Vol. 37 (2017)
No. 2

数 学 杂 志
J. of Math. (PRC)

ALMOST PERIODIC SOLUTION FOR A DYNAMICAL EQUATION WITH ALLEE EFFECTS ON TIME SCALES

WANG Li-li, HU Meng

(*School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China*)

Abstract: This paper is concerned with an equation representing dynamics of a renewable resource subjected to Allee effects on time scales. By using exponential dichotomy of linear system and contraction mapping fixed point theorem, sufficient conditions are established for the existence of unique positive almost periodic solution. Moreover, by constructing a suitable Lyapunov functional, we obtain sufficient conditions for the global exponential stability of the almost periodic solution.

Keywords: dynamical equation; Allee effect; almost periodic solution; global exponential stability; time scale

2010 MR Subject Classification: 34K14; 34N05

Document code: A **Article ID:** 0255-7797(2017)02-0283-08

1 Introduction

Mathematical ecological system became one of the most important topics in the study of modern applied mathematics. During the last decade, Allee effects received much attention from researchers, largely because of their potential role in extinctions of already endangered, rare or dramatically declining species. The Allee effect refers to a decrease in population growth rate at low population densities. There were several mechanisms that create Allee effects in populations; see, for example [1–6].

Mathematical component of the available literature deals with differential equations or difference equations. Notice that, in the real world, there are many species whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments [7, 8]. Therefore there is a need to establish correspondent dynamic models on new time scales.

A time scale is a nonempty arbitrary closed subset of reals. The theory of time scales was first introduced by Hilger in [9] , in order to unify continuous and discrete analysis. The study

* **Received date:** 2015-12-13 **Accepted date:** 2016-03-04

Foundation item: Supported by Key Project of Scientific Research in Colleges and Universities in Henan Province (15A110004; 16A110008); Basic and Frontier Technology Research Project of Henan Province (142300410113).

Biography: Wang Lili (1981–), female, born at Xinxiang, Henan, lecturer, major in differential equations.

of dynamic equations on time scales can combine the continuous and discrete situations, it unifies not only differential and difference equations, but also some other problems such as a mix of stop-start and continuous behaviors.

Although seasonality is known to have considerable impact on the species dynamics, to our knowledge there were few papers discussed the dynamics of a renewable resource subjected to Allee effects in a seasonally varying environment. Moreover, ecosystems are often disturbed by outside continuous forces in the real world, the assumption of almost periodicity of the parameters is a way of incorporating the almost periodicity of a temporally nonuniform environment with incommensurable periods (nonintegral multiples). In this paper, we introduce seasonality into the resource dynamic equation by assuming the involved coefficients to be almost periodic.

Motivated by the above statements, in the present paper, we shall study the following equation representing dynamics of a renewable resource x, that is subjected to Allee effects on time scales

$$x^{\Delta}(t) = a(t)x(t)(x(t) - b(t))(c(t) - x(t)), \tag{1.1}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an almost time scale; $a(t)$ represents time dependent intrinsic growth rate of the resource; the nonnegative functions $c(t)$ and $b(t)$ stand for seasonal dependent carrying capacity and threshold function of the species respectively satisfying $0 < b(t) < c(t)$. All the coefficients $a(t), b(t), c(t)$ are continuous, almost periodic functions. For the ecological justification of (1.1), one can refer to [5, 6].

For convenience, we introduce the notation

$$f^u = \sup_{t\in\mathbb{T}} f(t),\ \ f^l = \inf_{t\in\mathbb{T}} f(t),$$

where f is a positive and bounded function. Throughout this paper, we assume that the coefficients of equation (1.1) satisfy

$$\min\{a^l, b^l, c^l\} > 0,\ \ \max\{a^u, b^u, c^u\} < +\infty.$$

This is the first paper to study an almost equation representing dynamics of a renewable resource subjected to Allee effects on time scales. The aim of this paper is, by using exponential dichotomy of linear system and contraction mapping fixed point theorem, to obtain sufficient conditions for the existence of unique positive almost periodic solution of (1.1). We also investigate global exponential stability of the unique almost periodic solution by means of Lyapunov function.

2 Preliminaries

Let us first recall some basic definitions which can be found in [10].

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Lemma 2.1 (see [10]) If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Definition 2.1 [11] A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.2 [12] Let $x \in \mathbb{R}^n$, and $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$, the linear system

$$x^{\Delta}(t) = A(t)x(t), \ t \in \mathbb{T} \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant k, α, projection P and the fundamental solution matrix $X(t)$ of (2.1), satisfying

$$\begin{aligned} &|X(t)PX^{-1}(\sigma(s))|_0 \leq ke_{\ominus\alpha}(t, \sigma(s)), \ s, t \in \mathbb{T}, t \geq \sigma(s), \\ &|X(t)(I - P)X^{-1}(\sigma(s))|_0 \leq ke_{\ominus\alpha}(\sigma(s), t), \ s, t \in \mathbb{T}, t \leq \sigma(s), \end{aligned}$$

where $|\cdot|_0$ is a matrix norm on $\mathbb{T}$ $\big(A = (a_{ij})_{n\times m}$, then $|A|_0 = \big(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} |a_{ij}|^2\big)^{\frac{1}{2}}\big)$.

Considering the following almost periodic system

$$x^{\Delta}(t) = A(t)x(t) + f(t), \ t \in \mathbb{T}, \tag{2.2}$$

where $A(t)$ is an almost periodic matrix function, $f(t)$ is an almost periodic vector function.

Lemma 2.2 (see [12]) If the linear system (2.1) admits exponential dichotomy, then system (2.2) has a unique almost periodic solutions $x(t)$ as follows

$$x(t)=\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s-\int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{2.3}$$

where $X(t)$ is the fundamental solution matrix of (2.1).

Definition 2.3 The almost periodic solution x^* of equation(1.1) is said to be exponentially stable, if there exist positive constants $\alpha>0$, $\alpha\in\mathcal{R}^+$ and $N=N(t_0)\geq 1$ such that for any solution x of equation (1.1) satisfying

$$\|x-x^*\|\leq N|x(t_0)-x^*(t_0)|e_{-\alpha}(t,t_0), t\in[t_0,+\infty)_{\mathbb{T}}.$$

3 Main Results

Clearly, the trivial solution $x(t)\equiv 0$ is an almost periodic solution of (1.1). Since the study deals with resource dynamics, we are interested in the existence of positive almost periodic solutions of the considered equation.

First, we make the following assumptions:

(H1) $-abc\in\mathcal{R}^+$;

(H2) there exist two positive constants $L_1>L_2>0$, such that

$$\frac{4a^u(b^u+c^u)^3}{27\inf\limits_{t\in\mathbb{T}}\{abc\}}\leq L_1\leq b^l+c^l;\ \frac{2}{3}(b^l+c^l)\leq L_2\leq\frac{a^lL_1^2(b^l+c^l-L_1)}{\sup\limits_{t\in\mathbb{T}}\{abc\}};$$

(H3) $\frac{\lambda a^u}{\inf\limits_{t\in\mathbb{T}}\{abc\}}<1$, where $\lambda=\max\{|2(b^l+c^l)L_2-3L_1^2|,|2(b^u+c^u)L_1-3L_2^2|\}$.

Theorem 3.1 Assume that (H1)–(H3) hold, then equation (1.1) has a unique almost periodic solution.

Proof Let $Z=\{z|z\in C(\mathbb{T},\mathbb{R}), z \text{ is an almost periodic function}\}$ with the norm $\|z\|=\sup\limits_{t\in\mathbb{T}}|z(t)|$, then Z is a Banach space.

Equation (1.1) can be written as

$$x^{\Delta}(t)=-a(t)b(t)c(t)x(t)+a(t)x^2(t)[b(t)+c(t)-x(t)], t\in\mathbb{T}.$$

For $z\in Z$, we consider the almost periodic solution $x_z(t)$ of the nonlinear almost periodic differential equation

$$x^{\Delta}(t)=-a(t)b(t)c(t)x(t)+a(t)z^2(t)[b(t)+c(t)-z(t)], t\in\mathbb{T}. \tag{3.1}$$

Since $\inf\limits_{t\in\mathbb{T}}[a(t)b(t)c(t)]\geq a^lb^lc^l>0$, from Lemma 2.15 [12] and (H1), the linear equation

$$x^{\Delta}(t)=-a(t)b(t)c(t)x(t)$$

admits exponential dichotomy on $\mathbb{T}$.

Hence by Lemma 2.2, equation (3.1) has exactly one almost periodic solution

$$x_z(t)=\int_{-\infty}^{t} e_{-abc}(t,\sigma(s))a(s)z^2(s)[b(s)+c(s)-z(s)]\Delta s.$$

Define an operator $\Phi: Z\to Z$,

$$(\Phi z)(t)=\int_{-\infty}^{t} e_{-abc}(t,\sigma(s))a(s)z^2(s)[b(s)+c(s)-z(s)]\Delta s. \tag{3.2}$$

Obviously, z is an almost periodic solution of equation (1.1) if and only if z is the fixed point of operator Φ.

Let $\Omega=\{z|z\in Z, L_2\le z(t)\le L_1, t\in\mathbb{T}\}$.

Now, we prove that $\Phi\Omega\subset\Omega$. In fact, $\forall z\in\Omega$, we have

$$\begin{aligned}(\Phi z)(t) &= \int_{-\infty}^{t} e_{-abc}(t,\sigma(s))a(s)z^2(s)[b(s)+c(s)-z(s)]\Delta s\\ &\le \frac{a^u}{2}\int_{-\infty}^{t} e_{-abc}(t,\sigma(s))z(s)z(s)[2(b^u+c^u)-2z(s)]\Delta s\\ &\le \frac{4a^u(b^u+c^u)^3}{27}\int_{-\infty}^{t} e_{-abc}(t,\sigma(s))\Delta s\\ &\le \frac{4a^u(b^u+c^u)^3}{27\inf\limits_{t\in\mathbb{T}}\{abc\}}\le L_1.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}(\Phi z)(t) &= \int_{-\infty}^{t} e_{-abc}(t,\sigma(s))a(s)z^2(s)[b(s)+c(s)-z(s)]\Delta s\\ &\ge a^l\int_{-\infty}^{t} e_{-abc}(t,\sigma(s))z^2(s)[b^l+c^l-z(s)]\Delta s.\end{aligned} \tag{3.4}$$

Note that

$$\frac{2}{3}(b^l+c^l)\le L_2\le z(t)\le L_1, t\in\mathbb{T}.$$

Since the function $g(u)=u^2[b^l+c^l-u]$ is increasing on $u\in[0,\frac{2}{3}(b^l+c^l)]$ and decreasing on $u\in[\frac{2}{3}(b^l+c^l),+\infty]$, then we have $g(z(t))\ge g(L_1)$ for $t\in\mathbb{T}$, that is

$$z^2(t)[b^l+c^l-z(t)]\ge L_1^2(b^l+c^l-L_1), t\in\mathbb{T}.$$

Thus by (3.4), we obtain

$$\begin{aligned}(\Phi z)(t) &\ge a^lL_1^2(b^l+c^l-L_1)\int_{-\infty}^{t} e_{-abc}(t,\sigma(s))\Delta s\\ &\ge \frac{a^lL_1^2(b^l+c^l-L_1)}{\sup\limits_{t\in\mathbb{T}}\{abc\}}\ge L_2.\end{aligned} \tag{3.5}$$

It follows from (3.3) and (3.5) that

$$L_2 \leq (\Phi z)(t) \leq L_1. \tag{3.6}$$

In addition, for $z \in \Omega$, equation (3.1) has exactly one almost periodic solution

$$x_z(t) = \int_{-\infty}^{t} e_{-abc}(t, \sigma(s))a(s)z^2(s)[b(s) + c(s) - z(s)]\Delta s.$$

Since $x_z(t)$ is almost periodic, then $(\Phi z)(t)$ is almost periodic. This, together with (3.6), implies $\Phi z \in \Omega$. So we have $\Phi\Omega \subset \Omega$.

Next, we prove that Φ is a contraction mapping on Ω. In fact, in view of (H1)–(H3), for any $z_1, z_2 \in \Omega$,

$$\begin{aligned}
& \|\Phi z_1 - \Phi z_2\| \\
= \ & \sup_{t\in\mathbb{T}} |(\Phi z_1)(t) - (\Phi z_2)(t)| \\
= \ & \sup_{t\in\mathbb{T}} \bigg| \int_{-\infty}^{t} e_{-abc}(t, \sigma(s))a(s)z_1^2(s)[b(s) + c(s) - z_1(s)]\Delta s \\
& - \int_{-\infty}^{t} e_{-abc}(t, \sigma(s))a(s)z_2^2(s)[b(s) + c(s) - z_2(s)]\Delta s \bigg| \\
= \ & \sup_{t\in\mathbb{T}} \left\{ \int_{-\infty}^{t} e_{-abc}(t, \sigma(s))a(s)|(b(s) + c(s))(z_1^2(s) - z_2^2(s)) - (z_1^3(s) - z_2^3(s))|\Delta s \right\} \\
= \ & \sup_{t\in\mathbb{T}} \left\{ \int_{-\infty}^{t} e_{-abc}(t, \sigma(s))a(s)|z_1(s) - z_2(s)| \right. \\
& \left. |(b(s) + c(s))(z_1(s) + z_2(s)) - (z_1^2(s) + z_1(s)z_2(s) + z_2^2(s))|\Delta s \right\} \\
\leq \ & \lambda a^u \sup_{t\in\mathbb{T}} \left\{ \int_{-\infty}^{t} e_{-abc}(t, \sigma(s))\Delta s \right\} \|z_1 - z_2\| \\
\leq \ & \frac{\lambda a^u}{\inf\limits_{t\in\mathbb{T}}\{abc\}} \|z_1 - z_2\|,
\end{aligned}$$

where $\lambda = \max\{|2(b^l + c^l)L_2 - 3L_1^2|, |2(b^u + c^u)L_1 - 3L_2^2|\}$.

Since $\frac{\lambda a^u}{\inf_{t\in\mathbb{T}}\{abc\}} < 1$, this implies that Φ is a contraction mapping. Thus, Φ has exactly one fixed point z^* in Ω such that $\Phi(z^*) = z^*$. Otherwise, it is easy to verify that z^* satisfies equation (1.1). This means that equation (1.1) has a unique almost periodic solution in $z^*(t)$, and $L_2 \leq z^*(t) \leq L_1$. This completes the proof.

Next, we shall construct a suitable Lyapunov functional to study the global exponential stability of the almost periodic solution of (1.1).

Theorem 3.2 Assume that (H1)–(H3) hold. Suppose further that

(H4) $\gamma = 4(b^l c^l + L_2^2) - (b^u + c^u + L_1)^2 > 0$;

(H5) Let $0 < \alpha < \frac{a^l \gamma}{4}$, and $-\alpha \in \mathcal{R}^+$;

then equation (1.1) has a unique globally exponentially stable almost periodic solution.

Proof According to Theorem 3.1, we know that (1.1) has an almost periodic solution $x^*(t)$, and $L_2 \le x^*(t) \le L_1$. Suppose that $x(t)$ is arbitrary solution of (1.1) with initial condition $x(t_0) > 0, t_0 \in \mathbb{T}$. Now we prove $x^*(t)$ is globally exponentially stable.

Let $V(t) = |x(t) - x^*(t)|$. Calculating the upper right derivatives of $V(t)$ along the solution of equation (1.1), from (H4) and (H5), then

$$\begin{aligned}
D^+V^{\Delta}(t) &= \operatorname{sgn}(x(t) - x^*(t))(x^{\Delta}(t) - x^{*\Delta}(t)) \\
&= \operatorname{sgn}(x(t) - x^*(t))[-a(t)b(t)c(t) + a(t)(b(t) + c(t))(x(t) + x^*(t)) \\
&\quad -a(t)(x^2(t) + x(t)x^*(t) + x^{*2}(t))](x(t) - x^*(t)) \\
&= -a(t)[b(t)c(t) - (b(t) + c(t))(x(t) + x^*(t)) \\
&\quad +(x^2(t) + x(t)x^*(t) + x^{*2}(t))]|x(t) - x^*(t)| \\
&= -a(t)\left[\left(x(t) - \frac{b(t) + c(t) - x^*(t)}{2}\right)^2 - \frac{(b(t) + c(t) + x^*(t))^2}{4}\right. \\
&\quad \left. +b(t)c(t) + x^{*2}(t)\right]|x(t) - x^*(t)| \\
&\le -a^l\left[b^lc^l + L_2^2 - \frac{(b^u + c^u + L_1)^2}{4}\right]|x(t) - x^*(t)| \\
&\le -\alpha V(t).
\end{aligned} \tag{3.7}$$

Integrating (3.7) from t_0 to t, we get $V(t) \le e_{-\alpha}(t, t_0)V(t_0)$, that is

$$|x(t) - x^*(t)| \le e_{-\alpha}(t, t_0)|x(t_0) - x^*(t_0)|,$$

then $\|x - x^*\| \le |x(t_0) - x^*(t_0)|e_{-\alpha}(t, t_0)$.

From Definition 2.3, the almost periodic solution x^* of (1.1) is globally exponentially stable. This completes the proof.

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时标上具Allee效应的动力学方程的概周期解

王丽丽, 胡 猛

(安阳师范学院数学与统计学院, 河南 安阳 455000)

摘 要: 本文研究了时标上具Allee效应的可再生资源动力学方程的概周期解的存在性与稳定性. 利用线性系统指数二分性与压缩映射不动点定理, 得到了方程存在唯一概周期解的充分条件. 此外, 通过构建适当的Laypunov函数, 得到了概周期解是全局指数稳定的充分条件.

关 键 词: 动力学方程; Allee效应; 概周期解; 指数稳定; 时标

MR(2010)**主题分类号**: 34K14; 34N05 **中图分类号**: O175.12