we have

$$\|\chi_{B(x_1,r)}\|_{p(\cdot)} \sim \|\chi_{B(x_2,r)}\|_{p(\cdot)}.$$

Proof We consider two cases, by Lemma 2.3.

Case 1 $|B| \geq 1$.

$$\|\chi_{B(x_1,r)}\|_{p(\cdot)} \sim |B(x_1,r)|^{\frac{1}{p(\infty)}} \sim |B(x_2,r)|^{\frac{1}{p(\infty)}} \sim \|\chi_{B(x_2,r)}\|_{p(\cdot)}.$$

Case 2 $|B| \leq 1$.

$$\begin{aligned}&\|\chi_{B(x_1,r)}\|_{p(\cdot)} \sim |B(x_1,r)|^{\frac{1}{p(x')}} = |B(x_1,r)|^{\frac{1}{p(x')}} |B(x_2,r)|^{-\frac{1}{p(x'')}} \|\chi_{B(x_2,r)}\|_{p(\cdot)} \\ &\sim r^{\frac{1}{p(x')} - \frac{1}{p(x'')}} \|\chi_{B(x_2,r)}\|_{p(\cdot)} \leq C \|\chi_{B(x_2,r)}\|_{p(\cdot)},\end{aligned}$$

where we denote that $x' \in B(x_1, r)$ and $x'' \in B(x_2, r)$.

Indeed, since $x_2 \in 2B(x_1,r), x' \in B(x_1,r)$ and $x'' \in B(x_2,r)$ we note that $|x' - x''| \leq 4r$, we make use of local-Hölder continuity of $p(x)$ and get,

$$\left|\frac{1}{p'(x')} - \frac{1}{p'(x'')}\right| \log \frac{1}{r} \leq \frac{\log \frac{1}{r}}{\log(e + \frac{1}{|x'-x''|})} \leq \frac{\log \frac{1}{r}}{\log(e + \frac{1}{4r})} \leq C.$$

Lemma 2.5 Let $p_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ for $i = 1, 2$ and $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$, then for every ball $B = B(x, r) \subset \mathbb{R}^n$, we have

$$\|\chi_B\|_{p(\cdot)} \sim \|\chi_B\|_{p_1(\cdot)} \|\chi_B\|_{p_2(\cdot)}, \tag{2.1}$$

$$\|\chi_B\|_{p'(\cdot)} \sim r^{-n} \|\chi_B\|_{p_1'(\cdot)} \|\chi_B\|_{p_2'(\cdot)}. \tag{2.2}$$

Proof We will give the proof of inequality (2.2), the argument for inequality (2.1) is similar, we omit the details here. We consider two cases, by Lemma 2.3.

Case 1 $|B| \leq 1$.

$$\|\chi_B\|_{p_1'(\cdot)} \|\chi_B\|_{p_2'(\cdot)} \sim |B|^{\frac{1}{p_1'(x)}} |B|^{\frac{1}{p_2'(x)}} \sim r^n |B|^{\frac{1}{p'(x)}} \sim r^n \|\chi_B\|_{p'(\cdot)}.$$

Case 2 $|B| \geq 1$.

$$\|\chi_B\|_{p_1'(\cdot)} \|\chi_B\|_{p_2'(\cdot)} \sim |B|^{\frac{1}{p_1'(\infty)}} |B|^{\frac{1}{p_2'(\infty)}} \sim r^n |B|^{\frac{1}{p'(\infty)}} \sim r^n \|\chi_B\|_{p'(\cdot)}.$$

Lemma 2.6 Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all balls B and all measurable subsets $S = B(x_0, r_0) \subset B = B(x_1, r_1)$,

$$\frac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \le C \left(\frac{|S|}{|B|}\right)^{1-\frac{1}{p^-}}, \tag{2.3}$$

$$\frac{\|\chi_B\|_{p'(\cdot)}}{\|\chi_S\|_{p'(\cdot)}} \le C \left(\frac{|B|}{|S|}\right)^{1-\frac{1}{p^+}}. \tag{2.4}$$

Remark We can easily show that inequality (2.4) implies $\|\chi_{2B}\|_{p'(\cdot)} \le C\|\chi_B\|_{p'(\cdot)}$.

Proof We will prove inequality (2.3), the argument for inequality (2.4) is similar, we omit the details here. We consider three cases, by Lemma 2.3.

(1) $|S| < |B| < 1$, $\dfrac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \sim \dfrac{|S|^{\frac{1}{p'(x_S)}}}{|B|^{\frac{1}{p'(x_S)}}} |B|^{\frac{1}{p'(x_S)} - \frac{1}{p'(x_B)}} \le C \left(\dfrac{|S|}{|B|}\right)^{\frac{1}{(p')^+}} = C \left(\dfrac{|S|}{|B|}\right)^{1-\frac{1}{p^-}}$,
where we denote that $x_S \in S$ and $x_B \in B$.

Indeed, since $|x_B - x_S| \le 2r_1$, we make use of local-Hölder continuity of $p'(x)$ and get

$$\left|\frac{1}{p'(x_S)} - \frac{1}{p'(x_B)}\right| \log \frac{1}{r_1} \le \frac{\log \frac{1}{r_1}}{\log(e + \frac{1}{|x_S - x_B|})} \le \frac{\log \frac{1}{r_1}}{\log(e + \frac{1}{2r_1})} \le C.$$

(2) $|S| < 1 < |B|$, $\dfrac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \sim \dfrac{|S|^{\frac{1}{p'(x_S)}}}{|B|^{\frac{1}{p'(\infty)}}} \le \left(\dfrac{|S|}{|B|}\right)^{\frac{1}{(p')^+}} = \left(\dfrac{|S|}{|B|}\right)^{1-\frac{1}{p^-}}$.

(3) $1 \le |S| < |B|$, $\dfrac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \sim \dfrac{|S|^{\frac{1}{p'(\infty)}}}{|B|^{\frac{1}{p'(\infty)}}} \le \left(\dfrac{|S|}{|B|}\right)^{\frac{1}{(p')^+}} = \left(\dfrac{|S|}{|B|}\right)^{1-\frac{1}{p^-}}$.

Lemma 2.7 Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $1 < p^- \le p^+ < \infty$, $B = B(x_0, R)$ and $k < n - n/p_-$, then there exists a constant $C > 0$ such that

$$\int_{|x_0 - y| \le R} \frac{|f(y)|}{|x_0 - y|^k} dy \le C R^{-k} \|f\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)}.$$

Proof Using Lemma 2.1, we obtain

$$\int_{|x_0 - y| \le R} \frac{|f(y)|}{|x_0 - y|^k} dy = \sum_{i=-\infty}^{0} \int_{A_i} \frac{|f(y)|}{|x_0 - y|^k} dy \le C \sum_{i=-\infty}^{0} (2^i R)^{-k} \|f\|_{p(\cdot)} \|\chi_{B_i}\|_{p'(\cdot)}.$$

Lemma 2.6 gives

$$\begin{aligned}\int_{|x_0 - y| \le R} \frac{|f(y)|}{|x_0 - y|^k} dy &\le C \sum_{i=-\infty}^{0} (2^i R)^{-k} \|f\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \frac{\|\chi_{B_i}\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \\ &\le C \sum_{i=-\infty}^{0} R^{-k} (2^i)^{n - \frac{n}{p_-} - k} \|f\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \\ &\le C R^{-k} \|f\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)}.\end{aligned}$$

Lemma 2.8 Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ such that $1 < p^- \le p^+ < \infty$, $B = B(x_0, R)$ and $k > n - n/p_+$, then there exists a constant $C > 0$ such that

$$\int_{|x_0-y|>R} \frac{|f(y)|}{|x_0-y|^k} dy \le CR^{-k}\|f\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)}.$$

Proof Applying Lemma 2.1, we derive the estimate

$$\int_{|x_0-y|>R} \frac{|f(y)|}{|x_0-y|^k} dy = \sum_{i=1}^{\infty}\int_{A_i} \frac{|f(y)|}{|x_0-y|^k} dy \le C\sum_{i=1}^{\infty}(2^iR)^{-k}\|f\|_{p(\cdot)}\|A_i\|_{p'(\cdot)}.$$

Lemma 2.6 implies that

$$\int_{|x_0-y|>R} \frac{|f(y)|}{|x_0-y|^k} dy \le C\sum_{i=1}^{\infty}R^{-k}(2^i)^{n-\frac{n}{p_+}-k}\|f\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)} \le CR^{-k}\|f\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)}.$$

Lemma 2.9 Suppose $p_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ for $i = 1, 2$, $B = B(x_0, R)$ and $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$, then

$$M := \int_B\int_{2B}\int_{2B} \frac{|f_1(y_1)||f_2(y_2)|}{|(x-y_1, x-y_2)|^{2n-\beta}} dxdy_1dy_2 \le C|B|^{\frac{\beta}{n}}\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}\|\chi_B\|_{p'(\cdot)}.$$

Proof For $y_1, y_2 \in 2B$, one can obtain the following inequality in [4]

$$\int_B \frac{1}{|(x-y_1, x-y_2)|^{2n-\beta}} dx = \begin{cases} CR^{\beta-n}, & \text{if } n < \beta < 2n, \\ C\log\frac{8R}{|y_1-y_2|}, & \text{if } n = \beta, \\ \frac{C}{|y_1-y_2|^{n-\beta}}, & \text{if } 0 < \beta < n. \end{cases}$$

When $n < \beta < 2n$, using Lemma 2.1 and 2.5, we obtain

$$\begin{aligned} M &\le CR^{\beta-n}\int_{2B}|f_1(y_1)|dy_1\int_{2B}|f_2(y_2)|dy_2 \\ &\le CR^{\beta-n}\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}\|\chi_{2B}\|_{p_1'(\cdot)}\|\chi_{2B}\|_{p_2'(\cdot)} \\ &\le CR^{\beta}\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}\|\chi_B\|_{p'(\cdot)}. \end{aligned}$$

When $0 < \beta < n$, we write

$$\begin{aligned} M &\le C\int_{2B}\int_{2B} \frac{|f_1(y_1)||f_2(y_2)|}{|y_1-y_2|^{n-\beta}} dy_1dy_2 \\ &= C\sum_{i=-\infty}^{1}\sum_{j=-\infty}^{1}\int_{A_i}\int_{A_j} \frac{|f_1(y_1)||f_2(y_2)|}{|y_1-y_2|^{n-\beta}} dy_1dy_2 \\ &= C\sum_{i=-\infty}^{1}\left(\sum_{j=-\infty}^{i-2} + \sum_{j=i-1}^{i+1} + \sum_{j=i+2}^{1}\right)\int_{2B}\int_{2B} \frac{|f_1\chi_i(y_1)||f_2\chi_j(y_2)|}{|y_1-y_2|^{n-\beta}} dy_1dy_2 \\ &:= D_1 + D_2 + D_3, \end{aligned}$$

when $j > -1$ we define $D_3 = 0$.

First we estimate D_1.

For $y_1 \in A_i, y_2 \in A_j$, we have $|y_1 - y_2| \geq |y_1| - |y_2| > 2^{i-2}R$. Then

$$\begin{aligned} D_1 &\leq C \sum_{i=-\infty}^{1} (2^i R)^{\beta-n} \left(\sum_{j=-\infty}^{i-2} \int |f_1\chi_i(y_1)| dy_1 \int |f_2\chi_j(y_2)| dy_2 \right) \\ &= C \sum_{i=-\infty}^{1} (2^i R)^{\beta-n} \int |f_1\chi_i(y_1)| dy_1 \int_{B_{i-2}} |f_2(y_2)| dy_2. \end{aligned}$$

Now Lemma 2.1 yields

$$D_1 \leq C \sum_{i=-\infty}^{1} (2^i R)^{\beta-n} \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \|\chi_{B_i}\|_{p_1'(\cdot)} \|\chi_{B_i}\|_{p_2'(\cdot)}.$$

By Lemma 2.5, we get

$$\begin{aligned} D_1 &\leq C \sum_{i=-\infty}^{1} (2^i R)^{\beta} \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \|\chi_{B_i}\|_{p'(\cdot)} \\ &\leq C R^{\beta} \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \|\chi_B\|_{p'(\cdot)}. \end{aligned}$$

Next we estimate D_2.

Noting that $|y_1 - y_2| \leq |y_1| + |y_2| < C2^i R$ for $y_1 \in A_i, y_2 \in A_j$, using Lemmas 2.1 and 2.7, we have

$$\begin{aligned} D_2 &\leq C \sum_{i=-\infty}^{1} \sum_{j=i-1}^{i+1} \int |f_1\chi_i(y_1)| dy_1 \int \frac{|f_2\chi_j(y_2)|}{|y_1 - y_2|^{n-\beta}} dy_2 \\ &\leq C \sum_{i=-\infty}^{1} (2^i R)^{\beta-n} \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \|\chi_{B_i}\|_{p_1'(\cdot)} \|\chi_{B(y_1, 2^i R)}\|_{p_2'(\cdot)}. \end{aligned}$$

By Lemmas 2.4 and 2.5, we arrive at the inequality

$$\begin{aligned} D_2 &\leq C \sum_{i=-\infty}^{1} (2^i R)^{\beta} \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \|\chi_{B_i}\|_{p'(\cdot)} \\ &\leq C R^{\beta} \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \|\chi_B\|_{p'(\cdot)}. \end{aligned}$$

Finally, we estimate D_3.

We note $y_1 \in A_i, y_2 \in A_j$, $|y_1 - y_2| \geq |y_1| - |y_2| > 2^{j-2}R$ and derive

$$\begin{aligned} D_3 &\leq C \sum_{i=-\infty}^{1} \sum_{j=i+2}^{1} (2^j R)^{\beta-n} \int |f_1\chi_i(y_1)| dy_1 \int |f_2\chi_j(y_2)| dy_2 \\ &= C \sum_{j=-\infty}^{1} \sum_{i=-\infty}^{j-2} (2^j R)^{\beta-n} \int |f_1\chi_i(y_1)| dy_1 \int |f_2\chi_j(y_2)| dy_2 \\ &= C \sum_{j=-\infty}^{1} (2^j R)^{\beta-n} \int_{B_{j-2}} |f_1(y_1)| dy_1 \int |f_2\chi_j(y_2)| dy_2. \end{aligned}$$

Hence, we apply Lemma 2.1 and 2.5 and obtain

$$\begin{aligned} D_3 &\le C\sum_{j=-\infty}^{1}(2^jR)^{\beta-n}\|f_1\|_{p_1(\cdot)}\|\chi_{B_j}\|_{p_1'(\cdot)}\|f_2\|_{p_2(\cdot)}\|\chi_{B_j}\|_{p_2'(\cdot)}\\ &\le C\sum_{j=-\infty}^{1}(2^jR)^{\beta}\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}\|\chi_{B_j}\|_{p'(\cdot)}\\ &\le CR^{\beta}\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}\|\chi_B\|_{p'(\cdot)}. \end{aligned}$$

When $\beta=n$, the proof is similar. Therefore we omit the details. We use the following inequality

$$\sum_{i=-\infty}^{1}\left(\log\frac{8}{2^{i-2}}\right)2^{in}=\log 2\sum_{i=-\infty}^{1}(5-i)2^{in}\le C.$$

As long as we change the conclusion of Lemma 2.1 into the conclusion of Lemma 2.2 in the proof of Lemmas 2.7–2.9, we can obtain the corresponding conclusions in $L^{p(\cdot),\infty}$ space.

Corollary 2.1 Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ such that $1<p^-\le p^+<\infty$, $B=B(x_0,R)$, $p(x)\le p(\infty)$ for $|x|>r_0$ with $r_0>1$ and $k<n-n/p_-$, then there exists a constant $C>0$ such that

$$\int_{|x_0-y|\le R}\frac{|f(y)|}{|x_0-y|^k}dy\le CR^{-k}[f]_{p(\cdot),\infty}\|\chi_B\|_{p'(\cdot)}.$$

Corollary 2.2 Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ such that $1<p^-\le p^+<\infty$, $B=B(x_0,R)$, $p(x)\le p(\infty)$ for $|x|>r_0$ with $r_0>1$ and $k>n-n/p_+$, then there exists a constant $C>0$ such that

$$\int_{|x_0-y|>R}\frac{|f(y)|}{|x_0-y|^k}dy\le CR^{-k}[f]_{p(\cdot),\infty}\|\chi_B\|_{p'(\cdot)}.$$

Corollary 2.3 Let $p_i(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ such that $1<p_i^-\le p_i^+<\infty$, $p_i(x)\le p_i(\infty)$ for $i=1,2$ and $|x|>r_0$ with $r_0>1$. Suppose $B=B(x_0,R)$ and $\frac{1}{p(x)}=\frac{1}{p_1(x)}+\frac{1}{p_2(x)}$, then

$$M:=\int_B\int_{2B}\int_{2B}\frac{|f_1(y_1)||f_2(y_2)|}{|(x-y_1,x-y_2)|^{2n-\beta}}dxdy_1dy_2\le C|B|^{\frac{\beta}{n}}[f_1]_{p_1(\cdot),\infty}[f_2]_{p_2(\cdot),\infty}\|\chi_B\|_{p'(\cdot)}.$$

3 Proof of Theorems

We will give the proof of the Theorem 1.1 below. In Corollary 2.1–Corollary 2.3, we obtain the corresponding results in $L^{p(\cdot),\infty}$ space. The argument for Theorem 1.2 is similar, we omit the details here.

Proof of Theorem 1.1 We write

$$f_i(x)=f_i\chi_{2B}(x)+f_i\chi_{\mathbb{R}^n\setminus 2B}(x)\qquad(i=1,2).$$

And we need to estimate four terms $\tilde I_\beta(f_1\chi_{2B},f_2\chi_{2B})$, $\tilde I_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B},f_2\chi_{2B})$, $\tilde I_\beta(f_1\chi_{2B},f_2\chi_{\mathbb{R}^n\setminus 2B})$ and $\tilde I_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B},f_2\chi_{\mathbb{R}^n\setminus 2B})$.

First we estimate $\tilde{I}_\beta(f_1\chi_{2B}, f_2\chi_{2B})$.

Let $c = -\int\int_{|(y_1,y_2)|\geq 1} \frac{f_1\chi_{2B}(y_1)f_2\chi_{2B}(y_2)}{|(y_1,y_2)|^{2n-\beta}}dy_1dy_2$. By Lemma 2.9, we get

$$\int_B |\tilde{I}_\beta(f_1\chi_{2B}, f_2\chi_{2B})(x) - c|dx = \int_B |I_\beta(f_1\chi_{2B}, f_2\chi_{2B})(x)|dx$$
$$\leq C|B|^{\frac{\beta}{n}}\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}\|\chi_B\|_{p'(\cdot)}.$$

Hence, we arrive at the inequality

$$\frac{1}{|B|^{\frac{\beta}{n}}\|\chi_B\|_{p'(\cdot)}}\int_B |\tilde{I}_\beta(f_1\chi_{2B}, f_2\chi_{2B})(x) - c|dx \leq C\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}.$$

Next we estimate $\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{2B})$ and $\tilde{I}_\beta(f_1\chi_{2B}, f_2\chi_{\mathbb{R}^n\setminus 2B})$.

We only estimate $\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{2B})$ and the estimate for $\tilde{I}_\beta(f_1\chi_{2B}, f_2\chi_{\mathbb{R}^n\setminus 2B})$ is similar, we omit the details here.

Let $c = \tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{2B})(x_0)$, then for $x \in B$, we have

$$|\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{2B})(x) - c| \leq CR\int\int_{|x_0-y_1|>2R} \frac{|f_1\chi_{\mathbb{R}^n\setminus 2B}(y_1)f_2\chi_{2B}(y_2)|}{|(x_0-y_1, x_0-y_2)|^{2n-\beta+1}}dy_1dy_2$$
$$\leq CR\int_{|x_0-y_1|>2R}\frac{|f_1(y_1)|}{|x_0-y_1|^{2n-\beta+1}}dy_1\int |f_2\chi_{2B}(y_2)|dy_2.$$

Applying Lemma 2.8, 2.1 and 2.5, we obtain

$$|\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{2B})(x) - c| \leq CR^{-2n+\beta}\|f\|_{p_1(\cdot)}\|\chi_B\|_{p_1'(\cdot)}\|f\|_{p_2(\cdot)}\|\chi_{2B}\|_{p_2'(\cdot)}$$
$$\leq CR^{-n+\beta}\|f\|_{p_1(\cdot)}\|f\|_{p_2(\cdot)}\|\chi_B\|_{p'(\cdot)}.$$

Thus we get that

$$\frac{1}{|B|^{\frac{\beta}{n}}\|\chi_B\|_{p'(\cdot)}}\int_B |\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{2B})(x) - c|dx \leq C\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}.$$

Finally we estimate $\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{\mathbb{R}^n\setminus 2B})$.

Let $c = \tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{\mathbb{R}^n\setminus 2B})(x_0)$, then for $x \in B$,

$$|\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{\mathbb{R}^n\setminus 2B})(x) - c| \leq CR\int\int \frac{|f_1\chi_{\mathbb{R}^n\setminus 2B}(y_1)f_2\chi_{\mathbb{R}^n\setminus 2B}(y_2)|}{|(x_0-y_1, x_0-y_2)|^{2n-\beta+1}}dy_1dy_2.$$

By Lemmas 2.8 and 2.5, we have

$$|\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2(x) - c| \leq CR\int_{|x_0-y_1|>2R}\frac{|f_1(y_1)|}{|x_0-y_1|^{n-s_1}}dy_1\int_{|x_0-y_2|>2R}\frac{|f_2\chi_{2B}(y_2)|}{|x_0-y_2|^{n-s_2}}dy_2$$
$$\leq CR\|f\|_{p_1(\cdot)}\|\chi_{2B}\|_{p_1'(\cdot)}R^{s_1-n}\|f\|_{p_2(\cdot)}\|\chi_{2B}\|_{p_2'(\cdot)}R^{s_2-n}$$
$$\leq CR^{-n+\beta}\|f\|_{p_1(\cdot)}\|f\|_{p_2(\cdot)}\|\chi_B\|_{p'(\cdot)},$$

where we can take s_1 and s_2 such that $s_1 < n/p_1^+, s_2 < n/p_2^+$ and $s_1+s_2=\beta-1$.

Hence, we obtain

$$\frac{1}{|B|^{\frac{\beta}{n}}\|\chi_B\|_{p'(\cdot)}}\int_B |\tilde{I}_\beta(f_1\chi_{\mathbb{R}^n\setminus 2B}, f_2\chi_{\mathbb{R}^n\setminus 2B})(x)-c|dx \le C\|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}.$$

Consequently we prove Theorem 1.1.

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变指数空间上多线性分数次积分的 Lipschitz 光滑性

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摘要: 本文研究了多线性分数次积分算子在变指数空间的有界性. 利用多线性分数次积分转化为相对应的分数次积分的方法, 获得了它从变指数强和弱Lebesgue空间到变指数Lipschitz空间的有界性, 推广了先前的研究结果.

关键词: Lipschitz空间; 多线性分数次积分; 变指数

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