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ON CONVERGENCE CONDITIONS OF LEAST-SQUARES PROJECTION METHOD FOR OPERATOR EQUATIONS OF THE SECOND KIND

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Abstract: In this paper, we investigate the convergence conditions of least-squares projection method for compact operator equations of the second kind. By technics in functional analysis and Moore-Penrose inverse, we obtain 4 new mutually equivalent convergence conditions, which build the connections among several types of convergence conditions and provide us with more choices to examine the convergency of the approximation scheme. A simple and important example is also studied as an application of the theorem.

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1 Introduction

Operator equation was one of the principal tools in a large area of applied mathematics, and the literature discussing around this topic is vast. In this paper, we will limit our discussion on the compact operator equations of the second kind, which has the form

$$Tx := (I - K)x = b, \quad x \in X, \tag{1.1}$$

where $K : X \to X$ is a compact operator and $b \in X$ is given. Thanks to the First and the Second Riesz Theorem, we know $\dim \mathcal{N}(T) < \infty$ [6, Theorem 3.1, p.28], and $\mathcal{R}(T)$ was closed [6, Theorem 3.2, p.29]. We aim to obtain the best-approximate solution of (1.1), which is denoted as $x^\dagger := T^\dagger b$, where $T^\dagger$ is the Moore-Penrose inverse of T. Note that as $\mathcal{R}(T)$ is closed, $T^\dagger$ is naturally guaranteed to be bounded.

Due to the complexity of the specific problems that has form (1.1), it is difficult for us to find a universal solution to all the problems. A more promising strategy is finding the numerical solution, which involve approximating the abstract space and operator with finite freedoms. Let $\{X_n\}$ be a sequence of finite-dimensional subspaces of X such that

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots, \quad \overline{\bigcup_{n=0}^{\infty} X_n} = X, \tag{1.2}$$

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and for each $n \in \mathbb{N}$ set $T_n := TP_n$, where $P_n := P_{X_n}$ is the orthogonal projection from X onto X_n. Note that (1.2) implies

$$s\text{-}\lim_{n\to\infty} P_n = I_X, \quad s\text{-}\lim_{n\to\infty} P_{\mathcal{R}(T_n)} = P_{\mathcal{R}(T)}, \tag{1.3}$$

here we say $\{(X_n, T_n)\}_{n\in\mathbb{N}}$ is a LSA (least-squares approximation setting) for T, and all of our following discussions will be based on this setting. Our target is to find suitable LSA such that

$$s\text{-}\lim_{n\to\infty} T_n^\dagger = T^\dagger, \tag{1.4}$$

namely, $x^\dagger := T^\dagger b$ can be approximated by $x_n^\dagger := T_n^\dagger b$. There were many works touching upon this problem such as Du [1, 2], Groetsch [4], Groetsch-Neubauer [3], and Seidman [5]. Note that (1.4) does not naturally holds for equation (1.1), as Du's example [2, Example 2.10] shows. To guarantee the convergency of the approximation scheme $\{T_n^\dagger\}$ for $T^\dagger$. Groetsch [4, Proposition 0] and Du [2, Theorem 2.8 (d)] provide the following convergence conditions

$$(1.4) \Longleftrightarrow \sup_n \left\| T_n^\dagger T \right\| < +\infty \Longleftrightarrow (1.5),$$

where (1.5) is the stability condition of LSA $\{(X_n, T_n)\}_{n\in\mathbb{N}}$, that is,

$$\sup_n \left\| T_n^\dagger \right\| < +\infty. \tag{1.5}$$

However, as will show in a simple and important example, a direct examination of (1.5) could be difficult, but the examination of some other equivalent condition of (1.5) that we will soon give in our theorem can be very easy.

In this paper, we will give some equivalent characterizations for (1.4) (or (1.5)). These equivalent characterizations can not only increase our understanding on the convergence of this approximation scheme by offering different perspectives, but also provide us with some simple and 'easy to check' criteria to examine the convergence. To proceed, we need the following notation

$$\begin{aligned} &s\text{-}\lim_{n\to\infty} S_n := \{x \mid \text{ there is a sequence } \{x_n\} \text{ such that } S_n \ni x_n \to x\}, \\ &\text{w-}\widetilde{\lim_{n\to\infty}} S_n := \left\{ x \mid \text{ there is a sequence } \{x_n\} \text{ such that } \bigcup_{k=n}^{\infty} S_k \ni x_n \rightharpoonup x \right\}, \end{aligned}$$

where $\{S_n\}$ is a sequence of nonempty subsets of a Banach space. With the above notation the main result obtained in the paper is as below.

Theorem 1.1 For the compact operator equation (1.1) with LSA $\{(X_n, T_n)\}_{n\in\mathbb{N}}$, the following propositions are equivalent:

(a) (1.4) (or (1.5)) holds.

(b) There holds

$$s\text{-}\lim_{n\to\infty} \mathcal{G}\left(T_n^\dagger\right) = \text{w-}\widetilde{\lim_{n\to\infty}} \mathcal{G}\left(T_n^\dagger\right) = \mathcal{G}\left(T^\dagger\right).$$

(c) There holds

$$s\text{-}\lim_{n\to\infty}\mathcal{N}(T_n) = \text{w}\text{-}\widetilde{\lim_{n\to\infty}}\mathcal{N}(T_n) = \mathcal{N}(T).$$

(d) For any $b, b_n \in X$ with $\|b_n - b\| \to 0$ $(n \to \infty)$ there holds

$$s\text{-}\lim_{n\to\infty}T_n^{-1}\left(P_{\mathcal{R}(T_n)}b_n\right) = \text{w}\text{-}\widetilde{\lim_{n\to\infty}}T_n^{-1}\left(P_{\mathcal{R}(T_n)}b_n\right) = T^{-1}(Pb).$$

(e) There is a $n_* \in \mathbb{N}$ such that $X_{n_*} \supseteq \mathcal{N}(T)$.

In Section 2, we will give some lemmas and the proof of Theorem 1.1. In Section 3, we will study some examples to further explain the theorem.

2 Proofs

To prove Theorem 1.1 we need to prepare several lemmas.

Lemma 2.1 Let $T \in \mathcal{B}(X)$ with $\dim\mathcal{N}(T) < \infty$, and have LSA $\{(X_n, T_n)\}$.

(a) There hold

$$\mathcal{N}(T_n) = (\mathcal{N}(T)\cap X_n) \overset{\perp}{\oplus} X_n^{\perp}, \quad P_{\mathcal{N}(T_n)} = P_{\mathcal{N}(T)\cap X_n} + I - P_n.$$

(b) There is a $n_* \in \mathbb{N}$ such that

$$\begin{aligned} &\mathcal{N}(T)\cap X_n = \overline{\bigcup_{k=0}^{\infty}(\mathcal{N}(T)\cap X_k)} \quad \text{for } n \geq n_*, \\ &s\text{-}\lim_{n\to\infty}P_{\mathcal{N}(T_n)} = P_{\overline{\bigcup_{k=0}^{\infty}(\mathcal{N}(T)\cap X_k)}} = P_{\mathcal{N}(T)\cap X_{n_*}}. \end{aligned}$$

(c) If $\mathcal{R}(T)$ is closed and

$$\overline{\bigcup_{n=0}^{\infty}(\mathcal{N}(T)\cap X_n)} = \mathcal{N}(T), \tag{2.1}$$

then (1.5) holds.

Proof (a) It is clear that

$$\mathcal{N}(T_n) = \{x \in X \mid P_{X_n}x \in \mathcal{N}(T)\cap X_n\} = (\mathcal{N}(T)\cap X_n) \overset{\perp}{\oplus} X_n^{\perp},$$

and therefore

$$P_{\mathcal{N}(T_n)} = P_{\mathcal{N}(T)\cap X_n} + I - P_n.$$

(b) Since (1.2) and $\dim\mathcal{N}(T) < \infty$ hold, we have

$$\mathcal{N}(T)\cap X_n \subseteq \mathcal{N}(T)\cap X_{n+1} \ (\forall n), \quad \dim(\mathcal{N}(T)\cap X_n) \leq \dim\mathcal{N}(T) < \infty,$$

and therefore there is a $n_* \in \mathbb{N}$ such that

$$\mathcal{N}(T)\cap X_n = \overline{\bigcup_{k=0}^{\infty}(\mathcal{N}(T)\cap X_k)} \quad \text{for } n \geq n_*.$$

This implies that

$$s\text{-}\lim_{n\to\infty} P_{\mathcal{N}(T_n)} = s\text{-}\lim_{n\to\infty} P_{\mathcal{N}(T)\cap X_n} = P_{\overline{\bigcup\limits_{k=0}^{\infty}(\mathcal{N}(T)\cap X_k)}} = P_{\mathcal{N}(T)\cap X_{n_*}}.$$

(c) Assume that $\sup\limits_n \left\|T_n^\dagger\right\| = +\infty$. Then by the uniform boundedness principle, there is an $u \in X$ such that

$$\sup_n \left\|T_n^\dagger u\right\| = +\infty.$$

Hence there exists a subsequence $\left\{T_{n_k}^\dagger\right\}$ of $\left\{T_n^\dagger\right\}$ such that $\lim\limits_{k\to\infty} \left\|T_{n_k}^\dagger u\right\| = +\infty$. Due to (b), there is a $n_* \in \mathbb{N}$ such that

$$\mathcal{N}(T) \cap X_n = \overline{\bigcup_{k=0}^{\infty} \left(\mathcal{N}\left(T\right) \cap X_k\right)} \quad \text{for } n \geq n_*.$$

Hence it follows from (2.1) that $\mathcal{N}(T) \cap X_n = \mathcal{N}(T) \subseteq X_n$ for $n \geq n_*$, and therefore

$$\begin{aligned} \mathcal{N}\left(T_n\right)^\perp &= \left(\mathcal{N}\left(T\right) \cap X_n\right)^\perp \cap X_n \\ &= \mathcal{N}(T)^\perp \cap X_n \subseteq X_n \quad \text{for } n \geq n_*. \end{aligned}$$

Set $v_k := \frac{T_{n_k}^\dagger u}{\|T_{n_k}^\dagger u\|}$. Then $v_k \in \mathcal{N}\left(T_{n_k}\right)^\perp = X_{n_k} \cap \mathcal{N}(T)^\perp$ for k large enough, and therefore

$$\begin{cases} v_k = P_{\mathcal{N}(T)^\perp} v_k = T^\dagger T v_k \quad \text{for } k \text{ large enough}, \\ \|T v_k\| = \|T_{n_k} v_k\| = \dfrac{\left\|P_{\mathcal{R}\left(T_{n_k}\right)} u\right\|}{\|T_{n_k}^\dagger u\|} \to 0 \ (k \to \infty). \end{cases}$$

This with $T^\dagger \in \mathcal{B}\left(X\right)$ (by $\mathcal{R}\left(T\right)$ being closed) implies that

$$\lim_{k\to\infty} \|v_k\| = \lim_{n\to\infty} \left\|T^\dagger T v_k\right\| = 0,$$

that contradicts with $\|v_k\| = 1$.

Lemma 2.2 Let $T \in \mathcal{B}\left(X\right)$ have LSA $\{(X_n, T_n)\}$. Then

$$s\text{-}\lim_{n\to\infty} \mathcal{G}\left(T_n\right) = w\text{-}\widetilde{\lim_{n\to\infty}} \mathcal{G}\left(T_n\right) = \mathcal{G}\left(T\right).$$

Proof It is clear that $s\text{-}\lim\limits_{n\to\infty} \mathcal{G}\left(T_n\right) \subseteq w\text{-}\widetilde{\lim\limits_{n\to\infty}}\, \mathcal{G}\left(T_n\right)$. Hence, we need only to show that

$$w\text{-}\widetilde{\lim_{n\to\infty}} \mathcal{G}\left(T_n\right) \subseteq \mathcal{G}\left(T\right) \subseteq s\text{-}\lim_{n\to\infty} \mathcal{G}\left(T_n\right).$$

Let $(x, y) \in \mathcal{G}\left(T\right)$. Then $(x, T_n x) \in \mathcal{G}\left(T_n\right)$, and $(x, T_n x) \to (x, y)$ $(n \to \infty)$. Therefore $(x, y) \in s\text{-}\lim\limits_{n\to\infty} \mathcal{G}\left(T_n\right)$. This gives that $\mathcal{G}\left(T\right) \subseteq s\text{-}\lim\limits_{n\to\infty} \mathcal{G}\left(T_n\right)$. Let $(x, y) \in w\text{-}\widetilde{\lim\limits_{n\to\infty}}\, \mathcal{G}\left(T_n\right)$. Then there is a sequence $\{(x_n, y_n)\}$ such that $\cup_{k=n}^{\infty} \mathcal{G}\left(T_k\right) \ni (x_n, y_n) \rightharpoonup (x, y)$ $(n \to \infty)$. Thus there is a sequence $\{k_n\}$ such that

$$k_n \geq n, \quad x_n \rightharpoonup x \ \left(n \to \infty\right), \quad T_{k_n} x_n = y_n \rightharpoonup y \ \left(n \to \infty\right).$$

Note that for all $v \in X$ there holds

$$\begin{aligned} |\langle Tx - y, v\rangle| &\leq |\langle T(x - x_n), v\rangle| + |\langle (T - T_{k_n}) x_n, v\rangle| + |\langle y_n - y, v\rangle| \\ &= |\langle x - x_n, T^* v\rangle| + |\langle x_n, (I - P_{k_n}) T^* v\rangle| + |\langle y_n - y, v\rangle| . \end{aligned}$$

Thus we have $\langle Tx - y, v\rangle = 0 \; \forall v \in X,$ that is, $(x, y) \in \mathcal{G}(T)$. This gives that $w\text{-}\widetilde{\lim_{n\to\infty}}\mathcal{G}(T_n) \subseteq \mathcal{G}(T)$.

Lemma 2.3 Let H be a Hilbert space and $\{H_n\}$ a sequence of closed subspaces of H. Then

$$\{P_{H_n}\} \text{ is strongly convergent } \iff s\text{-}\lim_{n\to\infty} H_n = w\text{-}\widetilde{\lim_{n\to\infty}} H_n ;$$

in the case that $\{P_{H_n}\}$ is strongly convergent,

$$s\text{-}\lim_{n\to\infty} P_{H_n} = P_M , \quad \text{where } M := s\text{-}\lim_{n\to\infty} H_n .$$

Proof See [1, Lemma 2.13].

Next, we prove Theorem 1.1 as follows.

Proof of Theorem 1.1 Note that, by Lemma 2.1 (a), $P_{\mathcal{N}(T_n)} = P_{\mathcal{N}(T)\cap X_n} + I - P_n$, and there is a $n_* \in \mathbb{N}$ such that

$$s\text{-}\lim_{n\to\infty} P_{\mathcal{N}(T_n)} = P_{\mathcal{N}(T)\cap X_{n_*}} . \tag{2.2}$$

Therefore it follows that

$$\begin{aligned} T_n^\dagger - T^\dagger &= T_n^\dagger P_{\mathcal{R}(T)} - T_n^\dagger T_n T^\dagger + T_n^\dagger T_n T^\dagger - T^\dagger \\ &= T_n^\dagger T (I - P_n) T^\dagger + \left(P_{\mathcal{N}(T_n)^\perp} - P_{\mathcal{N}(T)^\perp}\right) T^\dagger \\ &= \left(T_n^\dagger T - I\right)(I - P_n) T^\dagger, \end{aligned}$$

that is,

$$T_n^\dagger - T^\dagger = \left(T_n^\dagger T - I\right)(I - P_n) T^\dagger. \tag{2.3}$$

Due to this with the uniform boundedness principle, it is clear that

$$(1.4) \iff \sup_n \left\|T_n^\dagger T\right\| < +\infty \iff (1.5). \tag{2.4}$$

(a) $\Longrightarrow$ (b) We need only to show that (a) implies that

$$w\text{-}\widetilde{\lim_{n\to\infty}}\mathcal{G}\left(T_n^\dagger\right) \subseteq \mathcal{G}\left(T^\dagger\right) \subseteq s\text{-}\lim_{n\to\infty}\mathcal{G}\left(T_n^\dagger\right). \tag{2.5}$$

Let (a) be valid, namely, (1.4) and (1.5) hold (by (2.4)). Then for all $x \in X$ there hold

$$\left\|T_n^\dagger T_n x - T^\dagger T x\right\| \leq \left\|T_n^\dagger\right\| \|(T_n - T)x\| + \left\|\left(T_n^\dagger - T^\dagger\right) Tx\right\| \to 0 \;\; (n \to 0),$$

$$\left\|T_n T_n^\dagger x - T T^\dagger x\right\| \leq \|T_n\| \left\|\left(T_n^\dagger - T^\dagger\right) x\right\| + \left\|(T_n - T) T^\dagger x\right\| \to 0 \;\; (n \to 0).$$

Thus we obtain that

$$\underset{n\to\infty}{\text{s-}\lim} P_{\mathcal{N}(T_n)^\perp} = P_{\mathcal{N}(T)^\perp}, \quad \underset{n\to\infty}{\text{s-}\lim} P_{\mathcal{R}(T_n)} = P_{\mathcal{R}(T)}. \tag{2.6}$$

Let $(y, x) \in \mathcal{G}\left(T^\dagger\right)$. Then

$$\left(T^\dagger y, P_{\mathcal{R}(T)} y\right) = \left(T^\dagger y, T T^\dagger y\right) = (x, Tx) \in \mathcal{G}(T).$$

By Lemma 2.2, $s\text{-}\lim\limits_{n\to\infty} \mathcal{G}(T_n) = \mathcal{G}(T)$, hence there is a squence $\{x_n\}$ such that

$$x_n \to x \ (n \to \infty), \quad T_n x_n \to Tx \ (n \to \infty).$$

This with (2.6) implies that

$$\begin{cases} T_n x_n \to Tx \ (n \to \infty), \\ P_{\mathcal{N}(T_n)^\perp} x_n \to P_{\mathcal{N}(T)^\perp} x \ (n \to \infty), \\ P_{\mathcal{R}(T_n)^\perp} y \to P_{\mathcal{R}(T)^\perp} y \ (n \to \infty), \end{cases}$$

and therefore

$$\begin{cases} T_n x_n + P_{\mathcal{R}(T_n)^\perp} y \to Tx + P_{\mathcal{R}(T)^\perp} y = y \ (n \to \infty), \\ T_n^\dagger \left(T_n x_n + P_{\mathcal{R}(T_n)^\perp} y\right) \to T^\dagger T x = T^\dagger y = x \ (n \to \infty). \end{cases}$$

Thus $(y, x) \in s\text{-}\lim\limits_{n\to\infty} \mathcal{G}\left(T_n^\dagger\right)$, we get

$$\mathcal{G}\left(T^\dagger\right) \subseteq \underset{n\to\infty}{s\text{-}\lim} \mathcal{G}\left(T_n^\dagger\right).$$

Let $(y, x) \in w\text{-}\widetilde{\lim\limits_{n\to\infty}} \mathcal{G}\left(T_n^\dagger\right)$. Then there is a sequence $\{(y_n, x_n)\}$ such that

$$\bigcup_{k=n}^{\infty} \mathcal{G}\left(T_k^\dagger\right) \ni (y_n, x_n) \rightharpoonup (y, x) \ (n \to \infty).$$

Hence, there is a sequence $\{k_n\}$ such that

$$k_n \ge n, \quad y_n \rightharpoonup y \ (n \to \infty), \quad T_{k_n}^\dagger y_n = x_n \rightharpoonup x \ (n \to \infty).$$

This with $T_{k_n}^\dagger y_n \in \mathcal{N}(T_{k_n})^\perp \subseteq X_{k_n}$ and (2.6) implies that

$$\begin{aligned} & x \in \underset{n\to\infty}{w\text{-}\widetilde{\lim}} \mathcal{N}(T_n)^\perp = \mathcal{N}(T)^\perp \quad \text{(by Lemma 2.3)}, \\ & T_{k_n} x_n = T_{k_n} T_{k_n}^\dagger y_n = T T_{k_n}^\dagger y_n \rightharpoonup Tx \ (n \to \infty), \end{aligned}$$

and for any $v \in X$,

$$\begin{aligned} \langle T_{k_n} x_n, v\rangle &= \left\langle P_{\mathcal{R}(T_{kn})} y_n, v\right\rangle = \left\langle y_n, P_{\mathcal{R}(T_{kn})} v\right\rangle \\ &= \left\langle y_n, \left(P_{\mathcal{R}(T_{kn})} - P_{\mathcal{R}(T)}\right) v\right\rangle + \left\langle y_n, P_{\mathcal{R}(T)} v\right\rangle \\ &\to \left\langle y, P_{\mathcal{R}(T)} v\right\rangle = \left\langle P_{\mathcal{R}(T)} y, v\right\rangle \ (n \to \infty). \end{aligned}$$

Thus we have

$$x \in \mathcal{N}(T)^{\perp},\ Tx = P_{\mathcal{R}(T)}y, \quad \text{that is,} \quad (y,x) \in \mathcal{G}\left(T^{\dagger}\right).$$

So we obtain that $\underset{n\to\infty}{w\text{-}\widetilde{\lim}}\,\mathcal{G}\left(T_n^{\dagger}\right) \subseteq \mathcal{G}\left(T^{\dagger}\right)$. Now (2.5) is proved.

(b) $\Longrightarrow$ (c) Let (b) be valid. By Lemma 2.3 it follows from (2.2) that

$$\underset{n\to\infty}{s\text{-}\lim}\,\mathcal{N}(T_n) = \underset{n\to\infty}{w\text{-}\widetilde{\lim}}\,\mathcal{N}(T_n) \subseteq \mathcal{N}(T).$$

Hence to prove (c) we need only to show that

$$\mathcal{N}(T) \cap \left(\underset{n\to\infty}{s\text{-}\lim}\,\mathcal{N}(T_n)\right)^{\perp} = \{0\}. \tag{2.7}$$

Assume that (2.7) is not valid. Then there is a $x_0 \in \mathcal{N}(T) \cap \left(\underset{n\to\infty}{s\text{-}\lim}\,\mathcal{N}(T_n)\right)^{\perp}$ with $\|x_0\| = 1$. Note that, by (2.2) with Lemma 2.3,

$$\left.\begin{aligned} \underset{n\to\infty}{s\text{-}\lim}\,P_{\mathcal{N}(T_n)} &= P_{\mathcal{N}(T)\cap X_{n_*}}, \\ \left(\underset{n\to\infty}{s\text{-}\lim}\,\mathcal{N}(T_n)\right)^{\perp} &= \left(\mathcal{N}(T) \cap X_{n_*}\right)^{\perp} \end{aligned}\right\} \text{ for some } n_* \in \mathbb{N}.$$

It follows that $x_0 \in \mathcal{N}(T) \cap \left(\mathcal{N}(T) \cap X_{n_*}\right)^{\perp}$, and that x_0 satisfies

$$T_n P_n x_0 = T P_n x_0 = T\left(P_n - I\right)x_0 \to 0 \ \ (n \to \infty),$$

and (noting $\mathcal{N}(T_n)^{\perp} = \left(\mathcal{N}(T) \cap X_n\right)^{\perp} \cap X_n \subseteq X_n$)

$$\begin{aligned} x_0 - T_n^{\dagger}\left(T_n P_n x_0\right) &= x_0 - P_{\mathcal{N}(T_n)^{\perp}} x_0 \\ &= P_{\mathcal{N}(T_n)} x_0 \to P_{\mathcal{N}(T)\cap X_{n_*}} x_0 = 0 \ \ (n \to \infty). \end{aligned}$$

So we have

$$(0, x_0) \in \underset{n\to\infty}{s\text{-}\lim}\,\mathcal{G}\left(T_n^{\dagger}\right) = \mathcal{G}\left(T^{\dagger}\right), \text{ that is, } x_0 = T^{\dagger}0 = 0.$$

This contradicts with $\|x_0\| = 1$.

(c) $\Longrightarrow$ (d) Let (c) be valid. For any $b, b_n \in X$ with $\|b_n - b\| \to 0$ $(n \to \infty)$ we need only to show that

$$\underset{n\to\infty}{w\text{-}\widetilde{\lim}}\,T_n^{-1}\left(P_{\mathcal{R}(T_n)} b_n\right) \subseteq T^{-1}\left(P_{\mathcal{R}(T)} b\right) \subseteq \underset{n\to\infty}{s\text{-}\lim}\,T_n^{-1}\left(P_{\mathcal{R}(T_n)} b_n\right). \tag{2.8}$$

Due to Lemma 2.3, (c) is equivalent to $\underset{n\to\infty}{\text{s-}\lim}\,P_{\mathcal{N}(T_n)} = P_{\mathcal{N}(T)}$. By Lemma 2.1 (a) and Lemma 2.3, the above equation is equivalent to (2.1). Then by Lemma 2.1 (c), we obtain (1.5), that is

$$M := \sup_n \left\|T_n^{\dagger}\right\| < +\infty. \tag{2.9}$$

Let $x \in w\text{-}\widetilde{\lim_{n\to\infty}} T_n^{-1}\left(P_{\mathcal{R}(T_n)}b_n\right)$. Then there is a squence $\{x_n\}$ with an integer sequence $\{k_n\}$ such that

$$k_n \geq n, \quad x_n \rightharpoonup x, \quad T_{k_n}x_n = P_{\mathcal{R}(T_{k_n})}b_{k_n} \to P_{\mathcal{R}(T)}b \ (n \to \infty).$$

Note that for all $v \in X$ there holds

$$\begin{aligned} \left|\left\langle Tx - P_{\mathcal{R}(T)}b, v\right\rangle\right| &\leq |\langle T(x - x_n), v\rangle| + |\langle (T - T_{k_n}) x_n, v\rangle| + \left|\left\langle T_{k_n}x_n - P_{\mathcal{R}(T)}b, v\right\rangle\right| \\ &\leq |\langle x - x_n, T^*v\rangle| + \|x_n\| \, \|(I - P_{k_n}) T^*v\| + \left|\left\langle T_{k_n}x_n - P_{\mathcal{R}(T)}b, v\right\rangle\right|. \end{aligned}$$

Hence we have $\left\langle Tx - P_{\mathcal{R}(T)}b, v\right\rangle = 0 \ \forall v \in X$, that is, $x \in T^{-1}\left(P_{\mathcal{R}(T)}b\right)$. This gives that

$$w\text{-}\widetilde{\lim_{n\to\infty}} T_n^{-1}\left(P_{\mathcal{R}(T_n)}b_n\right) \subseteq T^{-1}\left(P_{\mathcal{R}(T)}b\right).$$

Let $x \in T^{-1}\left(P_{\mathcal{R}(T)}b\right)$. Take $x_n := x + T_n^{\dagger}\left(P_{\mathcal{R}(T_n)}b_n - T_nx\right), \quad n \in \mathbb{N}$. Then

$$T_nx_n = T_nT_n^{\dagger}P_{\mathcal{R}(T_n)}b_n = P_{\mathcal{R}(T_n)}b_n, \quad \text{that is,} \quad x_n \in T_n^{-1}\left(P_{\mathcal{R}(T_n)}b_n\right),$$

and by use of (2.9),

$$\begin{aligned} \|x_n - x\| &= \left\|T_n^{\dagger}\left(P_{\mathcal{R}(T_n)}b_n - T_nx\right)\right\| \\ &\leq M\left(\|b_n - b\| + \left\|\left(P_{\mathcal{R}(T_n)} - P_{\mathcal{R}(T)}\right)b\right\| + \|(T - T_n)x\|\right) \\ &\to 0 \ (n \to \infty). \end{aligned}$$

That gives $x \in s\text{-}\lim_{n\to\infty} T_n^{-1}\left(P_{\mathcal{R}(T_n)}b_n\right)$. So we get

$$T^{-1}\left(P_{\mathcal{R}(T)}b\right) \subseteq s\text{-}\lim_{n\to\infty} T_n^{-1}\left(P_{\mathcal{R}(T_n)}b_n\right).$$

Thus we get (d).

(d) $\Longrightarrow$ (e) Let (d) be valid. It is clear that

$$s\text{-}\lim_{n\to\infty} T_n^{-1}(0) = w\text{-}\widetilde{\lim_{n\to\infty}} T_n^{-1}(0) = T^{-1}(0).$$

By (2.2) and Lemma 2.3, there is a $n_* \in \mathbb{N}$ such that

$$P_{\mathcal{N}(T)\cap X_{n_*}} = s\text{-}\lim_{n\to\infty} P_{\mathcal{N}(T_n)} = P_{\mathcal{N}(T)},$$

That gives (e).

(e) $\Longrightarrow$ (a) Let (e) hold. Then (2.1) is valid. Hence we have (1.5) by Lemma 2.1 (c), that is (a) holds.

3 Example

In Theorem 1.1, one thing worth to notice is condition (e), which claims that the strong convergence of the LSA $\{(X_n, T_n)\}_{n\in\mathbb{N}}$ is equivalent to

$$\exists \ n_* \in \mathbb{N} \text{ s.t. } X_{n_*} \supseteq \mathcal{N}(T).$$

Note that the examination of this condition does not involve any computation of operator norm or generalized-inverse, which are unavoidable in the examination of the stability condition (condition (a) in Theorem 1.1). Here we will look at a simple example to see how the condition (e) can be used in specific integral equation.

Example 1 Let $X := L^2[-\pi,\pi]$, and we choose the approximation space as

$$X_{n+1} := \mathrm{span}\{1,\ \sin x,\ \cos x,\ \sin 2x,\ \cos 2x, \cdots,\ \sin nx,\ \cos nx\}.$$

We consider the below integral equation of the second kind:

$$T\varphi := \frac{2}{3}\pi^3\varphi - \int xt\varphi(t)dt = f, \quad \varphi \in L^2[-\pi,\pi],$$

where $f \in L^2[-\pi,\pi]$ is known, and we want to get φ. Let $\{X_n, T_n\}$ be the LSA of the equation. It is easy to check that $\mathcal{N}(T) = \mathrm{span}\{x\}$, and for x, it has the Fourier series

$$x = 2\sin x - 2\times\frac{1}{2}\sin 2x + \cdots + 2\times\frac{(-1)^{n+1}}{n}\sin nx + \cdots \quad \text{in } L^2[-\pi,\pi].$$

As the non-zero coefficients in the series has infinite term, so it is obvious that there is no $n^* \in \mathbf{N}$ such that $\mathcal{N}(T) \subseteq X_{n^*}$, namely, the convergence condition (e) in Theorem 1.1 does not be satisfied. According to the Theorem 1.1, the stability condition fails in this case, and $s\text{-}\lim\limits_{n\to\infty} T_n^\dagger \neq T^\dagger$.

Here we look again on the stability condition (a), namely,

$$\sup_n \left\| T_n^\dagger \right\| < +\infty.$$

We notice that to examine this condition, we need to compute generalized-inverse and operator norm, the cost of which is almost equal to computing the minimal spectral of T_n. Thus, it is hard to find a unified method to achieve this task.

Condition (e) in Theorem 1.1 also give the clue to choose convergent approximation scheme. For Example 1, to guarantee the convergence, we choose the approximation space as

$$X_{n+1} = \mathrm{span}\left\{1,\ \frac{x}{\pi},\ \frac{1}{2}[3(\frac{x}{\pi})^2 - 1],\ \cdots\ \frac{\pi^n}{2^n n!}\frac{d^n}{dx^n}\{[(\frac{x}{\pi})^2 - 1]^n\}\right\}.$$

The above is the subspace spanned by the first $(n+1)$ terms of the sequence of Legendre polynomial on $[-\pi,\pi]$. Now the LSA $\{X_n, T_n\}$ possesses convergency as a result of $X_2 \supseteq \mathcal{N}(T)$.

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第二类算子方程最小二乘投影法的收敛性条件

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摘要: 本文研究针对第二类紧算子方程的最小二乘投影法的收敛条件. 通过泛函分析及广义逆理论, 得到了四个新的互相等价的收敛性条件, 这些条件建立起了几种不同收敛性之间的联系并为人们检验逼近框架的收敛性提供了更多地选择. 文中也给出了对一些简单且重要的例子的研究, 以作为主要定理应用的范例.

关键词: 收敛条件; 最小二乘投影法; 第二类算子方程

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