

RECOLLEMENT OF COHERENT FUNCTOR CATEGORIES OVER TRIANGULATED CATEGORIES

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Abstract: The relationship between recollement of triangulated category \mathcal{D} and recollement of its coherent functor category $\mathcal{A}(\mathcal{D})$ is studied. It is shown that the recollement of \mathcal{D} induces the prerecollement of $\mathcal{A}(\mathcal{D})$ and the necessary and sufficient condition for it to be a recollement is that the recollement of \mathcal{D} is split. Furthermore, we get the result that the recollement of \mathcal{D} could induce the prerecollement of $\mathcal{A}(\mathcal{D})$.

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1 Introduction

The notion of recollement of triangulated categories was introduced by Beilinson, Bernstein and Deligne [1] in connection with derived categories of sheaves of topological spaces in 1982. Besides the recollement of triangulated categories, MacPherson and Vilonen [2] introduced recollement of abelian categories, it first appeared as an inductive step in the construction of perverse sheaves. Recollements of abelian and triangulated categories play an important role in geometry of singular spaces. It is a basic problem of recollement that constructing a new recollement from the known recollement (see [3–5]).

The abelianization of a triangulated category is due to the work of Verdier and Freyd, however Krause (see [6, 7]) gave a slightly different construction which is based on coherent functors in the sense of Auslander [8].

Abelian category and triangulated category are two fundamental structures in represent theory of algebra. The author studied the relationship between torsion theory of triangulated category \mathcal{D} and that of its coherent functor category $\mathcal{A}(\mathcal{D})$ in [9].

In this paper, we mainly study the relationship between recollement of triangulated category \mathcal{D} and recollement of its coherent functor category $\mathcal{A}(\mathcal{D})$. We show that the recollement of \mathcal{D} induces the prerecollement of $\mathcal{A}(\mathcal{D})$. Furthermore, the necessary and sufficient

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condition for it to be a recollement is that the recollement of \mathcal{D} is split, i.e., for any $Y \in \mathcal{D}$, the triangles $i_!i^!Y \xrightarrow{f_1} Y \xrightarrow{g_1} j_*j^*Y \xrightarrow{h_1} i_!i^!Y[1]$ and $j_!j^!Y \xrightarrow{f_2} Y \xrightarrow{g_2} i_*i^*Y \xrightarrow{h_2} j_!j^!Y[1]$ induced by the adjunction morphisms satisfy $h_1 = 0$ and $h_2 = 0$. Because the stable category $\underline{\mathcal{A}(\mathcal{D})}$ is a triangulated category [7], we could also get a prerecollement of $\underline{\mathcal{A}(\mathcal{D})}$ relative to $\underline{\mathcal{A}(\mathcal{D}')}$ and $\underline{\mathcal{A}(\mathcal{D}'')}$.

2 Preliminaries

Throughout the paper, we assume that k is a field and all categories are k -bilinear Hom-finite additive categories with Krull-Schmidt property, i.e., any object can be decomposed into a direct sum of indecomposable objects, and such decomposition is unique up to isomorphisms.

First, we recall some useful definitions and results.

Definition 2.1 [10] Let $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be abelian categories. Then a prerecollement of \mathcal{A} relative to \mathcal{A}' and \mathcal{A}'' , diagrammatically expressed by

$$\begin{array}{ccccc}
 & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\
 \mathcal{A}' & \xrightarrow{i_*} & \mathcal{A} & \xrightarrow{j^*} & \mathcal{A}'' \\
 & \xleftarrow{i^!} & & \xleftarrow{j_*} &
 \end{array}$$

is given by six additive functors $i_* = i_! : \mathcal{A}' \rightarrow \mathcal{A}$; $j^* = j^! : \mathcal{A} \rightarrow \mathcal{A}''$; $i^*, i^! : \mathcal{A} \rightarrow \mathcal{A}'$; $j_*, j_! : \mathcal{A}'' \rightarrow \mathcal{A}$, which satisfy the following three conditions:

- (1) (i^*, i_*) , $(i_!, i^!)$, $(j_!, j^!)$ and (j^*, j_*) are adjoint pairs;
- (2) i_* , $j_!$ and j_* are full embeddings;
- (3) $j^*i_* = 0$.

If the prerecollement of \mathcal{A} relative to \mathcal{A}' and \mathcal{A}'' as above also satisfies

- (4) $\ker(j^*) = \text{Im}(i_*)$.

Then the prerecollement is called recollement.

Remark 2.2 [10] If $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ are abelian categories, and there exists a recollement as above, then

- (I) $i^*j_! = 0, i^!j_* = 0$;
- (II) The units and counits of adjunction give rise to exact sequences of natural transformations

$$\begin{array}{c}
 j_!j^! \rightarrow \text{id}_{\mathcal{A}} \rightarrow i_*i^* \rightarrow 0, \\
 0 \rightarrow i_!i^! \rightarrow \text{id}_{\mathcal{A}} \rightarrow j_*j^*.
 \end{array}$$

Definition 2.3 Suppose we are given triangulated categories $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ together with exact functors $i_* = i_! : \mathcal{D}' \rightarrow \mathcal{D}$, $j^* = j^! : \mathcal{D} \rightarrow \mathcal{D}''$, $i^*, i^! : \mathcal{D} \rightarrow \mathcal{D}'$, and $j_!, j_* : \mathcal{D}'' \rightarrow \mathcal{D}$ which satisfy the following four conditions:

- (a) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^* = j^!, j_*)$ are adjoint triples;
- (b) $i^!j_* = 0$ (and, by adjointness, j^*i_* and $i^*j_!$ are zero morphisms);
- (c) $i_*, j_!, j_*$ are full embeddings (and thus $i^*i_* \cong i^!i_* \cong \text{id}_{\mathcal{D}'}$ and $j^*j_! \cong j^*j_* \cong \text{id}_{\mathcal{D}''}$);

(d) any object X in \mathcal{D} determines distinguished triangles

$$i_!i^!X \rightarrow X \rightarrow j_*j^*X \rightarrow i_!i^!X[1]$$

and

$$j_!j^!X \rightarrow X \rightarrow i_*i^*X \rightarrow j_!j^!X[1],$$

here the morphisms $i_!i^!X \rightarrow X, X \rightarrow j_*j^*X$ are the adjunction morphisms. Then we say that \mathcal{D} admits recollement relative to \mathcal{D}' and \mathcal{D}'' , and diagrammatically expressed by

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}' & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}'' \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array} .$$

If $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ satisfy (a), (b) and (c), then we call \mathcal{D} admits prercollement relative to \mathcal{D}' and \mathcal{D}'' .

Let \mathcal{C} be an additive category. We consider functors $F : \mathcal{C}^{op} \rightarrow Ab$ into the category of abelian groups and call a sequence $F' \rightarrow F \rightarrow F''$ of functors exact if the induced sequence $F'X \rightarrow FX \rightarrow F''X$ of abelian groups is exact for all X in \mathcal{C} .

Definition 2.4 The recollement of \mathcal{D} in definition is called split. If for any $Y \in \mathcal{D}$, the triangles $i_!i^!Y \xrightarrow{f_1} Y \xrightarrow{g_1} j_*j^*Y \xrightarrow{h_1} i_!i^!Y$ (see [1]) and $j_!j^!Y \xrightarrow{f_2} Y \xrightarrow{g_2} i_*i^*Y \xrightarrow{h_2} j_!j^!Y$ (see [1]) induced by the adjunction morphisms satisfy $h_1 = 0$ and $h_2 = 0$.

Definition 2.5 [6] A functor F is said to be coherent if there exists an exact sequence (called presentation)

$$\text{Hom}_{\mathcal{C}}(-, X) \rightarrow \text{Hom}_{\mathcal{C}}(-, Y) \rightarrow F \rightarrow 0.$$

The morphisms between two coherent functors form a small set by Yoneda’s lemma, and the coherent functors $F : \mathcal{C}^{op} \rightarrow Ab$ form an additive category with cokernels. We denote this category by $\mathcal{A}(\mathcal{C})$. The Yoneda functor $h_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{A}(\mathcal{C})$ which sends an object X to $\text{Hom}_{\mathcal{C}}(-, X)$ is fully faithful.

The following results are due to the work of Krause which is crucial to our construction of recollement.

Lemma 2.6 [6] Let \mathcal{T} be a triangulated category. Then

(1) the category $\mathcal{A}(\mathcal{T})$ is abelian and the Yoneda functor $h_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{A}(\mathcal{T})$ is cohomological;

(2) given a cohomological functor $H : \mathcal{T} \rightarrow \mathcal{A}$ to an abelian category, there is (up to a unique isomorphism) a unique exact functor $\bar{H} : \mathcal{A}(\mathcal{T}) \rightarrow \mathcal{A}$ such that $H = \bar{H} \circ h_{\mathcal{T}}$;

(3) given an exact functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ between triangulated categories, there is (up to a unique isomorphism) a unique exact functor $\mathcal{A}(F) : \mathcal{A}(\mathcal{T}) \rightarrow \mathcal{A}(\mathcal{T}')$ such that $h_{\mathcal{T}'} \circ F = \mathcal{A}(F) \circ h_{\mathcal{T}}$.

Lemma 2.7 [6] Let $F : \mathcal{T} \rightarrow \mathcal{T}'$ and $G : \mathcal{T}' \rightarrow \mathcal{T}$ be exact functors between triangulated categories. Then

(1) F is fully faithful if and only if $\mathcal{A}(F)$ is fully faithful;

- (2) if F induces an equivalence $\mathcal{T}/\ker(F) \simeq \mathcal{T}'$, then $\mathcal{A}(F)$ induces an equivalence $\mathcal{A}(\mathcal{T})/(\ker \mathcal{A}(F)) \simeq \mathcal{A}(\mathcal{T}')$;
- (3) F preserves small (co)products if and only if $\mathcal{A}(F)$ preserves small (co)products;
- (4) F is left adjoint to G if and only if $\mathcal{A}(F)$ is left adjoint to $\mathcal{A}(G)$.

Lemma 2.8 [7] Let \mathcal{T} be a triangulated category, then $\mathcal{A}(\mathcal{T})$ is a Frobenius abelian category.

3 Main Results

In this section, we assume that $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ are triangulated categories. First, we give a new proof of the following result, although it appeared already in [11].

Lemma 3.1 Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be three triangulated categories, and if there exists a recollement

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j!} & \\ \mathcal{D}' & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}'' \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

Then the following is also a prerecollement of abelian categories:

$$\begin{array}{ccccc} & \xleftarrow{\mathcal{A}(i^*)} & & \xleftarrow{\mathcal{A}(j!)} & \\ \mathcal{A}(\mathcal{D}') & \xrightarrow{\mathcal{A}(i_*)} & \mathcal{A}(\mathcal{D}) & \xrightarrow{\mathcal{A}(j^*)} & \mathcal{A}(\mathcal{D}'') \\ & \xleftarrow{\mathcal{A}(i^!)} & & \xleftarrow{\mathcal{A}(j_*)} & \end{array}$$

Proof From Lemma 2.7 (4), $(\mathcal{A}(i^*), \mathcal{A}(i_*))$, $(\mathcal{A}(i!), \mathcal{A}(i^!))$, $(\mathcal{A}(j!), \mathcal{A}(j^!))$ and $(\mathcal{A}(j^*), \mathcal{A}(j_*))$ are adjoint pairs.

We claim that $\mathcal{A}(G \circ F) \cong \mathcal{A}(G) \circ \mathcal{A}(F)$ for all $F : \mathcal{D} \rightarrow \mathcal{D}', G : \mathcal{D}' \rightarrow \mathcal{D}''$. According to (3) of Lemma 2.6, there are two exact functors $\mathcal{A}(F) : \mathcal{A}(\mathcal{D}) \rightarrow \mathcal{A}(\mathcal{D}')$ and $\mathcal{A}(G) : \mathcal{A}(\mathcal{D}') \rightarrow \mathcal{A}(\mathcal{D}'')$ such that $h_{\mathcal{D}'} \circ F = \mathcal{A}(F) \circ h_{\mathcal{D}}$ and $h_{\mathcal{D}''} \circ G = \mathcal{A}(G) \circ h_{\mathcal{D}'}$, so $h_{\mathcal{D}''} \circ (G \circ F) = \mathcal{A}(G) \circ h_{\mathcal{D}'} \circ F = (\mathcal{A}(G) \circ \mathcal{A}(F)) \circ h_{\mathcal{D}}$.

However, there is an exact functor $\mathcal{A}(G \circ F) : \mathcal{A}(\mathcal{D}) \rightarrow \mathcal{A}(\mathcal{D}'')$ satisfying $h_{\mathcal{D}''} \circ G \circ F = \mathcal{A}(G \circ F) \circ h_{\mathcal{D}}$, so we get that $\mathcal{A}(G \circ F) \cong \mathcal{A}(G) \circ \mathcal{A}(F)$ by the uniqueness.

Then we could get that $\mathcal{A}(j^*) \circ \mathcal{A}(i_*) \cong \mathcal{A}(j^* i_*) = 0$, and then condition (3) of Definition 2.1 holds.

Similarly, $\mathcal{A}(\text{id}_{\mathcal{D}}) \cong \text{id}_{\mathcal{A}(\mathcal{D})}$. Because of the isomorphisms $\mathcal{A}(i^*) \mathcal{A}(i_*) \cong \mathcal{A}(i^* \circ i_*) \cong \text{id}_{\mathcal{A}(\mathcal{D})}$ and so on, condition (2) also holds.

We prove our main results in the following.

Theorem 3.2 Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be three triangulated categories, and if there exists a recollement

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j!} & \\ \mathcal{D}' & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}'' \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

Then the prerecollement of abelian categories defined above is a recollement if and only if the given recollement of triangulated categories is split.

Proof If

$$\begin{array}{ccccc} & \xleftarrow{\mathcal{A}(i^*)} & & \xleftarrow{\mathcal{A}(j_!)} & \\ \mathcal{A}(\mathcal{D}') & \xrightarrow{\mathcal{A}(i_*)} & \mathcal{A}(\mathcal{D}) & \xrightarrow{\mathcal{A}(j^*)} & \mathcal{A}(\mathcal{D}'') \\ & \xleftarrow{\mathcal{A}(i^!)} & & \xleftarrow{\mathcal{A}(j_*)} & \end{array}$$

is a recollement of abelian categories, then

$$\begin{aligned} \mathcal{A}(j_!)\mathcal{A}(j^!) &\rightarrow \text{id}_{\mathcal{A}(\mathcal{D})} \rightarrow \mathcal{A}(i_*)\mathcal{A}(i^*) \rightarrow 0, \\ 0 &\rightarrow \mathcal{A}(i_!)\mathcal{A}(i^!) \rightarrow \text{id}_{\mathcal{A}(\mathcal{D})} \rightarrow \mathcal{A}(j_*)\mathcal{A}(j^*) \end{aligned}$$

are exact. Because $\mathcal{A}(j_!)\mathcal{A}(j^!)(\text{Hom}_{\mathcal{D}}(-, Y)) = \text{Hom}_{\mathcal{D}}(-, j_!j^!Y)$ etc. hold for all $\text{Hom}_{\mathcal{D}}(-, Y) \in \mathcal{A}(\mathcal{D})$, it is easy to see that the above two exact sequences are as follows:

$$\begin{aligned} \text{Hom}_{\mathcal{D}}(-, j_!j^!Y) &\rightarrow \text{Hom}_{\mathcal{D}}(-, Y) \rightarrow \text{Hom}_{\mathcal{D}}(-, i_*i^*Y) \rightarrow 0, \\ 0 &\rightarrow \text{Hom}_{\mathcal{D}}(-, i_!i^!Y) \rightarrow \text{Hom}_{\mathcal{D}}(-, Y) \rightarrow \text{Hom}_{\mathcal{D}}(-, j_*j^*Y), \end{aligned}$$

where the morphisms are also induced by the adjunction morphisms. But from

$$i_!i^!Y \xrightarrow{f_1} Y \xrightarrow{g_1} j_*j^*Y \xrightarrow{h_1} i_!i^!Y[1] \quad \text{and} \quad j_!j^!Y \xrightarrow{f_2} Y \xrightarrow{g_2} i_*i^*Y \xrightarrow{h_2} j_!j^!Y[1],$$

we know that $\text{Hom}_{\mathcal{D}}(-, h_2)$ and $\text{Hom}_{\mathcal{D}}(-, h_1[-1])$ are zero, so $h_1 = 0, h_2 = 0$.

Conversely, we only need to prove that $\ker(\mathcal{A}(j^*)) \subseteq \text{Im}(\mathcal{A}(i_*))$, since $\ker(\mathcal{A}(j^*)) \supseteq \text{Im}(\mathcal{A}(i_*))$.

Given any $G \in \mathcal{A}(\mathcal{D})$ which satisfies $\mathcal{A}(j^*)G = 0$. First, there exists an exact sequence

$$\text{Hom}_{\mathcal{D}}(-, X) \xrightarrow{\alpha} \text{Hom}_{\mathcal{D}}(-, Y) \xrightarrow{\beta} G \rightarrow 0,$$

where $\alpha = \text{Hom}_{\mathcal{D}}(-, f)$. So $f : X \rightarrow Y$ determines a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in \mathcal{D} , and then $j^*X \xrightarrow{j^*f} j^*Y \xrightarrow{j^*g} j^*Z \xrightarrow{j^*h} j^*X[1]$ is a triangle in \mathcal{D}'' . We get that

$$\text{Hom}_{\mathcal{D}''}(-, j^*X) \xrightarrow{\text{Hom}_{\mathcal{D}''}(-, j^*f)} \text{Hom}_{\mathcal{D}''}(-, j^*Y) \rightarrow \mathcal{A}(j^*)G = 0 \rightarrow 0$$

is also exact, since $\mathcal{A}(j^*)$ is exact and $\mathcal{A}(j^*)(\text{Hom}_{\mathcal{D}}(-, X)) = \text{Hom}_{\mathcal{D}''}(-, j^*X)$ etc.. So

$$\text{Hom}_{\mathcal{D}}(-, j_!j^*X) \xrightarrow{\text{Hom}_{\mathcal{D}}(-, j_!j^*f)} \text{Hom}_{\mathcal{D}}(-, j_!j^*Y) \rightarrow 0$$

is also exact.

Because

$$0 \rightarrow \text{Hom}_{\mathcal{D}}(-, j_!j^!X) \rightarrow \text{Hom}_{\mathcal{D}}(-, X) \rightarrow \text{Hom}_{\mathcal{D}}(-, i_*i^*X) \rightarrow 0$$

and

$$0 \rightarrow \text{Hom}_{\mathcal{D}}(-, j_!j^!Y) \rightarrow \text{Hom}_{\mathcal{D}}(-, Y) \rightarrow \text{Hom}_{\mathcal{D}}(-, i_*i^*Y) \rightarrow 0,$$

there exists the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}_{\mathcal{D}}(-, j_! j^! X) & \longrightarrow & \text{Hom}_{\mathcal{D}}(-, X) & \longrightarrow & \text{Hom}_{\mathcal{D}}(-, i_* i^* X) \longrightarrow 0, \\
 & & \downarrow \text{Hom}_{\mathcal{D}}(-, j_! j^! f) & & \downarrow \text{Hom}_{\mathcal{D}}(-, f) & & \downarrow \text{Hom}_{\mathcal{D}}(-, i_* i^* f) \\
 0 & \longrightarrow & \text{Hom}_{\mathcal{D}}(-, j_! j^! Y) & \longrightarrow & \text{Hom}_{\mathcal{D}}(-, Y) & \longrightarrow & \text{Hom}_{\mathcal{D}}(-, i_* i^* Y) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & G & \longrightarrow & H
 \end{array}$$

where H is the cokernel of $\text{Hom}_{\mathcal{D}}(-, i_* i^* f) = \mathcal{A}(i_*)(\text{Hom}_{\mathcal{D}'}(-, i^* f))$. From ‘‘Snake lemma’’, $G \cong H$. $H \in \text{Im } \mathcal{A}(i_*)$, so $\ker(\mathcal{A}(j^*)) \subseteq \text{Im}(\mathcal{A}(i_*))$.

From Lemma 2.8, $\mathcal{A}(\mathcal{D})$, $\mathcal{A}(\mathcal{D}')$ and $\mathcal{A}(\mathcal{D}'')$ are all Frobenius abelian categories, so the stable categories of them are triangulated categories. We get the following corollary

Corollary 3.3 Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be three triangulated categories, and if there exists a recollement

$$\begin{array}{ccccc}
 & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\
 \mathcal{D}' & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}'' \\
 & \xleftarrow{i^!} & & \xleftarrow{j_*} &
 \end{array}$$

Then the following is a prerecollement of triangulated categories:

$$\begin{array}{ccccc}
 & \xleftarrow{\mathcal{A}(i^*)} & & \xleftarrow{\mathcal{A}(j_!)} & \\
 \underline{\mathcal{A}(\mathcal{D}')} & \xrightarrow{\mathcal{A}(i_*)} & \underline{\mathcal{A}(\mathcal{D})} & \xrightarrow{\mathcal{A}(j^*)} & \underline{\mathcal{A}(\mathcal{D}'')} \\
 & \xleftarrow{\mathcal{A}(i^!)} & & \xleftarrow{\mathcal{A}(j_*)} &
 \end{array}$$

Proof First, the functors in the recollement are actually triangulated functors, we only check $\mathcal{A}(i^*)$. For any triangle in $\underline{\mathcal{A}(\mathcal{D})}$, without generality, we assume that it is standard triangle

$$F \xrightarrow{a} G \xrightarrow{b} N \xrightarrow{c} \Omega F,$$

then there exists a commutative diagram in $\mathcal{A}(\mathcal{D})$:

$$\begin{array}{ccccc}
 F & \xrightarrow{d} & h_X & \xrightarrow{e} & \Omega F \\
 \downarrow a & & \downarrow f & & \parallel \\
 G & \xrightarrow{b} & N & \xrightarrow{c} & \Omega F,
 \end{array}$$

where the left square is a pushout, and the two rows are short exact sequences. $\mathcal{A}(i^*)$ is exact in $\mathcal{A}(\mathcal{D})$, so the following is also a pushout since it is commutative and the two rows are short exact sequences.

$$\begin{array}{ccccc}
 \mathcal{A}(i^*)F & \xrightarrow{\mathcal{A}(i^*)d} & \mathcal{A}(i^*)h_X & \xrightarrow{\mathcal{A}(i^*)e} & \mathcal{A}(i^*)(\Omega F) \\
 \downarrow \mathcal{A}(i^*)a & & \downarrow \mathcal{A}(i^*)f & & \parallel \\
 \mathcal{A}(i^*)G & \xrightarrow{\mathcal{A}(i^*)b} & \mathcal{A}(i^*)N & \xrightarrow{\mathcal{A}(i^*)c} & \mathcal{A}(i^*)(\Omega F).
 \end{array}$$

Therefore

$$\mathcal{A}(i^*)F \xrightarrow{\mathcal{A}(i^*)a} \mathcal{A}(i^*)G \xrightarrow{\mathcal{A}(i^*)b} \mathcal{A}(i^*)N \xrightarrow{\mathcal{A}(i^*)c} \mathcal{A}(i^*)(\Omega F)$$

is also a triangle in $\underline{\mathcal{A}(\mathcal{D}')}$.

For Definition 2.3 (a), we only need to prove

$$\text{Hom}_{\underline{\mathcal{A}(\mathcal{D}')}}(\mathcal{A}(i^*)F, G) \cong \text{Hom}_{\underline{\mathcal{A}(\mathcal{D})}}(F, \mathcal{A}(i_*)G).$$

In fact, let

$$\eta : \text{Hom}_{\underline{\mathcal{A}(\mathcal{D}')}}(\mathcal{A}(i^*)F, G) \xrightarrow{\sim} \text{Hom}_{\underline{\mathcal{A}(\mathcal{D})}}(F, \mathcal{A}(i_*)G),$$

if $\alpha : \mathcal{A}(i^*)F \rightarrow G$ factors through $\text{Hom}_{\mathcal{D}' }(-, X')$ for some $X' \in \mathcal{D}'$, then

$$\begin{array}{ccc} \mathcal{A}(i^*)F & \xrightarrow{\alpha} & G \\ \downarrow \alpha_1 & \nearrow \alpha_2 & \\ \text{Hom}_{\mathcal{D}' }(-, X') & & \end{array}$$

According to $i^*i_* \cong \text{id}_{\mathcal{D}'}$, there exists an object $X \in \mathcal{D}$ such that $i^*X = X'$, then

$$\mathcal{A}(i^*) \text{Hom}_{\mathcal{D}}(-, X) = \text{Hom}_{\mathcal{D}' }(-, X'),$$

and $\alpha_1 = \mathcal{A}(i^*)(\alpha')$. So

$$\begin{array}{ccc} \mathcal{A}(i^*)F & \xrightarrow{\alpha} & G \\ \downarrow \mathcal{A}(i^*)(\alpha') & \nearrow \alpha_2 & \\ \mathcal{A}(i^*) \text{Hom}_{\mathcal{D}}(-, X) & & \end{array}$$

By the naturality of η , we have

$$\begin{array}{ccc} F & \xrightarrow{\eta(\alpha)} & \mathcal{A}(i_*)G \\ \downarrow \alpha' & \nearrow \eta(\alpha_2) & \\ \text{Hom}_{\mathcal{D}}(-, X) & & \end{array}$$

So $\eta(\alpha)$ factors through projective object. The converse is similar, so

$$\text{Hom}_{\underline{\mathcal{A}(\mathcal{D}')}}(\mathcal{A}(i^*)F, G) \cong \text{Hom}_{\underline{\mathcal{A}(\mathcal{D})}}(F, \mathcal{A}(i_*)G).$$

(b) is trivial.

For (c), we only need to prove that $\underline{\mathcal{A}(i_*)}$ is a full embedding, the others are similar. Since $\mathcal{A}(i^*)\mathcal{A}(i_*) = \text{id}_{\underline{\mathcal{A}(\mathcal{D}')}}$, $\underline{\mathcal{A}(i_*)}$ is a full embedding. It is easy to see that $\underline{\mathcal{A}(i_*)}$ is full, we only need to check that it is faithful. If $\text{Hom}_{\mathcal{D}' }(F, G) \rightarrow \text{Hom}_{\mathcal{D}}(\mathcal{A}(i_*)F, \mathcal{A}(i_*)G)$ maps α to $\mathcal{A}(i_*)(\alpha) = 0$, then $\mathcal{A}(i_*)(\alpha)$ factors through some $\text{Hom}_{\mathcal{D}}(-, X)$, i.e.,

$$\begin{array}{ccc} \mathcal{A}(i_*)F & \xrightarrow{\mathcal{A}(i_*)(\alpha)} & \mathcal{A}(i_*)G \\ \downarrow \alpha_1 & \nearrow \alpha_2 & \\ \text{Hom}_{\mathcal{D}}(-, X) & & \end{array}$$

So

$$\begin{array}{ccc} \mathcal{A}(i^*)\mathcal{A}(i_*)F & \xrightarrow{\eta^{-1}(\mathcal{A}(i_*)\alpha)} & G \\ \downarrow \mathcal{A}(i^*)(\alpha_1) & \nearrow \eta^{-1}\alpha_2 & \\ \mathcal{A}(i^*)\text{Hom}_{\mathcal{D}}(-, X) & & \end{array}$$

From $F \cong \mathcal{A}(i^*)\mathcal{A}(i_*)F$, we know that $\eta^{-1}(\mathcal{A}(i_*)\alpha)$ corresponds to α . We also get that $\mathcal{A}(i^*)\text{Hom}_{\mathcal{D}}(-, X) \cong \text{Hom}_{\mathcal{D}'}(-, i^*X)$, so $\alpha = 0$ in $\underline{\mathcal{A}(\mathcal{D}'})$.

References

- [1] Beilinson A, Bernstein J, Deligne P. Faisceaux pervers[J]. Astérisque, 1982, 100: 5–171.
- [2] MacPherson R, Vilonen K. Elementary construction of perverse sheaves[J]. Inv. Math., 1986, 84(2): 403–436.
- [3] Lin Zengqiang, Lin Yanan. One-point extension and recollement[J]. Sci. China Ser. A, 2008, 51(3): 376–382.
- [4] Lin Yanan, Wang Mingxiong. From recollement of triangulated categories to recollement of abelian categories[J]. Sci. China Ser. A, 2010, 53(4): 1111–1116.
- [5] Han Yang. Recollements and Hochschild theory[J]. J. Algebra, 2014, 397(1): 535–547.
- [6] Krause H. Localization theory for triangulated categories[A]. Triangulated categories[C]. Cambridge: Cambridge Univ. Press, 2010: 161–235.
- [7] Krause H. Derived categories, resolutions, and Brown representability[A]. Interactions between homotopy theory and algebra[C]. Providence, RI: Amer. Math. Soc., 2007: 101–139.
- [8] Auslander M. Coherent functors[M]. Berlin: Springer-Verlag, 1966.
- [9] Lin Ji. Torsion theory of triangulated categories and abelian categories[J]. J. Math., 2014, 34(6): 1134–1140.
- [10] Franjou V, Pirashvili T. Comparison of abelian categories recollements[J]. Doc. Math., 2004, 9: 41–56.
- [11] Wang Mingxiong. Recollements of Comma categories and category $\mathcal{A}(\mathcal{S})$ [J]. J. Xiamen Univ (Nat. Sci.), 2010, 49(5): 593–596.
- [12] Hartshorne R. Coherent functors[J]. Advances Math., 1998, 140(1): 44–94.

三角范畴的coherent函子范畴的recollement

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摘要: 文章研究了三角范畴 \mathcal{D} 及其coherent函子范畴 $\mathcal{A}(\mathcal{D})$ 的recollement之间的关系. 利用 \mathcal{D} 的recollement可以诱导 $\mathcal{A}(\mathcal{D})$ 的prerecollement, 文章证明了该prerecollement是recollement的充分必要条件是 \mathcal{D} 的recollement是可裂的; 并且 \mathcal{D} 的recollement可以诱导 $\mathcal{A}(\mathcal{D})$ 的prerecollement.

关键词: recollement; 三角范畴; coherent函子范畴; abel范畴

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