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OPTIMAL EXISTENCE OF SYMMETRIC POSITIVE SOLUTIONS FOR A FOURTH-ORDER SINGULAR BOUNDARY VALUE PROBLEM

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Abstract: In this paper, we study a fourth-order singular boundary value problem. Using the Leggett-Williams fixed point theorem together with constructing a special cone, we establish optimal existence of symmetric positive solutions for a fourth-order singular boundary value problem under certain conditions, which generalizes optimal existence of symmetric positive solutions to singular boundary value problem.

Keywords: symmetric positive solutions; boundary value problem; cone

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1 Introduction

We consider existence of symmetric positive solutions for a fourth-order singular boundary value problem:

$$\begin{cases} x^{(4)}(t) + f(t, x(t)) = 0,\ 0 < t < 1, \\ x(0) = x(1) = x^{'}(0) = x^{'}(1) = 0, \end{cases} \tag{1}$$

which describes the deformations of an elastic beam with both endpoints fixed, where $f : (0,1) \times (0,+\infty) \to (0,+\infty)$ is conditions and $f(t,x) = f(1-t,x)$ for each $(0,1) \times (0,+\infty)$. $f(t,x(t))$ may be singular at $t = 0$ and/or $t = 1$.

Here symmetric positive solutions for a fourth-order singular boundary value problem (1) satisfying $x(t) = x(1-t)$ and $x(t) > 0, t \in (0,1)$.

Boundary value problems arise in a variety of different areas of applied mathematics and physics (see [1, 2] and the references therein). Recently many authors studied the existence of positive solutions for four-order singular boundary value problems for example [3–13] and the references therein. Most of these results are obtained via transforming the four-order boundary value problems into a second-order boundary value problems, and then

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applying the Leray-Schauder continuation method, the topologial degree theory, the fixed point theorems on cones, the critical point theory, or the lower and upper solution method. However results about the existence of symmetric positive solutions to singular boundary value problem (1) are few. Motivated by the results in [9, 11] we try to establish optimal existence of symmetric positive solutions to problem (1) by applying Leggett-Williams fixed point theorem.

2 Preliminary

We consider problem (1) in a Banach space $C[0,1]$ equipped with the norm $\|x\| = \max\limits_{0\le t\le 1}|x(t)|$. A function $x(t)\in C[0,1]$ is said to be a concave function if $x(\tau t_1+(1-\tau)t_2)\ge \tau x(t_1)+(1-\tau)x(t_2)$ for all $t_1,t_2,\tau\in[0,1]$. We denote

$$C^+[0,1]=\{x\in C[0,1]: x(t)\ge 0, t\in[0,1]\}.$$

Let K be a cone of $C[0,1]$ and m,n be constants, $0<m<n$. Define

$$K_r=\{x\in K \mid \|x\|<r\}, \overline{K_r}=\{x\in K\mid \|x\|\le r\},$$
$$K(u,m,n)=\{x\in K\mid m\le u(x), \|x\|\le n\}.$$

Let $G(t,s)$ be the Green's function of the corresponding boundary value problem (1), i.e.,

$$G(t,s)=\begin{cases}\frac{1}{6}t^2(1-s)^2[(s-t)+2(1-t)s], & 0\le t\le s\le 1,\\ \frac{1}{6}s^2(1-t)^2[(t-s)+2(1-s)t], & 0\le s\le t\le 1,\end{cases}$$

and $G(\tau(s),s)=\max\limits_{0\le t\le 1}G(t,s)$, where

$$\tau(s)=\begin{cases}\frac{1}{3-2s}, & 0\le s\le \frac{1}{2},\\ \frac{2s}{1+2s}, & \frac{1}{2}\le s\le 1.\end{cases}$$

After a simple calculation, we get

(I) $\displaystyle\int_0^1 G(\tau(s),s)ds=\max_{0\le t\le 1}\int_0^1 G(t,s)ds=\max_{0\le t\le 1}\frac{1}{24}t^2(1-t)^2=\frac{1}{384}$;

(II) $G(1-t,1-s)=G(t,s)$;

(III) $\displaystyle\min_{0\le c\le 1}\frac{G(\frac{1}{4},c)}{G(\frac{1}{2},c)}=\frac{1}{4}$;

(IV) (see [9]) $q(t)G(\tau(s),s)\le G(t,s)\le G(\tau(s),s), q(t)=\min\{t^2,(1-t)^2\}, t\in[0,1]$.

Lemma 2.1 (see [14]) Let $A:K\to K$ be a completely continuous operator, u be a nonnegative continuous concave function on K, and satisfies $u(x)\le\|x\|$ for all $x\in\overline{K_r}$. In addition, assume that there exist $0<d<m<n\le r$ satisfy the following conditions:

(i) $\{x\in K(u,m,n)\mid u(x)>m\}\ne\varnothing$, and $u(Ax)>m$ for $x\in K(u,m,n)$;

(ii) $\|Ax\|\le d$ for $x\in\overline{K_d}$;

(iii) $u(Ax)>m$ for $x\in K(u,m,r)$ and $\|Ax\|>n$;

then A has at least three fixed points x_1,x_2,x_3 on $\overline{K_r}$ satisfy $\|x_1\|<d, m<u(x_2)$, and $\|x_3\|>d$ for $u(x_3)<m$.

3 Main Results

Theorem 3.1 Suppose the following conditions hold:

(H1) $f \in C((0,1)\times[0,+\infty),[0,+\infty)), f(t,x) \le g(t)h(x), g \in C((0,1),[0,+\infty)), h \in C([0,+\infty),[0,+\infty))$;

(H2) $0 < \int_0^1 G(\tau(s),s)g(s)ds < +\infty$;

(H3) There exist $0 < d < m < \frac{r}{2}$ such that

1) $h(x) \le d[\int_0^1 G(\tau(s),s)g(s)ds]^{-1}$ for $0 \le x \le d$;

2) $f(t,x) > 16m[\int_0^1 G(\tau(s),s)ds]^{-1} = 6144m$ for $m \le x \le 2m$;

3) $h(x) < r[\int_0^1 G(\tau(s),s)g(s)ds]^{-1}$ for $0 \le x \le r$;

then problem(1) has triple symmetric positive solutions x_1, x_2, x_3 satisfy $\|x_1\| < d, m < u(x_2)$, and $\|x_3\| > d$ for $u(x_3) < m$.

Proof Denote $K = \{x \in C^+[0,1] : x(t)$ is convex function and $x(t) = x(1-t), t \in [0,1]\}$, then K is a cone of $C^+[0,1]$.

Let $u(x) = \min\limits_{\frac14 \le t \le \frac34} x(t)$ for $x \in K$. then $u(x) = x(\frac14) \le x(\frac12) = \|x\|$. It is well known that $x(t)$ is a positive solution of problem (1) if only if $x(t)$ is a positive solution of the equation $x(t) = \int_0^1 G(t,s)f(s,x(s))ds$.

Define operator $A : K \to K$ by $Ax(t) = \int_0^1 G(t,s)f(s,x(s))ds$. Obviously $Ax(t) \ge 0, (Ax)''(t) < 0$ for $0 < t < 1$, and for $x \in K$,

$$
\begin{aligned}
&Ax(1-t) = \int_0^1 G(1-t,s)f(s,x(s))ds\\
=\ & \int_0^{1-t} G(1-t,s)f(s,x(s))ds + \int_{1-t}^1 G(1-t,s)f(s,x(s))ds\\
=\ & -\int_1^t G(1-t,1-r)f(1-r,x(1-r))dr - \int_t^0 G(1-t,1-r)f(1-r,x(1-r))dr\\
=\ & \int_t^1 G(t,r)f(r,x(r))dr + \int_0^t G(t,r)f(r,x(r))dr = Ax(t)
\end{aligned}
$$

consequently $Ax \in K$, that is $A : K \to K$. By Arzela-Ascoli theorem, we can prove $A : K \to K$ is completely continuous.

From (H1) and 3) in (H3), for any $x \in \overline{K_r}$, we know that

$$
\begin{aligned}
\|Ax\| &\le \int_0^1 G(\tau(s),s)g(s)h(x(s))ds\\
&\le \int_0^1 G(\tau(s),s)g(s)ds \cdot r[\int_0^1 G(\tau(s),s)g(s)ds]^{-1} = r.
\end{aligned}
$$

So $A(\overline{K_r}) \subset \overline{K_r}$. Choose $x(t) = 2m, 0 < t < 1$, then $x(t) \in K(u, m, 2m)$, and $u(x) = u(2m) > m$. Then $\{x \in K(u, m, 2m) \mid u(x) > m\} \neq \varnothing$. And for $x \in K(u, m, 2m), u(x) = x(\frac{1}{4}) \geq m$. Hence $m \leq x(s) \leq 2m, \frac{1}{4} \leq s \leq \frac{3}{4}$. Thus for any $x \in K(u, m, 2m)$, from 2) in (H3), we obtain

$$\begin{aligned} u(Ax) &= \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} Ax(t) = \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \int_0^1 G(t,s)f(s,x(s))ds \\ &\geq \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \int_0^1 q(t)G(\tau(s),s)f(s,x(s))ds \\ &\geq \frac{1}{16}\int_0^1 G(\tau(s),s)f(s,x(s))ds \\ &\geq \frac{1}{16}\cdot 6144m\int_0^1 G(\tau(s),s)ds = m. \end{aligned}$$

Thus condition (i) of Lemma 2.1 holds.

Next from (H1) and 1) in (H3), for any $x \in \overline{K_d}$, we have

$$\begin{aligned} \|Ax\| &\leq \max_{0\leq t\leq 1}\int_0^1 G(t,s)g(s)h(x(s))ds \\ &\leq \int_0^1 G(\tau(s),s)g(s)ds\cdot d[\int_0^1 G(\tau(s),s)g(s)ds]^{-1} = d. \end{aligned}$$

So $A : \overline{K_d} \to \overline{K_d}$. Thus condition (ii) of Lemma 2.1 follows.

Finally we prove $u(Ax) > m$ for $x \in K(u, m, r)$ and $\|Ax\| > 4m$.

From 2) in (H3), for $x \in K(u, m, r)$ and $\|Ax\| > 4m$, we know that

$$\begin{aligned} u(Ax) &= Ax(\frac{1}{4}) = \int_0^1 G(\frac{1}{4},s)f(s,x(s))ds \\ &= \int_0^1 \frac{G(\frac{1}{4},s)}{G(\frac{1}{2},s)}G(\frac{1}{2},s)f(s,x(s))ds \\ &\geq \min_{0\leq c\leq 1}\frac{G(\frac{1}{4},c)}{G(\frac{1}{2},c)}\int_0^1 G(\frac{1}{2},s)f(s,x(s))ds \\ &= \frac{1}{4}Ax(\frac{1}{2}) = \frac{1}{4}\|Ax\| > m. \end{aligned}$$

Therefore condition (iii) of Lemma 2.1 holds too. The proof is completed.

Remark Theorem 3.1 also holds when nonlinearity $f(t, x(t))$ is nonsingular at $t = 0$ and $t = 1$.

4 Example

Example 4.1 The following boundary value problem:

$$\begin{cases} x^{(4)}(t) + \frac{h(x)}{t^2(1-t)^2} = 0,\ 0 < t < 1, \\ x(0) = x(1) = x^{'}(0) = x^{'}(1) = 0 \end{cases} \tag{2}$$

has triple symmetric positive solutions, where

$$h(x) = \begin{cases} 4x^2, & 0 \le x < 2, \\ \frac{5760}{2x-1}, & x \ge 2. \end{cases}$$

Proof Let $f(t,x) = h(x)g(t), g(t) = \frac{1}{t^2(1-t)^2}$. Obviously $g(t)$ is signular at $t = 0$ and $t = 1$. $h(x) \in C[0,+\infty)$. So (H1) holds.

Since

$$\begin{aligned} G(\tau(s),s) &= \begin{cases} \frac{2s^2(1-s)^3}{3(3-2s)^2}, & 0 \le s \le \frac{1}{2}, \\ \frac{2s^3(1-s)^2}{3(1+2s)^2}, & \frac{1}{2} \le s \le 1. \end{cases} \\ \int_0^1 G(\tau(s),s)g(s)ds &= \frac{2}{3}\int_0^{\frac{1}{2}} \frac{1-s}{(3-2s)^2}ds + \frac{2}{3}\int_{\frac{1}{2}}^1 \frac{s}{(1+2s)^2}ds \\ &= \frac{1}{3}(\ln 3 - \ln 2) - \frac{1}{12} \approx 0.05 > 0, \end{aligned}$$

then (H2) holds.

1) In (H3) followings from $[\int_0^1 G(\tau(s),s)g(s)ds]^{-1} \approx 20$, we may take $d = \frac{1}{4}$ then

$$h(x) = 4x^2 \le 4d^2 = \frac{1}{4} < d[\int_0^1 G(\tau(s),s)g(s)ds]^{-1} \approx \frac{20}{4} \text{ for } 0 \le x \le d = \frac{1}{4}.$$

2) In (H3) is immediate, since we may take $m = 2$ then

$$\begin{aligned} f(t,x) &= h(x)g(t) = \frac{1}{t^2(1-t)^2}\frac{5760}{2x-1} \\ &\ge 16\frac{5760}{2x-1} > 16h(4) > 2 \times 6144 = 6144m, \ \ 2 \le x \le 4, 0 < t < 1. \end{aligned}$$

3) In (H3) is immediate, since we may take $r = 100 > 2m = 4$ then

$$\begin{aligned} \max_{0 \le x \le 100} h(x) &\le h(2) = \frac{5760}{3} \\ &< 100[\int_0^1 G(\tau(s),s)g(s)ds]^{-1} \approx 2000, \ \ 0 \le x \le r = 100. \end{aligned}$$

Thus from Theorem 3.1, we know that problen (2) has triple symmetric positive solutions x_1, x_2, x_3 satisfy $\|x_1\| < \frac{1}{4}$, $2 < u(x_2)$, and $\|x_3\| > \frac{1}{4}$ for $u(x_3) < 2$.

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一类四阶奇异边值问题对称正解的最优存在性

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摘要: 本文研究了一类四阶奇异边值问题. 通过建立一个特定的锥, 利用Leggett-Williams 不动点定理, 从而在一定的条件下得到一类四阶奇异边值问题对称正解的最优存在性, 推广了奇异边值问题对称正解的最优存在性的结果.

关键词: 对称正解; 边值问题; 锥

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