GORENSTEIN FLAT (COTORSION) DIMENSIONS
AND HOPF ACTIONS

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Abstract: In this paper, we study the relationship of Gorenstein flat (cotorsion) dimensions between $A$-Mod and $A\#H$-Mod. Using the properties of separable functors, we get that (1) Let $A$ be a right coherent ring, assume that $A\#H/A$ is separable and $\varphi : A \to A\#H$ is a splitting monomorphism of $(A, A)$-bimodules, then $l.Gwd(A) = l.Gwd(A\#H)$; (2) Assume that $A\#H/A$ is separable and $\varphi : A \to A\#H$ is a splitting monomorphism of $(A, A)$-bimodules, then $l.Gcd(A) = l.Gcd(A\#H)$, which generalized the results in skew group rings.

Keywords: coherent ring; Gorenstein flat module; Gorenstein cotorsion module

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1 Introduction

The (Gorenstein) homological properties and representation dimensions for skew group algebras, or more generally, for smash products and crossed products were discussed by several authors, for example in [4, 13, 14, 16, 17, 20]. In [13], López-Ramos studied the relationship of Gorenstein injective (projective) dimensions between $A$-Mod and $A\#H$-Mod. He showed that under some conditions, $glGid(A) < \infty$ if and only if $glGid(A\#H) < \infty$ (resp. $glGpd(A) < \infty$ if and only if $glGpd(A\#H) < \infty$).

The aim of this paper is to study the relationship of Gorenstein flat (cotorsion) dimensions between $A$-Mod and $A\#H$-Mod. First we prove that over a right coherent ring $A$, if $A\#H/A$ is separable and $\varphi : A \to A\#H$ is a splitting monomorphism of $(A, A)$-bimodules, $l.Gwd(A) = l.Gwd(A\#H)$. Then we study the relationship of Gorenstein cotorsion dimensions between $A$-Mod and $A\#H$-Mod. We prove that if $A\#H/A$ is separable and $\varphi : A \to A\#H$ is a splitting monomorphism of $(A, A)$-bimodules, $l.Gcd(A) = l.Gcd(A\#H)$.

Next we recall some notions and facts required in the following.

Throughout this paper, $H$ always denotes a finite-dimensional Hopf algebra over $k$ with comultiplication $\Delta : H \otimes H \to H$, counit $\varepsilon : H \to k$ and antipode $S : H \to H$. A $k$-algebra $A$ is called a left $H$-module algebra if $A$ is a left $H$-module such that $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ and $h \cdot 1_A = \varepsilon(h)1_A$ for all $a, b \in A$ and $h \in H$. 

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Let $A$ be a left $H$-module algebra, the smash product algebra (or semidirect product) of $A$ with $H$, denoted by $A\#H$, is the vector space $A \otimes H$, whose elements are denoted by $a\#h$ instead of $a \otimes h$, with multiplication given by $(a\#h)(b\#l) = \Sigma a(h_{(1)} \cdot b)\#h_{(2)}l$ for $a, b \in A$ and $h, l \in H$. The unit of $A\#H$ is $1 \# 1$ and we usually view $ab$ as $a\#h$ and $ha$ as $(1\#h)(a\#1)$. In this paper, $A$-Mod and $A\#H$-Mod denote the categories of left $A$-modules and left $A\#H$-modules, respectively.

The notion of separable functor was introduced in [15]. Consider categories $C$ and $D$, a covariant functor $F : C \to D$ is said to be separable if for all $M, N \in C$ there are maps $\varphi_{M,N}^F : \text{Hom}_D(F(M), F(N)) \to \text{Hom}_C(M, N)$ satisfying the following conditions.

1. For $\alpha \in \text{Hom}_C(M, N)$, we have $\varphi_{M,N}^F(F(\alpha)) = \alpha$.
2. Given $M', N' \in C$, $\alpha \in \text{Hom}_C(M, M')$, $\beta \in \text{Hom}_C(N, N')$, $f \in \text{Hom}_D(F(M), F(N))$ and $g \in \text{Hom}_D(F(M'), F(N'))$ such that the following diagram commutes

$$
\begin{array}{c}
F(M) \xrightarrow{f} F(N) \\
\downarrow F(\alpha) \downarrow F(\beta) \\
F(M') \xrightarrow{g} F(N'),
\end{array}
$$

then the following diagram is also commutative

$$
\begin{array}{ccc}
M & \xrightarrow{\varphi_{M,N}^F(f)} & N \\
\downarrow \alpha & & \downarrow \beta \\
M' & \xrightarrow{\varphi_{M',N'}^F(g)} & N'.
\end{array}
$$

Let $\varphi : A \to A\#H$ denote the inclusion map. We can associate to $\varphi$ the restriction of scalars functor $A(-) : A\#H$-Mod $\to$ $A$-Mod, the induction functor $A\#H \otimes_A - = \text{Ind}(-) : A$-Mod $\to$ $A\#H$-Mod and the coinduction functor $\text{Hom}_A(A\#H, -) : A$-Mod $\to$ $A\#H$-Mod. It is well known that $A\#H \otimes_A -$ is left adjoint to $A(-)$ and that $\text{Hom}_A(A\#H, -)$ is right adjoint to $A(-)$. Since $H$ is a finite-dimensional Hopf algebra, by [6, Theorem 5], the functor $A\#H \otimes_A -$ is isomorphic to $\text{Hom}_A(A\#H, -)$. So we have a double adjunctions ($A\#H \otimes_A -$, $A(-)$) and ($A(-)$, $A\#H \otimes_A -$). Now we consider the separability of functors $A(-)$ and $A\#H \otimes_A -$. From [15, Proposition 1.3], we have the following

1. $A(-)$ is separable if and only if $A\#H/A$ is separable.
2. $A\#H \otimes_A - = \text{Ind}(-)$ is separable if and only if $\varphi$ splits as an $A$-bimodule map.

A left $R$-module $M$ is called Gorenstein flat [7] if there exists an exact sequence

$$
\cdots \to F^{-2} \to F^{-1} \to F^0 \to F^1 \to \cdots
$$

of flat left $R$-modules such that $M = \ker(F^0 \to F^1)$ and which remains exact whenever $E \otimes_R -$ is applied for any injective right $R$-module $E$. We will say that $M$ has Gorenstein flat dimension less than or equal to $n$ [10] if there exists an exact sequence

$$
0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0
$$
with every $F_i$ being Gorenstein flat. If no such finite sequence exists, define $Gfd_R(M) = \infty$; otherwise, if $n$ is the least such integer, define $Gfd_R(M) = n$. In [3] left weak Gorenstein global dimension of $R$ was defined as $l.Gwd(R) = \sup\{Gfd_R(M) | M \text{ is any left } R\text{-module}\}$. A left $R$-module $M$ is called Gorenstein cotorsion [8] if $\text{Ext}^1_R(N, M) = 0$ for all Gorenstein flat left $R$-modules $N$. We will say that $M$ has Gorenstein cotorsion dimension less than or equal to $n$ [12] if there exists an exact sequence

$$0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^n \rightarrow 0$$

with every $C^i$ being Gorenstein cotorsion. The left global Gorenstein cotorsion dimension $l.Gcd(R)$ of $R$ is defined as the supremum of the Gorenstein cotorsion dimensions of left $R$-modules.

### 2 Gorenstein Flat Modules and Actions of Finite-Dimensional Hopf Algebras

In this paper, $\varphi : A \rightarrow A#H$ always denotes the inclusion map. If $M \in A#H$-Mod, then $AM$ will denote the image of $M$ by the restriction of the scalars functor $A(-) : A#H$-Mod$\rightarrow$A-Mod.

**Lemma 2.1** (see [11, Corollary 3.6A]) Let $\eta : R \rightarrow S$ be a ring homomorphism such that $S$ becomes a flat left $R$-module under $\eta$. Then, for any injective module $M_S$, the right $R$-module $M$ (obtained by pullback along $\eta$) is also injective.

**Remark 2.2** Let $\varphi : A \rightarrow A#H$ be the inclusion map. Since $A#H$ is free as a left $A$-module, then from Lemma 2.1 we know that for any injective right $A#H$-module $M$, the right $A$-module $M$ (obtained by pullback along $\varphi$) is also injective.

**Proposition 2.3** (1) If $M \in A$-Mod is Gorenstein flat, then $A#H \otimes_A M$ is Gorenstein flat as a left $A#H$-module.

(2) If $M \in A#H$-Mod is Gorenstein flat, then $AM$ is Gorenstein flat as a left $A$-module.

**Proof**

(1) Since $M$ is a Gorenstein flat left $A$-module, we have an exact sequence

$$\mathfrak{F} \equiv \cdots \rightarrow F^{-2} \rightarrow F^{-1} \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

of flat left $A$-modules such that $M = \ker(F^0 \rightarrow F^1)$ and which remains exact whenever $E \otimes_A -$ is applied for any injective right $A$-module $E$.

Since $A#H$ is free as a right $A$-module by [5, Proposition 6.1.7] and $A#H \otimes_A -$ preserves flat modules, we get that $A#H \otimes_A \mathfrak{F}$ is an exact sequence of flat left $A#H$-modules and

$$A#H \otimes_A M = \ker(A#H \otimes_A F^0 \rightarrow A#H \otimes_A F^1).$$

Finally, let $E'$ be any injective right $A#H$-module. Then $E' \otimes_{A#H} (A#H \otimes_A \mathfrak{F}) \cong (E' \otimes_{A#H} A#H) \otimes_A \mathfrak{F}$ is exact since $E' \otimes_{A#H} A#H \cong E'$ (as right $A$-modules) is injective by Remark 2.2. Thus $A#H \otimes_A M$ is Gorenstein flat.
(2) Let $M \in A^\#H$-Mod be Gorenstein flat, then we have an exact sequence
\[ \mathfrak{S}' \equiv \cdots \rightarrow F'_{r-2} \rightarrow F'_{r-1} \rightarrow F'_{r0} \rightarrow F'_{r1} \rightarrow \cdots \]
of flat left $A^\#H$-modules such that $M = \ker(F'_{r0} \rightarrow F'_{r1})$ and which remains exact whenever $E \otimes_{A^\#H} -$ is applied for any injective right $A^\#H$-module $E$. Then $A^\#H$ is a right coherent ring. Consider the adjoint pair $(\cdot \otimes_A (-), (- \otimes_A \cdot))$ of flat left $A$-modules since the functor $\cdot(-)$ is exact and preserves flat modules.

Finally, let $E'$ be any injective right $A$-module. Then
\[ E' \otimes_A (A^\#H) \cong E' \otimes_A (A^\#H \otimes_A \mathfrak{S}') \cong (E' \otimes_A A^\#H) \otimes_A \mathfrak{S}' \quad (*). \]
Since $H$ is a finite-dimensional Hopf algebra, by [6, Theorem 5], we can easily get that $E' \otimes_A A^\#H$ is injective as a right $A^\#H$-module. By $(*)$ we know that $E' \otimes_A (A^\#H)$ is exact. Therefore $A^\#H$ is Gorenstein flat.

**Proposition 2.4** Assume that $A^\#H/A$ is separable and $\varphi : A \rightarrow A^\#H$ is a splitting monomorphism of $(A, A)$-bimodules. Then $A$ is a right coherent ring if and only if $A^\#H$ is a right coherent ring.

**Proof** Let $\{F_i\}_{i \in I}$ be a family of flat left $A^\#H$-modules, then $\varphi(F_i)$ is flat as a left $A$-module for every $i$. If we consider the adjoint pair $(A^\#H \otimes_A (-), A(-))$, we know that $A(-)$ preserves inverse limits. Thus $A(\prod F_i) \cong \prod A(F_i)$. Since $A$ is a right coherent ring, $A(\prod F_i) \cong \prod A(F_i)$ is flat as a left $A$-module. Then, we get that $\prod F_i$ is a flat left $A^\#H$-module. Thus $A^\#H$ is a right coherent ring.

Conversely, let $\{F_i\}_{i \in I}$ be a family of flat left $A$-modules, since $A^\#H \otimes_A -$ preserves flat modules, we know that $A^\#H \otimes_A F_i$ is flat as a left $A^\#H$-module for every $i$. If we consider the adjoint pair $(A(-), A^\#H \otimes_A -)$, we know that $A^\#H \otimes_A -$ preserves inverse limits. Thus

\[ A^\#H \otimes_A (\prod F_i) \cong \prod A^\#H \otimes_A F_i. \]
Since $A^\#H$ is a right coherent ring, $A^\#H \otimes_A (\prod F_i) \cong \prod A^\#H \otimes_A F_i$ is flat as a left $A^\#H$-module. Then, we get that $A(\prod F_i)$ is a flat left $A$-module. Since $\varphi : A \rightarrow A^\#H$ is a splitting monomorphism of $(A, A)$-bimodules, we get that the functor $A^\#H \otimes_A -$ is separable by [15, Proposition 1.3]. Consider the adjoint pair $(A^\#H \otimes_A -, A(-))$, by [9, Proposition 5] we know that the natural map $\eta_M : M \rightarrow (A^\#H \otimes_A M)$ is a split monomorphism for every $M \in A$-Mod. Then $\prod F_i$ is a direct summand of $A(\prod F_i)$. Hence $\prod F_i$ is flat as a left $A$-module since the class of flat modules is closed under direct summands. Thus $A$ is a right coherent ring.

Next we consider the relationship of the left weak Gorenstein global dimensions in $A$-Mod and $A^\#H$-Mod when $A$ is a right coherent.

**Theorem 2.5** Let $A$ be a right coherent ring. Assume that $A^\#H/A$ is separable and $\varphi : A \rightarrow A^\#H$ is a splitting monomorphism of $(A, A)$-bimodules. Then $l.Gwd(A) = l.Gwd(A^\#H)$.

**Proof** For every $n$, we need to show that $Gfd_A(M) \leq n$ for every left $A$-module $M$ if and only if $Gfd_{A^\#H}(N) \leq n$ for every left $A^\#H$-module $N$. 
Suppose that $l.Gwd(A\#H) = n$ and let $M$ be any $A$-module. From Proposition 2.3 we know that $A\#H \otimes_A -$ and $A(-)$ both preserve Gorenstein flat modules. Thus

$$Gfd_{A\#H}(A\#H \otimes_A M) \leq n, \quad Gfd_A(A\#H \otimes_A M) \leq n.$$ 

Since $A\#H \otimes_A -$ is separable, $M$ is a direct summand of $A(A\#H \otimes_A M)$. Since $A$ is a right coherent ring, by [2, Propositions 2.2 and 2.10] we know that $Gfd_A(M) \leq n$.

Since $A\#H/A$ is separable, $A(-)$ is separable by [15, Proposition 1.3]. Similarly, we can prove that if $l.Gwd(A) \leq n$ then $l.Gwd(A\#H) \leq n$.

**Lemma 2.6** (1) If $N \in A$-Mod is Gorenstein cotorsion, then $A\#H \otimes_A N$ is Gorenstein cotorsion as a left $A\#H$-module.

(2) If $N \in A\#H$-Mod is Gorenstein cotorsion, then $A N$ is Gorenstein cotorsion as a left $A$-module.

(3) Let $M \in A\#H$-Mod and $A\#H/A$ be separable. Then $M$ is Gorenstein cotorsion as a left $A\#H$-module if and only if $A M$ is Gorenstein cotorsion as a left $A$-module.

**Proof** (1) Let $N$ be any Gorenstein cotorsion left $A$-module and $F$ any Gorenstein flat left $A\#H$-module. For $F$ we have an exact sequence $0 \to K \to P \to F \to 0 \ (*)$ of left $A\#H$-modules with $P$ projective. Since $A(-)$ is exact and preserves Gorenstein flat and projective modules, we have an exact sequence $0 \to_A K \to_A P \to_A F \to 0$ with $A P$ projective and $A F$ Gorenstein flat. Hence we have the following commutative diagram:

$$\begin{array}{cccccc}
0 & \to & \text{Hom}_{A\#H}(F, A\#H \otimes_A N) & \to & \text{Hom}_{A\#H}(P, A\#H \otimes_A N) & \to & \text{Hom}_{A\#H}(K, A\#H \otimes_A N) \\
\sigma_1 \downarrow & & \sigma_2 \downarrow & & \sigma_3 \downarrow & & \\
0 & \to & \text{Hom}_A(A F, N) & \to & \text{Hom}_A(A P, N) & \to & \text{Hom}_A(A K, N) & \to & 0.
\end{array}$$

Note that $\sigma_1$, $\sigma_2$ and $\sigma_3$ are isomorphisms by adjoint isomorphism. Hence

$$\text{Hom}_{A\#H}(P, A\#H \otimes_A N) \to \text{Hom}_{A\#H}(K, A\#H \otimes_A N)$$

is an epimorphism.

Applying the functor $\text{Hom}_{A\#H}(-, A\#H \otimes_A N)$ to $(*), we get a long exact sequence

$$0 \to \text{Hom}_{A\#H}(F, A\#H \otimes_A N) \to \text{Hom}_{A\#H}(P, A\#H \otimes_A N) \to \text{Hom}_{A\#H}(K, A\#H \otimes_A N) \to$$

$$\to Ext^1_{A\#H}(F, A\#H \otimes_A N) \to Ext^1_{A\#H}(P, A\#H \otimes_A N) = 0.$$ 

Since

$$\text{Hom}_{A\#H}(P, A\#H \otimes_A N) \to \text{Hom}_{A\#H}(K, A\#H \otimes_A N)$$

is an epimorphism, we know that $Ext^1_{A\#H}(F, A\#H \otimes_A N) = 0$ for any Gorenstein flat left $A\#H$-module $F$. Hence $A\#H \otimes_A N$ is a Gorenstein cotorsion left $A\#H$-module.

(2) Similarly, using the adjoint pair $(A\#H \otimes_A -, A(-))$ we can prove that $A(-)$ preserves Gorenstein cotorsion modules.
3) The “only if” part can be gotten directly by (2).

Conversely, if \( A \) is Gorenstein cotorsion as a left \( A \)-module, then by (1) we know that \( A \# H \otimes_A M \) is Gorenstein cotorsion. Since \( A \# H/A \) is separable, the functor \( A(-) \) is separable by [15, Proposition 1.3]. Consider the adjoint pair \( (A(-), A \# H \otimes_A -) \), by [9, Proposition 5] we know that the natural map \( \eta_M : M \to A \# H \otimes_A A M \) is a split monomorphism for every left \( A \# H \)-module \( M \). Then \( M \) is a direct summand of \( A \# H \otimes_A A M \). Hence \( M \) is Gorenstein cotorsion as a left \( A \# H \)-module since the class of Gorenstein cotorsion modules is closed under direct summands.

**Proposition 2.7** Let \( M \in A \# H \)-Mod and \( N \in A \)-Mod. Then

1. \( \text{Gcd}_A(A M) \leq \text{Gcd}_{A \# H}(M) \).
2. \( \text{Gcd}_{A \# H}(A \# H \otimes_A N) \leq \text{Gcd}_A(N) \).

**Proof** (1) Assume that \( \text{Gcd}_{A \# H}(M) = n < \infty \), then there exists an exact sequence of left \( A \# H \)-modules

\[
0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to 0
\]

with every \( C^i \) being Gorenstein cotorsion. By Lemma 2.6, \( A(-) \) preserves Gorenstein cotorsion modules, we have an exact sequence of left \( A \)-modules

\[
0 \to A M \to A C^0 \to A C^1 \to \cdots \to A C^n \to 0
\]

with every \( C^i \) being Gorenstein cotorsion. Thus \( \text{Gcd}_A(A M) \leq \text{Gcd}_{A \# H}(M) \).

(2) Similarly, using Lemma 2.6, we can get that

\[
\text{Gcd}_{A \# H}(A \# H \otimes_A N) \leq \text{Gcd}_A(N).
\]

**Theorem 2.8** Assume that \( A \# H/A \) is separable and \( \varphi : A \to A \# H \) is a splitting monomorphism of \( (A, A) \)-bimodules, then \( l.\text{Gcd}_A(A) = l.\text{Gcd}_A(A \# H) \).

**Proof** Let \( M \) be any left \( A \)-module. Since \( \varphi : A \to A \# H \) is a splitting monomorphism of \( (A, A) \)-bimodules, \( M \) is a direct summand of \( A(A \# H \otimes_A M) \). Hence

\[
\text{Gcd}_A(M) \leq \text{Gcd}_A(A(A \# H \otimes_A M)).
\]

By Proposition 2.7,

\[
\text{Gcd}_A(A(A \# H \otimes_A M)) \leq \text{Gcd}_{A \# H}(A \# H \otimes_A M) \leq l.\text{Gcd}(A \# H).
\]

Thus \( l.\text{Gcd}(A) \leq l.\text{Gcd}(A \# H) \).

Let \( N \) be any left \( A \# H \)-module. Since \( A \# H/A \) is separable, \( N \) is a direct summand of \( A \# H \otimes_A A N \). Hence

\[
\text{Gcd}_{A \# H}(N) \leq \text{Gcd}_{A \# H}(A \# H \otimes_A A N).
\]

By Proposition 2.7,

\[
\text{Gcd}_{A \# H}(A \# H \otimes_A A N) \leq \text{Gcd}_A(A N) \leq l.\text{Gcd}(A).
\]
Thus $l.Gcd(A \# H) \leq l.Gcd(A)$.

**Corollary 2.9** Let $A$ be a $k$-algebra and $G$ a finite group with $|G|^{-1} \in k$. Then $l.Gcd(A) = l.Gcd(A * G)$.

**Proof** By the definition of the skew group ring, we know that $A$ is a left $H$-module algebra and $A * G = A \# H$, where $H = kG$. Since $G$ a finite group with $|G|^{-1} \in k$, $H$ is semisimple. Then from [19], we know that $A \# H/A$ is separable. By [1, Lemma 4.5], we know that $A$ is a direct summand of $A \# H$ as $(A, A)$-bimodule. By Theorem 2.8 we immediately get the desired result.

**References**


Gorenstein平坦(余挠)维数和Hopf作用

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摘要: 设$H$是域$k$上的有限维Hopf代数. $A$是左$H$-模代数. 本文研究了Gorenstein平坦(余挠)维数在$A$-模范畴和$A\#H$-模范畴之间的关系. 利用可分函子的性质, 证明了(1) 设$A$是右凝聚环, 若$A\#H/A$可分且$\varphi:A\to A\#H$是可积的$(A, A)$-双模同态, 则$l.Gwd(A) = l.Gwd(A\#H)$; (2) 若$A\#H/A$可分且$\varphi:A\to A\#H$是可积的$(A, A)$-双模同态, 则$l.Gcd(A) = l.Gcd(A\#H)$, 推广了斜群环上的结果.

关键词: 凝聚环; Gorenstein平坦模; Gorenstein余挠模
