

HOPF BIFURCATION ANALYSIS OF A PREDATOR-PREY SYSTEM WITH NON-SELECTIVE HARVESTING AND TIME DELAY

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Abstract: In this paper, we mainly study the Hopf bifurcation and the stability of modified predator-prey biological economic system with nonselective harvesting and time delay. By using the stability and bifurcation theory of differential-algebraic system, the conditions for stability of the positive equilibrium point are obtained, let time delay as bifurcation parameter, the existence of Hopf bifurcation and direction of Hopf bifurcation are obtained. We have improved the Leslie-Gower predator-prey system, make the system which we established more practical, so the conclusions are made more scientific.

Keywords: stability; Hopf bifurcation; time delay; non-selective; predator-prey system; periodic solution

2010 MR Subject Classification: 34D20; 34K18; 34C23

Document code: A

Article ID: 0255-7797(2017)02-0257-14

1 Introduction

In recent years, the increasingly serious problem of environmental degradation and resource shortage, made the analysis and modeling of biological systems more interested. The predator-prey system played a crucial role among the relationships between the biological population, and it naturally attracted much attention both for mathematicians and biologists, especially on predator-prey systems with or without time delay. As we know, delay differential equation models exhibit much more complicated dynamics than differential equation models without delay, see [1–12]. A lot of researchers studied the dynamics of predator-prey models with harvesting and obtained many dynamic behaviors, such as stability of equilibrium, Hopf bifurcation, periodic solution, Bogdanov-Takens bifurcation, Neimark-Sacker bifurcation, and so on, see [10–15].

In [16], Lucas studied the dynamic properties of the following Leslie-Gower predator-prey system

$$\begin{cases} \dot{x} = x(a - y), \\ \dot{y} = y(d - k\frac{y}{x}), \end{cases} \quad (1.1)$$

* **Received date:** 2014-11-16

Accepted date: 2015-02-26

Foundation item: Supported by the Funding Program of Higher School Outstanding Youth Scientific and Technological Innovation Team in Hubei of China (T201412).

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where x and y denote prey and predator population densities at time t , respectively, a , d , and k are positive constants that represent the prey intrinsic growth rate, predator mortality rate, and the maximum value of the per capita reduction rate of x due to y , respectively.

At present, economic profit is a very important factor for merchants, governments and even every citizen, so it is necessary to research biological economic systems, which are often described by differential-algebraic equations or differential difference-algebraic equations.

In 1954, Gordon [13] studied the effect of the harvest effort on ecosystem from an economic perspective and proposed the following economic theory:

Net Economic Revenue (NER) = Total Revenue (TR) - Total Cost (TC).

This provides theoretical evidence for the establishment of differential-algebraic equation.

Based on the economic theory as mentioned above and system (1.1), Liu and Fu [12] considered the following Leslie-Gower predator-prey system

$$\begin{cases} \dot{x} = x(a - y - E), \\ \dot{y} = y(d - k\frac{y}{x}), \\ 0 = E(px - c) - v. \end{cases} \quad (1.2)$$

They investigated the Hopf bifurcation of the above system without considering the effect of time delay and the harvesting of predator.

As is known to all, delay differential equation models exhibit much more complicated dynamics than ordinary differential equation models, see [1–12], as was pointed by Kuang [17] that any model of species dynamics without delays is an approximation at best. When we considered the model with non-selective harvesting, namely at the same time there are also the harvesting of predator and harvesting of the prey in the model, it will be more in line with the actual situation of the predator-prey systems.

Motivated by the above discussion, in this paper, by choosing the time delay as a bifurcation parameter and consider the predator-prey systems with non-selective harvesting, we investigate a modified Leslie-Gower predator-prey systems with non-selective harvesting and time delay described by the following system

$$\begin{cases} \frac{dx}{dt} = x(t)(r_1 - by(t - \tau) - E(t)), \\ \frac{dy}{dt} = y(t)(r_2 - k\frac{y(t)}{x(t)} - E(t)), \\ 0 = E(t)(p_1x(t) + p_2y(t) - c_1 - c_2) - m, \end{cases} \quad (1.3)$$

where $p_1 > 0$ and $p_2 > 0$ are harvesting reward per unit harvesting effort for unit prey and predator, respectively; c_1 and c_2 are harvesting cost per unit harvesting effort for prey and predator, respectively; m is the economic profit per unit harvesting effort.

In this paper, we mainly discuss the effects of economic profit on the dynamics of system (1.3) in the region $R_+^3 = \{(x, y, E) | x > 0, y > 0, E > 0\}$.

For convenience, we let

$$\begin{aligned} f(X(t), E(t)) &= \begin{pmatrix} f_1(X(t), E(t)) \\ f_2(X(t), E(t)) \end{pmatrix} = \begin{pmatrix} x(t)(r_1 - by(t - \tau) - E(t)) \\ y(t)(r_2 - k\frac{y(t)}{x(t)} - E(t)) \end{pmatrix}, \\ g(X(t), E(t)) &= E(t)(p_1x(t) + p_2y(t) - c_1 - c_2) - m, \end{aligned}$$

where $X_t = (x, y)^T$.

The rest of the paper is arranged as follows: in Section 2, the local stability of the positive equilibrium points are investigated by the corresponding characteristic equation of system (1.3). In Section 3, by using the normal form and Hopf bifurcation theorem, we study the Hopf bifurcation of the nonnegative equilibrium depending on the parameter where we show that the positive equilibrium loses its stability and system (1.3) exhibits Hopf bifurcation in the second section. In Section 4, the theoretical result is supplied by a numerical example. Finally, this paper ends with a brief discussion.

2 Local Stability Analysis of System

In this section, we discuss the local stability of a positive equilibrium for system (1.3). Now, we try to find all possible positive equilibrium points of system (1.3). A point $Y_0 = (x_0, y_0, E_0)$ is an equilibrium point of system (1.3) if and only if Y_0 satisfy the following equations

$$\begin{cases} x(t)(r_1 - by(t - \tau) - E(t)) = 0, \\ y(t)(r_2 - k\frac{y(t)}{x(t)} - E(t)) = 0, \\ E(t)(p_1x(t) + p_2y(t) - c_1 - c_2) - m = 0. \end{cases} \quad (2.1)$$

From (2.1), we can easy get E_0 satisfy

$$E^3 + \gamma_1 E^2 + \gamma_2 E + \gamma_3 = 0, \quad (2.2)$$

where

$$\begin{aligned} \gamma_1 &= \frac{bc_1 + bc_2 - kp_1 - p_2r_1 - p_2r_2}{p_2}, \\ \gamma_2 &= \frac{kp_1r_1 + r_1p_2r_2 + bm - bc_1c_2 - bc_2r_2}{p_2}, \quad \gamma_3 = \frac{-bmr_2}{p_2} < 0. \end{aligned}$$

Based on the root and coefficient relationship of equation and $\gamma_3 < 0$, we can find at least one positive root E_0 , so system (1.3) has at least one positive equilibrium point $Y_0 = (x_0, y_0, E_0) = (\frac{k(r_1 - E_0)}{b(r_2 - E_0)}, \frac{r_1 - E_0}{b}, E_0)$, where $r_1 > E_0$, $r_2 > E_0$.

Now, we derive the formula for determining the properties of the positive equilibrium point of system (1.3). As in [13], first we consider the local parametric ψ of the third equation of system (1.3), which is defined as follows

$$[x, y, \bar{E}]^T = \psi(Z(t)) = N_0^T + U_0 + V_0h(Z(t)), \quad g(\psi(Z(t))) = 0,$$

where

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$Z(t) = (y_1, y_2)^T, \quad N_0 = (x_0, y_0, \bar{E}_0),$$

$$h(Z(t)) = h(h_1(y_1(t), y_2(t)), h_2(y_1(t), y_2(t)), h_3(y_1(t), y_2(t))) : R^2 \longrightarrow R$$

is a smoothing mapping, that is

$$x(t) = x_0 + y_1(t), y(t) = y_0 + y_2(t), \bar{E}(t) = \bar{E}_0 + h_3(y_1(t), y_2(t)).$$

Then we can obtain the parametric system of system (1.3) as follows

$$\begin{cases} \frac{dy_1(t)}{dt} = (y_1(t) + x_0)(r_1 - b(y_2(t) + y_0) - (\bar{E}_0 + h_3(y_1(t), y_2(t)))), \\ \frac{dy_2(t)}{dt} = (y_2(t) + y_0)(r_2 - k \frac{(y_2(t) + y_0)}{(y_1(t) + x_0)} - (\bar{E}_0 + h_3(y_1(t), y_2(t)))). \end{cases} \quad (2.3)$$

Noticing that $g(\psi(Z(t))) = 0$, so we can get the linearized system of parametric system (2.3) at $(0,0)$ as follows

$$\begin{cases} \frac{dy_1(t)}{dt} = -bx_0y_2(t - \tau), \\ \frac{dy_2(t)}{dt} = \frac{ky_0^2}{x_0^2}y_1(t) - \frac{ky_0}{x_0}y_2(t). \end{cases} \quad (2.4)$$

From (2.4), we can obtain the characteristic equation of the linearized system of parametric system (2.2) at $(0,0)$ as follows

$$\lambda^2 + \frac{ky_0}{x_0}\lambda + b\frac{ky_0^2}{x_0}e^{-\lambda\tau} = 0. \quad (2.5)$$

By eq. (2.5), when $\tau = 0$, it is obvious that $\frac{ky_0}{x_0} > 0$ and $\frac{ky_0^2}{x_0} > 0$, then, two roots of eq. (2.5) has always negative real parts, i.e., the positive equilibrium point of system (1.3) is locally asymptotically stable.

Now, based on the above discussion, we study the local stability around the positive equilibrium point for system (1.3) and the existence of Hopf bifurcation occurring at the positive equilibrium point when $\tau > 0$.

If $i\omega$ is a root of eq. (2.5), and substituting $i\omega$ (ω is a positive real number) into eq. (2.5), and separating the real and imaginary parts, two transcendental equations can be obtained as follows

$$\omega^2 = \frac{ky_0^2}{x_0} \cos(\omega\tau), \quad (2.6)$$

$$\frac{ky_0}{x_0}\omega = \frac{ky_0^2}{x_0} \sin(\omega\tau). \quad (2.7)$$

Since $\sin(\omega\tau)^2 + \cos(\omega\tau)^2 = 1$ and adding (2.6) and (2.7), we obtain

$$\omega^4 + \left(\frac{ky_0}{x_0}\right)^2\omega^2 - \left(\frac{ky_0^2}{x_0}\right)^2 = 0. \quad (2.8)$$

From (2.8), $(\frac{ky_0}{x_0})^2 > 0$ and $-(\frac{kby_0^2}{x_0})^2 < 0$, we can easy find that eq. (2.5) has a unique positive root ω_0 , that is

$$\omega_0 = \sqrt{\frac{-(\frac{ky_0}{x_0})^2 + \sqrt{(\frac{ky_0}{x_0})^2 + 4(\frac{kby_0^2}{x_0})^2}}{2}}. \quad (2.9)$$

Substituting ω_0 into (2.6) and solving for τ , we get

$$\tau_n = \frac{1}{\omega_0} \arccos\left(\frac{x_0\omega_0^2}{kby_0^2}\right) + \frac{2n\pi}{\omega_0}, n = 0, 1, 2, \dots. \quad (2.10)$$

Thus when $\tau = \tau_n$, the characteristic equation (2.5) has a pair of purely imaginary roots $i\omega_0$.

Lemma 2.1 Denote by $\lambda_n(\tau) = \eta_n(\tau) + i\omega_n(\tau)$ the root of (2.5) such that $\eta_n(\tau_n) = 0$, $\omega_n(\tau_n) = \omega_0$, $n = 0, 1, 2, \dots$. Then the following transversality condition $\eta'_n(\tau_n)$ is satisfied.

Proof Differentiating eq. (2.5) with respect to τ , we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{[2\lambda x_0 + ky_0 - kby_0^2\tau e^{-\lambda\tau}]}{kby_0^2\tau e^{-\lambda\tau}} = -\frac{1}{\lambda^2} - \frac{\tau}{\lambda} - \frac{1}{\lambda^2 + \frac{ky_0}{x_0}\lambda}.$$

Noting that

$$\text{sign}\{\text{Re}\left(\frac{d\lambda}{dt}\right)\}_{\lambda=i\omega} = \text{sign}\{\text{Re}\left(\frac{d\lambda}{dt}\right)^{-1}\}_{\lambda=i\omega} = \text{sign}\left\{\frac{2\omega_0^2 + (\frac{ky_0}{x_0})^2}{\omega_0^2(\omega_0^2 + \frac{ky_0}{x_0})}\right\} > 0.$$

The proof is completed.

From the above analysis and [17, 18], we have the following results.

Theorem 2.1 (i) For system (1.3), its positive equilibrium point Y_0 is locally asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$.

(ii) System (1.3) undergoes Hopf bifurcation at the positive equilibrium point Y_0 for $\tau = \tau_n, n = 0, 1, 2, \dots$.

3 Direction and the Stability of Hopf Bifurcation

In this section, we investigate the direction of bifurcation and the stability of bifurcation periodic orbits from the positive equilibrium point Y_0 of system (1.3) at $\tau = \tau_0$ by using the normal form approach theory and center manifold theory introduced by Hassard [15].

Now, we re-scale the time by

$$t = \frac{t}{\tau}, \bar{y}_1 = y_1, \bar{y}_2 = y_2, \tau = \tau_0 + \mu, \bar{Z} = (\bar{y}_1, \bar{y}_2)$$

for simplicity, we continue to use Z said $\bar{Z}$, then the parametric system (2.3) of system (1.3) is equivalent to the following functional differential equation (FDE) system in $C = C([-1, 0], \mathbb{R}^2)$,

$$\dot{Z}(T) = L_\mu Z(T) + f(\mu, Z_t), \quad (3.1)$$

where $Z(T) = (y_1(t), y_2(t))^T$, and $L_\mu : C \rightarrow \mathbb{R}$, $f : \mathbb{R} \times C \rightarrow \mathbb{R}$ are given, respectively, by

$$L_\mu(\phi) = (\tau_0 + \mu) \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix} \phi^T(0) + (\tau_0 + \mu) \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \phi^T(-1),$$

where

$$a_{12} = -bx_0, \quad a_{21} = \frac{ky_0^2}{x_0^2}, \quad a_{22} = -\frac{ky_0}{x_0},$$

and $f(\mu, \phi) = (\tau_0 + \mu) \begin{pmatrix} f_{11} \\ f_{22} \end{pmatrix}$, where

$$\begin{aligned} f_{11} &= -b\phi_1(0)\phi_2(-1), \\ f_{22} &= -\frac{ky_0^2}{x_0^3}\phi_1^2(0) + \frac{2ky_0}{x_0}\phi_1(0)\phi_2(0) - \frac{k}{x_0}\phi_2^2(0) + \cdots, \end{aligned}$$

and $\phi = (\phi_1, \phi_2) \in C$. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\theta \in [-1, 0]$ such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C.$$

In fact, choosing

$$\eta(\theta, \mu) = (\tau_0 + \mu) \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix} \delta(\theta) + (\tau_0 + \mu) \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \delta(\theta + 1),$$

where $\delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ 1, & \theta = 0. \end{cases}$ For $\phi \in C^1([-1, 0], \mathbb{R}^2)$, define

$$A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta), & \theta = 0. \end{cases} \quad (3.2)$$

Then system (3.1) can be rewritten as

$$\dot{Z} = A(\mu)Z_t + R(\mu)Z_t. \quad (3.3)$$

For $\psi \in C^1([0, 1], (\mathbb{R}^2)^*)$, the adjoint operator A^* of A as

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^0 d\eta^T(s, 0)\psi(-s), & s = 0, \end{cases} \quad (3.4)$$

where η^T is the transpose of the matrix η .

For $\phi \in C^1([-1, 0], \mathbb{R}^2)$ and for $\psi \in C^1([0, 1], (\mathbb{R}^2)^*)$, in order to normalize the eigenvectors of operator A and adjoint operator A^* , we define a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi, \quad (3.5)$$

where $\eta(\theta) = \eta(\theta, 0)$. It is easy to verify that $A(0)$ and A^* are a pair of adjoint operators.

By the discussion in Section 2, we know that $\pm i\omega\tau_0$ are eigenvalues of $A(0)$. Thus they are also eigenvalues of A^* . Next we calculate the eigenvector $q(\theta)$ of A associated to the eigenvalue $i\omega\tau_0$ and the eigenvector $q^*(s)$ of A^* associated to the eigenvalue $-i\omega\tau_0$. Then it is not difficult to show that

$$q(\theta) = (1, \beta)^T, q^*(s) = G(\beta^*, 1)e^{i\omega\tau_0 s},$$

where

$$\beta = -\frac{i\omega}{bx_0 e^{-i\omega\tau_0}}, \quad \beta^* = \frac{i\omega - \frac{k y_0}{x_0}}{bx_0 e^{-i\omega\tau_0}}, \quad \bar{G} = (\beta + \bar{\beta}^* - bx_0\tau_0\beta\bar{\beta}^* e^{-i\omega\tau_0})^{-1}.$$

Moreover, $\langle q^*(s), q(\theta) \rangle = 1$ and $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

In the reminder of this section, we use the same notations as those in [15]. We first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Define

$$\dot{z}(t) = \langle q^*, Z_t \rangle, W(t, \theta) = Z_t - 2\text{Re}z(t)q(\theta). \quad (3.6)$$

On the center manifold C_0 , we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots. \quad (3.7)$$

In fact, z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q and $\bar{q}^*$. Note that W is real if z_t is real. We consider only real solutions. For the solution $z_t \in C_0$, since $\mu = 0$ and (3.3), we have

$$\begin{aligned} \dot{z} &= i\omega\tau_0 z + \langle q^*(\theta), f(0, W(z, \bar{z}, \theta) + 2\text{Re}[z(t)q(\theta)]) \rangle, \\ &= i\omega\tau_0 z + \bar{q}^* f(0, W(z, \bar{z}, 0) + 2\text{Re}[z(t)q(\theta)]), \end{aligned} \quad (3.8)$$

rewrite it as

$$\dot{z} = i\omega\tau_0 z + g(z, \dot{z}), \quad (3.9)$$

where

$$g(z, \dot{z}) = g_{20}(\theta) \frac{z^2}{2} + g_{11}(\theta) z\bar{z} + g_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots. \quad (3.10)$$

From (3.3) and (3.8), we have

$$\dot{W} = \dot{Z}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2\text{Re}\{\bar{q}^* f(z, \bar{z})q(\theta)\}, & -1 \leq \theta < 0, \\ AW - 2\text{Re}\{\bar{q}^* f(z, \bar{z})q(\theta)\} + f, & \theta = 0. \end{cases} \quad (3.11)$$

Rewrite (3.11) as

$$\dot{W} = AW + H(z, \bar{Z}, \theta), \quad (3.12)$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots. \quad (3.13)$$

Substituting the corresponding series into (3.12) and comparing the coefficient, we obtain

$$(A - 2i\omega\tau_0)W_{20}(\theta) = -H_{20}(\theta), AW_{11}(\theta) = -H_{11}(\theta). \quad (3.14)$$

Notice that

$$q(\theta) = (1, \beta)^T e^{i\omega\tau_0\theta}, \quad q^*(0) = G(\beta^*, 1),$$

and (3.6) we obtain

$$\begin{aligned} y_{1t}(0) &= z + \bar{z} + W_{20}^1(0) \frac{z^2}{2} + W_{11}^1(0) z\bar{z} + W_{02}^1(0) \frac{\bar{z}^2}{2} + \cdots, \\ y_{2t}(0) &= \beta z + \bar{\beta} \bar{z} + W_{20}^2(0) \frac{z^2}{2} + W_{11}^2(0) z\bar{z} + W_{02}^2(0) \frac{\bar{z}^2}{2} + \cdots, \\ y_{1t}(-1) &= \beta z e^{-i\omega\tau_0} + \bar{\beta} \bar{z} e^{i\omega\tau_0} + W_{20}^1(-1) \frac{z^2}{2} + W_{11}^1(-1) z\bar{z} + W_{02}^1(-1) \frac{\bar{z}^2}{2} + \cdots, \\ y_{2t}(-1) &= \beta z e^{-i\omega\tau_0} + \bar{\beta} \bar{z} e^{i\omega\tau_0} + W_{20}^2(-1) \frac{z^2}{2} + W_{11}^2(-1) z\bar{z} + W_{02}^2(-1) \frac{\bar{z}^2}{2} + \cdots. \end{aligned}$$

According to (3.8) and (3.9), we know that

$$g(z, \bar{z}) = \bar{q}^*(0) f_0(z, \bar{z}) = \bar{G}\tau_0(\bar{\beta}^*, 1) \begin{pmatrix} f_{11}^0 \\ f_{22}^0 \end{pmatrix}, \quad (3.15)$$

where

$$\begin{aligned} f_{11}^0 &= -by_{1t}(0)y_{2t}(-1), \\ f_{22}^0 &= -\frac{ky_0^2}{x_0^3} y_{1t}(0)^2 + \frac{2ky_0}{x_0} y_{1t}(0)y_{2t}(0) - \frac{k}{x_0} y_{2t}(0)^2 + \cdots. \end{aligned}$$

By (3.7), it follows that

$$\begin{aligned} g(z, \bar{z}) &= \bar{G}\tau_0 \{ -b\bar{\beta}^* [z + \bar{z} + W_{20}^1(0) \frac{z^2}{2} + W_{11}^1(0) z\bar{z} + W_{02}^1(0) \frac{\bar{z}^2}{2}] \\ &\quad \times [\beta z e^{-i\omega\tau_0} + \bar{\beta} \bar{z} e^{i\omega\tau_0} + W_{20}^2(-1) \frac{z^2}{2} + W_{11}^2(-1) z\bar{z} + W_{02}^2(-1) \frac{\bar{z}^2}{2}] \\ &\quad - \frac{ky_0^2}{x_0^3} [z + \bar{z} + W_{20}^1(0) \frac{z^2}{2} + W_{11}^1(0) z\bar{z} + W_{02}^1(0) \frac{\bar{z}^2}{2}]^2 \\ &\quad + \frac{2ky_0}{x_0} [z + \bar{z} + W_{20}^1(0) \frac{z^2}{2} + W_{11}^1(0) z\bar{z} + W_{02}^1(0) \frac{\bar{z}^2}{2}] \\ &\quad \times [\beta z + \bar{\beta} \bar{z} + W_{20}^2(0) \frac{z^2}{2} + W_{11}^2(0) z\bar{z} + W_{02}^2(0) \frac{\bar{z}^2}{2}] \\ &\quad - \frac{k}{x_0} [\beta z + \bar{\beta} \bar{z} + W_{20}^2(0) \frac{z^2}{2} + W_{11}^2(0) z\bar{z} + W_{02}^2(0) \frac{\bar{z}^2}{2}]^2 + \cdots \}. \end{aligned}$$

That is,

$$\begin{aligned}
g(z, \bar{z}) = & \bar{G}\tau_0\{z^2[-b\beta\bar{\beta}^*e^{-i\omega\tau_0} - \frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}\beta - \frac{k}{x_0}\beta^2] \\
& + z\bar{z}[-b\bar{\beta}^*(\bar{\beta}e^{i\omega\tau_0} + \beta e^{-i\omega\tau_0}) - 2\frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}(\bar{\beta} + \beta) - 2\frac{k}{x_0}\beta\bar{\beta}] \\
& + \bar{z}^2[-b\bar{\beta}^*\bar{\beta}e^{i\omega\tau_0} - \frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}\bar{\beta} - \frac{k}{x_0}\bar{\beta}^2] \\
& + z^2\bar{z}[(-b\beta\bar{\beta}^*e^{i\omega\tau_0} - 2\frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}\beta)W_{11}^1(0) + (\frac{2ky_0}{x_0} - 2\frac{k}{y_0})W_{11}^2(0) \\
& + (\frac{-b\beta\bar{\beta}^*e^{i\omega\tau_0}}{2} - \frac{ky_0^2}{x_0^3} + \frac{\frac{2ky_0}{x_0}\bar{\beta}}{2})W_{20}^1(0) + (\frac{\frac{2ky_0}{x_0}}{2} - \frac{k}{x_0}\bar{\beta})W_{20}^2(0) \\
& - b\bar{\beta}^*W_{11}^2(-1) + \frac{-b\bar{\beta}^*}{2}W_{20}^2(-1)] + \dots\}.
\end{aligned}$$

Comparing the coefficients with (3.10), it follows that

$$\begin{aligned}
g_{20} &= 2\bar{G}\tau_0[-b\beta\bar{\beta}^*e^{-i\omega\tau_0} - \frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}\beta - \frac{k}{x_0}\beta^2], \\
g_{11} &= \bar{G}\tau_0[-b\bar{\beta}^*(\bar{\beta}e^{i\omega\tau_0} + \beta e^{-i\omega\tau_0}) - 2\frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}(\bar{\beta} + \beta) - 2\frac{k}{x_0}\beta\bar{\beta}], \\
g_{02} &= [-b\bar{\beta}^*\bar{\beta}e^{i\omega\tau_0} - \frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}\bar{\beta} - \frac{k}{x_0}\bar{\beta}^2], \\
g_{21} &= [(-b\beta\bar{\beta}^*e^{i\omega\tau_0} - 2\frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}\beta)W_{11}^1(0) + (\frac{-b\beta\bar{\beta}^*e^{i\omega\tau_0}}{2} - \frac{ky_0^2}{x_0^3} + \frac{\frac{2ky_0}{x_0}\bar{\beta}}{2})W_{20}^1(0) \\
& + (\frac{2ky_0}{x_0} - 2\frac{k}{y_0})W_{11}^2(0) + (\frac{\frac{2ky_0}{x_0}}{2} - \frac{k}{x_0}\bar{\beta})W_{20}^2(0) - b\bar{\beta}^*W_{11}^2(-1) + \frac{-b\bar{\beta}^*}{2}W_{20}^2(-1)].
\end{aligned}$$

Now we compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (3.11) and (3.15), we have that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -2\text{Re}\{\bar{q}^*(0)f(z, \bar{z})q(\theta)\} = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \quad (3.16)$$

Comparing the coefficients with (3.13), we can obtain that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \quad (3.17)$$

Substituting the above equalities into (3.14), it follows that

$$\begin{cases} \dot{W}_{20}(\theta) = 2i\omega W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \\ \dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta). \end{cases} \quad (3.18)$$

Solving (3.18), we have

$$\begin{cases} W_{20}(\theta) = \frac{i\bar{g}_{20}}{\tau_0\omega}q(0)e^{i\omega\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega\tau_0}\bar{q}(0)e^{-i\omega\tau_0\theta} + Ee^{2i\omega\tau_0\theta}, \\ W_{11}(\theta) = -\frac{i\bar{g}_{11}}{\tau_0\omega}q(0)e^{i\omega\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega\tau_0}\bar{q}(0)e^{-i\omega\tau_0\theta} + F. \end{cases} \quad (3.19)$$

In what follows, we seek appropriate E and F in (3.19). From (3.11) and (3.15), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_0 \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad (3.20)$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_0 \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad (3.21)$$

where

$$d_1 = -b\beta^* e^{-i\omega\tau_0}, d_2 = -\frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}\beta - \frac{k}{x_0}\beta^2,$$

$$d_3 = -bRe(\beta^* e^{i\omega\tau_0}), d_4 = -\frac{ky_0^2}{x_0^3} + \frac{2ky_0}{x_0}Re(\beta) - \frac{k}{x_0}\beta\bar{\beta}.$$

Substituting (3.19)–(3.21) into (3.14) and noting that

$$(i\omega\tau_0 I - \int_{-1}^0 e^{i\omega\tau_0\theta} d\eta(\theta))q(0) = 0,$$

$$(-i\omega\tau_0 I - \int_{-1}^0 e^{-i\omega\tau_0\theta} d\eta(\theta))q(0) = 0.$$

We obtain

$$\begin{pmatrix} 2i\omega & bx_0 e^{-2i\omega\tau_0} \\ -\frac{ky_0^2}{x_0^2} & 2i\omega + \frac{ky_0}{x_0} \end{pmatrix} E = 2 \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad (3.22)$$

$$\begin{pmatrix} 0 & bx_0 \\ -\frac{ky_0^2}{x_0^2} & \frac{ky_0}{x_0} \end{pmatrix} F = 2 \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}. \quad (3.23)$$

It is easy to obtain E and F from (3.22) and (3.23), that is

$$E^{(1)} = \frac{(2i\omega + \frac{ky_0}{x_0})d_1 - bx_0 e^{-2i\omega\tau_0}}{\frac{2ky_0 i\omega}{x_0} + \frac{bky_0^2}{x_0} e^{-2i\omega\tau_0} - 4\omega^2},$$

$$E^{(2)} = \frac{\frac{ky_0^2}{x_0^2} d_3 + 2i\omega d_4}{\frac{2ky_0 i\omega}{x_0} + \frac{bky_0^2}{x_0} e^{-2i\omega\tau_0} - 4\omega^2},$$

$$F^{(1)} = \frac{\frac{ky_0}{x_0} d_3 - bx_0 d_4}{\frac{bky_0^2}{x_0}}, F^{(2)} = \frac{d_3}{bx_0}.$$

Therefore we can compute the following quantities

$$\mathcal{C}_1(0) = \frac{i}{2\tau_0\omega_0} (g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\text{Re}\mathcal{C}_1(0)}{\text{Re}\lambda'(\tau_0)}, t_2 = -\frac{\text{Im}\mathcal{C}_1(0) + \mu_2 \text{Im}\lambda'(\tau_0)}{\tau_0\omega_0}, \beta_2 = 2\text{Re}\mathcal{C}_1(0),$$

which determine the direction of Hopf bifurcation and stability of bifurcated periodic solutions of system (1.3) at the critical value τ_0 .

Theorem 3.1 (i) The direction of Hopf bifurcation is determined by the sign of μ_2 : the Hopf bifurcation is supercritical if $\mu_2 > 0$ and the Hopf bifurcation is subcritical if $\mu_2 < 0$.

(ii) The stability of bifurcated periodic solution is determined by β_2 : the periodic solution are stable if $\beta_2 > 0$ and unstable if $\beta_2 < 0$.

(iii) The period of bifurcation periodic solution is determined by t_2 : the period increase if $t_2 > 0$, decrease if $t_2 < 0$.

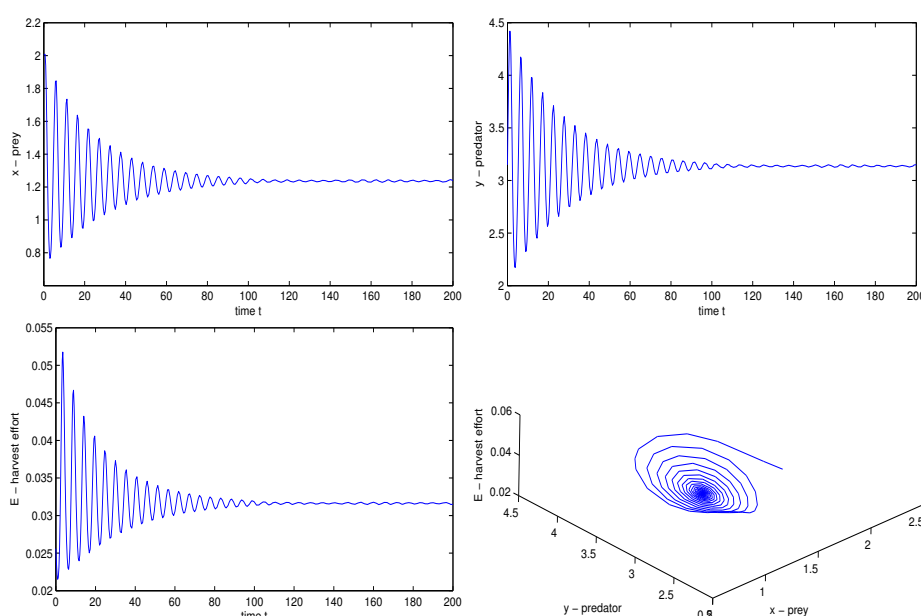


Figure 1: When $\tau = 0.62 < \tau_0$ and with initial conditions $x_0 = 2.0053$, $y_0 = 3.1480$, $E_0 = 0.0256$, that show the positive equilibrium point Y_0 is locally asymptotically stable.

4 Numerical Simulations

As an example we consider the differential-algebraic predator-prey system (1.3) with the parameters $r_1 = 1.6$, $r_2 = 1.3$, $b = k = m = 0.5$, $p_1 = 7$, $p_2 = 6$, $c_1 = 5$, $c_2 = 3$, that is,

$$\begin{cases} \frac{dx}{dt} = x(t)(1.6 - 0.5y(t - \tau) - E(t)), \\ \frac{dy}{dt} = y(t)(1.3 - 0.5\frac{y(t)}{x(t)} - E(t)), \\ 0 = E(t)(7x(t) + 6y(t) - 5 - 3) - 0.5. \end{cases} \quad (4.1)$$

And by the discussions in Section 2 and Section 3, we determine the stability of the positive equilibrium point and Hopf bifurcation. Here, for convenience, we only discuss one of the positive equilibrium point Y_0 of system (4.1), and others positive equilibrium points of system (4.1) can be similar studied. we can easily get $Y_0 = (2.0053, 3.1480, 0.0256)$, and by computing, we get $\omega_0 = 0.9942$, $\tau_0 = 0.6473$. So by Theorem 2.1, the equilibrium point Y_0 is asymptotically stable when $\tau \in [0, \tau_0) = [0, 0.6473)$ and unstable when $\tau > 0.6473$.

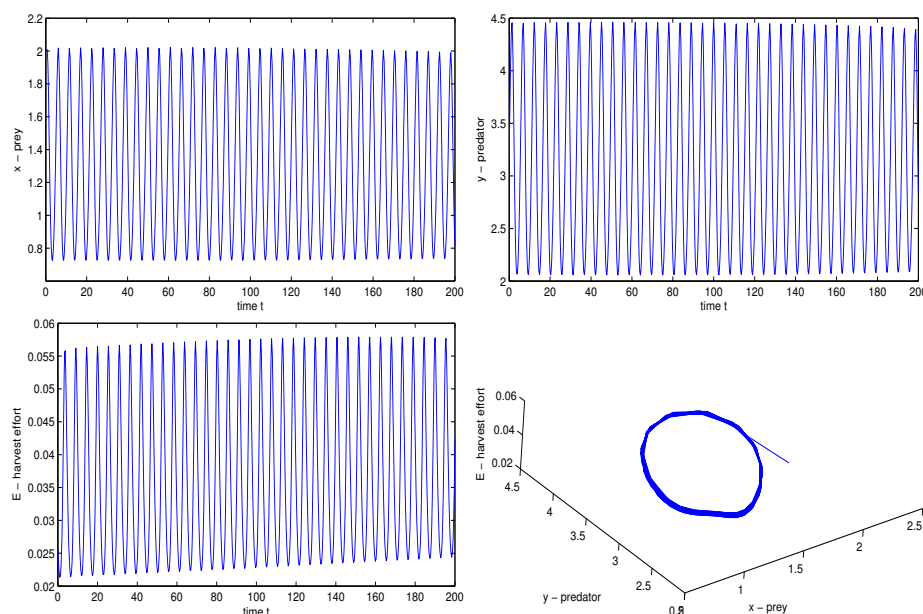


Figure 2: When $\tau = 0.682 > \tau_0$, with the same initial conditions above that shows the bifurcating periodic solutions from the positive equilibrium point Y_0 .

When $\tau = 0$, we can easily show that the positive equilibrium point

$$Y_0 = (2.0053, 3.1480, 0.0256)$$

is asymptotically stable.

By the theory of Hassard [15], as it is discussed in former section, we also determine the direction of Hopf bifurcation and the other properties of bifurcating periodic solution. By computing, we can obtain the following values $\mathcal{C}_1(0) = 0.5303 - 0.4428i$, $\lambda'(\tau_0) = 1.6352 + 1.1431i$, it follows that $\mu_2 = -0.3243 < 0$, $\beta_2 = 1.0607 > 0$, $t_2 = 1.2643 > 0$, from which and Theorem 3.1 we conclude that the Hopf bifurcation of system(4.1) occurring at $\tau_0 = 0.6473$ is subcritical and the bifurcating periodic solution exists when τ cross τ_0 to the left and the bifurcating periodic solution is unstable.

By Theorem 3.1, the positive equilibrium point Y_0 of system (4.1) is locally asymptotically stable when $\tau = 0.62 < \tau_0$ as is illustrated by computer simulation in Fig.1. And periodic solutions occur from Y_0 when $\tau = 0.682 > \tau_0$ as is illustrated by computer simulation in Fig.2.

5 Discussion

Nowadays, economic profit is a very important factor for governments, merchants, and even citizen, and the harvested biological resources in the predator-prey systems are usually sold as commodities in the market in order to achieve the economic interest. So modelling and qualitative analysis for bio-economic system are necessary.

Compared with most other researches on dynamics of predator-prey population, see [1, 5, 12, 18], the main contribution of this paper lies in the following aspect. The predator-prey system we consider incorporate delay and non-selective harvesting, which could make our model more realistic and the analysis result in this paper is more scientific. So our paper provide a new ideal and a efficacious method for the qualitative analysis of the Hopf bifurcation of the differential-algebraic biological economic system. In addition, stage structure, diffusion effects, disease effects may be incorporated into our bio-economic system, which would make the bio-economic system exhibit much more complicated dynamics.

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一类带无选择性捕获和时滞的捕食食饵系统的Hopf分支分析

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摘要: 本文主要研究了一个改进的带时滞和无选择捕获函数的捕食-食饵生态经济系统的稳定性和Hopf分支. 利用微分代数系统的稳定性理论和分支理论, 得到了系统正平衡点稳定性的条件, 以及当时滞 τ 作为分支参数时系统产生Hopf分支的条件. 对Leslie-Gower捕食-食饵模型进行了一定程度的完善, 使得建立的模型更符合实际情况, 因此得到的结论也更加科学.

关键词: 稳定性; Hopf分支; 时滞; 无选择性; 捕食食饵系统; 周期解

MR(2010)主题分类号: 34D20 ; 34K18 ; 34C23

中图分类号: O29; O193