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REGULAR SPACE-LIKE HYPERSURFACES IN THE DE SITTER SPACE $\mathbb{S}_1^{M+1}$ WITH PARALLEL BLASCHKE TENSORS

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Abstract: In this paper, we introduce two conformal non-homogeneous coordinate systems. Modeled on the de Sitter space $\mathbb{S}_1^{m+1}$, we cover the conformal space $\mathbb{Q}_1^{m+1}$. The conformal geometry of regular space-like hypersurfaces in $\mathbb{Q}_1^{m+1}$ can be treated as in the Möbius geometry of hypersurfaces in the sphere $\mathbb{S}^{m+1}$. As a result, we give a complete classification of the regular space-like hypersurfaces with parallel Blaschke tensors.

Keywords: conformal form; parallel Blaschke tensor; conformal metric; conformal second fundamental form; maximal hypersurfaces; constant scalar curvature

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1 Introduction

Let $\mathbb{R}_s^{s+m}$ be the $(s+m)$-dimensional pseudo-Euclidean space which is the real vector space $\mathbb{R}^{s+m}$ equipped with the non-degenerate inner product $\langle\cdot,\cdot\rangle_s$ given by

$$\langle X,Y\rangle_s=-X_1\cdot Y_1+X_2\cdot Y_2,\quad X=(X_1,X_2),\,Y=(Y_1,Y_2)\in\mathbb{R}^s\times\mathbb{R}^m\equiv\mathbb{R}^{s+m},$$

where the dot "·" is the standard Euclidean inner product either on $\mathbb{R}^s$ or on $\mathbb{R}^m$.

Denote by $\mathbb{RP}^{m+2}$ the real projection space of dimension $m+2$. Then the so called conformal space $\mathbb{Q}_1^{m+1}$ is defined as (see [1])

$$\mathbb{Q}_1^{m+1}=\{[\xi]\in\mathbb{RP}^{m+2};\langle\xi,\xi\rangle_2=0\},$$

while, for any $a>0$, the de Sitter space $\mathbb{S}_1^{m+1}(a)$ and the anti-de Sitter space $\mathbb{H}_1^{m+1}\left(-\frac{1}{a^2}\right)$ are defined respectively by

$$\mathbb{S}_1^{m+1}(a)=\{\xi\in\mathbb{R}_1^{m+2};\langle\xi,\xi\rangle_1=a^2\},\quad \mathbb{H}_1^{m+1}\left(-\frac{1}{a^2}\right)=\{\xi\in\mathbb{R}_2^{m+2};\langle\xi,\xi\rangle_2=-a^2\}.$$

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Then $\mathbb{S}_1^{m+1}(a)$, $\mathbb{H}_1^{m+1}\left(-\frac{1}{a^2}\right)$ and the Lorentzian space $\mathbb{R}_1^{m+1}$ are called Lorentzian space forms. Denote $\mathbb{S}_1^{m+1} = \mathbb{S}_1^{m+1}(1)$ and $\mathbb{H}_1^{m+1} = \mathbb{H}_1^{m+1}(-1)$. Define three hyperplanes as follows

$$\begin{aligned}\pi =& \{[x] \in \mathbb{Q}_1^{m+1}; x_1 = x_{m+2}\},\\ \pi_+ =& \{[x] \in \mathbb{Q}_1^{m+1}; x_{m+2} = 0\},\\ \pi_- =& \{[x] \in \mathbb{Q}_1^{m+1}; x_1 = 0\}.\end{aligned}$$

Then there are three conformal diffeomorphisms from the Lorentzian space forms into the conformal space

$$\begin{aligned}&\sigma_0 : \mathbb{R}_1^{m+1} \to \mathbb{Q}_1^{m+1}\backslash\pi, \quad u \longmapsto \left[(\langle u,u\rangle_1 + 1, 2u, \langle u,u\rangle_1 - 1)\right],\\ &\sigma_1 : \mathbb{S}_1^{m+1} \to \mathbb{Q}_1^{m+1}\backslash\pi_+, \quad u \longmapsto \left[(1,u)\right],\\ &\sigma_{-1} : \mathbb{H}_1^{m+1} \to \mathbb{Q}_1^{m+1}\backslash\pi_-, \quad u \longmapsto \left[(u,1)\right].\end{aligned} \tag{1.1}$$

Therefore $\mathbb{Q}_1^{m+1}$ is the common conformal compactification of $\mathbb{R}_1^{m+1}$, $\mathbb{S}_1^{m+1}$ and $\mathbb{H}_1^{m+1}$.

In the reference [1], Nie at al. successfully set up a unified framework of conformal geometry for both regular surfaces and hypersurfaces in Lorentzian space forms by introducing the conformal space $\mathbb{Q}_1^{m+1}$ and some basic conformal invariants, including the conformal metric g, the conformal form Φ, the Blaschke tensor A and the conformal second fundamental form B. Later, all of these were generalized to regular submanifolds of higher codimensions (see [2]). Under this framework, several characterization or classification theorems were obtained for hypersurfaces with some special conformal invariants, see for example (see [1, 3]). The achievement of these certainly proves the efficiency of the above framework. In particular, as the main theorems, regular hypersurfaces with parallel conformal second fundamental forms, and conformal isotropic submanifolds were classified in [1] and [2], respectively. Note that, a regular submanifold in the conformal space $\mathbb{Q}_1^{m+1}$ with vanishing conformal form is called conformal isotropic if its Blaschke tensor A is parallel to the conformal metric. For the later use, we rewrite these two theorems applied in the special case of space-like hypersurfaces as follows.

Theorem 1.1 [1] Let $x : M^m \to \mathbb{Q}_1^{m+1}$ be a regular space-like hypersurface with parallel conformal second fundamental form. Then x is locally conformal equivalent to one of the following hypersurfaces

1. $\mathbb{H}^k \times \mathbb{R}^{m-k} \subset \mathbb{R}_1^{m+1}$, $\quad k = 1, \cdots, m-1$; or
2. $\mathbb{S}^{m-k}(a) \times \mathbb{H}^k\left(-\frac{1}{a^2-1}\right) \subset \mathbb{S}_1^{m+1}$, $a > 1, k = 1, \cdots, m-1$; or
3. $\mathbb{H}^k\left(-\frac{1}{a^2}\right) \times \mathbb{H}^{m-k}(-\frac{1}{1-a^2}) \subset \mathbb{H}_1^{m+1}$, $0 < a < 1, k = 1, \cdots, m-1$; or
4. $WP(p,q,a) \subset \mathbb{R}_1^{m+1}$ for some constants p, q, a, as indicated in Example 3.1.

Theorem 1.2 [2] Any regular, space-like and conformal isotropic hypersurface in $\mathbb{Q}_1^{m+1}$ is conformal equivalent to a maximal, space-like and regular hypersurface in $\mathbb{R}_1^{m+1}$, $\mathbb{S}_1^{m+1}$ or $\mathbb{H}_1^{m+1}$ with constant scalar curvature.

We remark that a Möbius classification of umbilic-free hypersurfaces in the unit sphere with parallel Möbius second fundamental forms was established in [4]. By the way, for complete hypersurfaces in $H^{n+1}(-1)$ with constant scalar curvature, two rigidity theorems were proved in [5].

Motivated by the above theorems, we aims in the present paper at a complete classification of regular space-like hypersurfaces in $\mathbb{Q}_1^{m+1}$ with parallel Blaschke tensors. To this end, we would like to make a direct use of the ideas and technics with which we previously studied the Möbius geometry of umbilc-free hypersurfaces in the unit sphere(see [6–9]). So we firstly define two conformal non-homogeneous coordinate systems (with the coordinate maps $\overset{(1)}{\Psi}$, $\overset{(2)}{\Psi}$, respectively) covering the conformal space $\mathbb{Q}_1^{m+1}$, which are modeled on the de Sitter space $\mathbb{S}_1^{m+1}$, so that the conformal geometry of the hypersurfaces in $\mathbb{Q}_1^{m+1}$ corresponds right to that of the hypersurfaces in the de Sitter space. It follows that the conformal geometry of regular hypersurfaces in each of $\mathbb{H}_1^{m+1}$ and $\mathbb{R}_1^{m+1}$ is made unified with that in $\mathbb{S}_1^{m+1}$. This shows that we only need to consider and study the conformal invariants of the hypersurfaces in $\mathbb{S}_1^{m+1}$ which plays the same role as the unit sphere does in the Möbius geometry of umbilic-free submanifolds. With this consideration, we only focus here on the study of the conformal invariants of regular space-like hypersurfaces in the de Sitter space $\mathbb{S}_1^{m+1}$. As a result, we are able to establish a complete classification for all the regular space-like hypersurfaces with parallel Blaschke tensors.

Note that the above two conformal non-homogeneous coordinate maps $\overset{(1)}{\Psi}$ and $\overset{(2)}{\Psi}$ are conformal equivalent where both of them are defined. Therefore we can use Ψ to denote either one of $\overset{(1)}{\Psi}$ and $\overset{(2)}{\Psi}$. By this, the main theorem of the present paper is stated as follows.

Theorem 1.3 Let $x : M^m \to \mathbb{S}_1^{m+1}$, $m \geq 2$, be a regular space-like hypersurface. If the Blaschke tensor A of x is parallel, then one of the following holds.

1. x is conformal isotropic and thus is locally conformal equivalent to a maximal space-like regular hypersurface in $\mathbb{S}_1^{m+1}$ with constant scalar curvature, or the conformal image under $\Psi \circ \sigma_{-1}$ of a maximal regular hypersurface in $\mathbb{H}_1^{m+1}$ with constant scalar curvature, or the conformal image under $\Psi \circ \sigma_0$ of a maximal regular hypersurface in $\mathbb{R}_1^{m+1}$ with constant scalar curvature;

2. x is of parallel conformal second fundamental form B and thus is locally conformal equivalent to

(a) the image under $\Psi \circ \sigma_0$ of $\mathbb{H}^k \times \mathbb{R}^{m-k} \subset \mathbb{R}_1^{m+1}$, $\quad k = 1, \cdots m-1$; or

(b) $\mathbb{S}^{m-k}(a) \times \mathbb{H}^k\left(-\frac{1}{a^2-1}\right) \subset \mathbb{S}_1^{m+1}$, $a > 1, k = 1, \cdots m-1$; or

(c) the image under $\Psi \circ \sigma_{-1}$ of $\mathbb{H}^k\left(-\frac{1}{a^2}\right) \times \mathbb{H}^{m-k}(-\frac{1}{1-a^2}) \subset \mathbb{H}_1^{m+1}$, $0 < a < 1, k = 1, \cdots m-1$; or

(d) $WP(p,q,a) \subset \mathbb{R}_1^{m+1}$ for some constants p, q, a.

3. x is non-isotropic with a non-parallel conformal second fundamental form B and is locally conformal equivalent to

(a) one of the maximal hypersurfaces as indicated in Example 3.2; or

(b) one of the non-maximal hypersurfaces as indicated in Example 3.3.

Remark 1.1 It is directly verified in Section 3 that each of the regular space-like hypersurfaces stated in the above theorem has a parallel Blaschke tensor.

2 Necessary Basics on Regular Space-Like Hypersurfaces

This section provides some basics of the conformal geometry of regular space-like hypersurfaces in the Lorentzian space forms. The main idea comes originally from the work of Wang on the Möbius geometry of umbilic-free submanifolds in the unit sphere (see [10]), and much of the detail can be found in a series of papers by Nie at al (see for example [1–3]).

Let $x : M^m \to \mathbb{S}^{m+1}_1 \subset \mathbb{R}^{m+2}_1$ be a regular space-like hypersurface in $\mathbb{S}^{m+1}_1$. Denote by h the (scalar-valued) second fundamental form of x with components h_{ij} and $H = \frac{1}{m}\mathrm{tr}\, h$ the mean curvature. Define the conformal factor $\rho > 0$ and the conformal position Y of x, respectively, as follows

$$\rho^2 = \frac{m}{m-1}\left(|h|^2 - m|H|^2\right), \quad Y = \rho(1, x) \in \mathbb{R}^1_1 \times \mathbb{R}^{m+2}_1 \equiv \mathbb{R}^{m+3}_2. \tag{2.1}$$

Then $Y(M^m)$ is clearly included in the light cone $\mathbb{C}^{m+2} \subset \mathbb{R}^{m+3}_2$, where

$$\mathbb{C}^{m+2} = \{\xi \in \mathbb{R}^{m+3}_2; \langle \xi, \xi\rangle_2 = 0, \xi \neq 0\}.$$

The positivity of ρ implies that $Y : M^m \to \mathbb{R}^{m+3}_2$ is an immersion of M^m into the $\mathbb{R}^{m+3}_2$. Clearly, the metric $g := \langle dY, dY\rangle_2 \equiv \rho^2\langle dx, dx\rangle_1$ on M^m, induced by Y and called the conformal metric, is invariant under the pseudo-orthogonal group $O(m + 3, 2)$ of linear transformations on $\mathbb{R}^{m+3}_2$ reserving the Lorentzian product $\langle\cdot, \cdot\rangle_2$. Such kind of things are called the conformal invariants of x.

Definition 2.1 (see [1–3]) Let $x, \tilde{x} : M^m \to \mathbb{S}^{m+1}_1$ be two regular space-like hypersurfaces with $Y, \tilde{Y}$ their conformal positions, respectively. If there exists some $\mathbb{T} \in O(m + 3, 2)$ such that $\tilde{Y} = \mathbb{T}(Y)$, then $x, \tilde{x}$ are called conformal equivalent to each other.

For any local orthonormal frame field $\{e_i\}$ and the dual $\{\theta^i\}$ on M^m with respect to the standard metric $\langle dx, dx\rangle_1$, define

$$E_i = \rho^{-1}e_i, \quad \omega^i = \rho\theta^i. \tag{2.2}$$

Then $\{E_i\}$ is a local orthonormal frame field with respect to the conformal metric g with $\{\omega^i\}$ its dual coframe. Let n be the time-like unit normal of x. Define $\xi = (-H, -Hx + n)$, then $\langle \xi, \xi\rangle_2 = -1$. Let Δ denote the Laplacian with respect to the conformal metric g. Define $N : M^m \to \mathbb{R}^{m+3}_2$ by

$$N = -\frac{1}{m}\Delta Y - \frac{1}{2m^2}\langle \Delta Y, \Delta Y\rangle_2 Y. \tag{2.3}$$

Then it holds that

$$\langle \Delta Y, Y\rangle_2 = -m, \quad \langle Y, Y\rangle_2 = \langle N, N\rangle_2 = 0, \quad \langle Y, N\rangle_2 = 1. \tag{2.4}$$

Furthermore, $\{Y, N, Y_i, \xi,\ 1 \le i \le m\}$ forms a moving frame in $\mathbb{R}^{m+3}_2$ along Y, with respect to which the equations of motion is as follows

$$\begin{cases} dY = \sum Y_i\omega^i, \\ dN = \sum \psi_i Y_i + \phi\xi, \\ dY_i = -\psi_i Y - \omega_i N + \sum \omega_{ij}Y_j + \tau_i\xi, \\ d\xi = \phi Y + \sum \tau_i Y_i. \end{cases} \tag{2.5}$$

By the exterior differentiation of (2.5) and using Cartan's lemma, we can write

$$\phi = \sum_i \Phi_i\omega^i, \ \ \psi_i = \sum_j A_{ij}\omega^j, \ \ A_{ij} = A_{ji}, \ \ \tau_i = \sum_j B_{ij}\omega^j, \ \ B_{ij} = B_{ji}. \tag{2.6}$$

Then the conformal form Φ, the Blaschke tensor A and the conformal second fundamental form B defined by

$$\Phi = \sum_i \Phi_i\omega^i, \quad A = \sum_{i,j} A_{ij}\omega^i\omega^j, \quad B = \sum_{i,j} B_{ij}\omega^i\omega^j$$

are all conformal invariants. By a long but direct computation, we find that

$$\begin{aligned} A_{ij} = & -\langle Y_{ij}, N\rangle_2 = -\rho^{-2}((\log\rho)_{,ij} - e_i(\log\rho)e_j(\log\rho) + h_{ij}H) \\ & - \frac{1}{2}\rho^{-2}\left(|\bar\nabla\log\rho|^2 - |H|^2 - 1\right)\delta_{ij}, \end{aligned} \tag{2.7}$$

$$B_{ij} = -\langle Y_{ij}, \xi\rangle_2 = \rho^{-1}(h_{ij} - H\delta_{ij}), \tag{2.8}$$

$$\Phi_i = -\langle \xi, dN\rangle_2 = -\rho^{-2}[(h_{ij} - H\delta_{ij})e_j(\log\rho) + e_i(H)], \tag{2.9}$$

where $Y_{ij} = E_j(Y_i)$, $\bar\nabla$ is the Levi-Civita connection of the standard metric $\langle\cdot,\cdot\rangle_1$, and the subscript $_{,ij}$ denotes the covariant derivatives with respect to $\bar\nabla$. The differentiation of (2.5) also gives the following integrability conditions

$$\Phi_{ij} - \Phi_{ji} = \sum(B_{ik}A_{kj} - B_{kj}A_{ki}), \tag{2.10}$$

$$A_{ijk} - A_{ikj} = B_{ij}\Phi_k - B_{ik}\Phi_j, \tag{2.11}$$

$$B_{ijk} - B_{ikj} = \delta_{ij}\Phi_k - \delta_{ik}\Phi_j, \tag{2.12}$$

$$R_{ijkl} = \sum(B_{ik}B_{jl} - B_{il}B_{jk}) + A_{il}\delta_{jk} - A_{ik}\delta_{jl} + A_{jk}\delta_{il} - A_{jl}\delta_{ik}, \tag{2.13}$$

where A_{ijk}, B_{ijk}, Φ_{ij} are respectively the components of the covariant derivatives of A, B, Φ, and R_{ijkl} is the components of the Riemannian curvature tensor of the conformal metric g. Furthermore, by (2.1) and (2.8) we have

$$\operatorname{tr}B = \sum B_{ii} = 0, \quad |B|^2 = \sum(B_{ij})^2 = \frac{m-1}{m}, \tag{2.14}$$

and by (2.13) we find the Ricci curvature tensor

$$R_{ij} = \sum B_{ik}B_{kj} + \delta_{ij}\operatorname{tr}A + (m-2)A_{ij}, \tag{2.15}$$

which implies that
$$\operatorname{tr} A = \frac{1}{2m}(m^2\kappa - 1) \tag{2.16}$$
with κ being the normalized scalar curvature of g.

It is easily seen [1] that the conformal position vector Y defined above is exactly the canonical lift of the composition map $\bar{x} = \sigma_1 \circ x : M^m \to \mathbb{Q}_1^{m+1}$, implying that the conformal invariants g, Φ, A, B defined above are the same as those of $\bar{x}$ introduced by Nie at al. in [1].

On the other hand, the conformal space $\mathbb{Q}_1^{m+1}$ is clearly covered by the following two open sets
$$\begin{aligned} U_1 &= \left\{[y] \in \mathbb{Q}_1^{m+1}; y = (y_1, y_2, y_3) \in \mathbb{R}_1^1 \times \mathbb{R}_1^1 \times \mathbb{R}^{m+1} \equiv \mathbb{R}_2^{m+3}, y_1 \neq 0\right\}, \\ U_2 &= \left\{[y] \in \mathbb{Q}_1^{m+1}; y = (y_1, y_2, y_3) \in \mathbb{R}_1^1 \times \mathbb{R}_1^1 \times \mathbb{R}^{m+1} \equiv \mathbb{R}_2^{m+3}, y_2 \neq 0\right\}. \end{aligned} \tag{2.17}$$
Define the following two diffeomorphisms
$$\overset{(\alpha)}{\Psi} : U_\alpha \to \mathbb{S}_1^{m+1}, \quad \alpha = 1, 2 \tag{2.18}$$
by
$$\overset{(1)}{\Psi}([y]) = y_1^{-1}(y_2, y_3) \text{ for } [y] \in U_1,\ y = (y_1, y_2, y_3); \tag{2.19}$$
$$\overset{(2)}{\Psi}([y]) = y_2^{-1}(y_1, y_3) \text{ for } [y] \in U_2,\ y = (y_1, y_2, y_3). \tag{2.20}$$
Then with respect to the conformal structure on $\mathbb{Q}_1^{m+1}$ introduced in [1] and the standard metric on $\mathbb{S}_1^{m+1}$, both $\overset{(1)}{\Psi}$ and $\overset{(2)}{\Psi}$ are conformal.

Now for a regular space-like hypersurface $\bar{x} : M^m \to \mathbb{Q}_1^{m+1}$ with the canonical lift
$$Y : M^m \to \mathbb{C}^{m+2} \subset \mathbb{R}_2^{m+3},$$
write $Y = (Y_1, Y_2, Y_3) \in \mathbb{R}_1^1 \times \mathbb{R}_1^1 \times \mathbb{R}^{m+1}$. Then we have the following two composed hypersurfaces
$$\overset{(\alpha)}{x} := \overset{(\alpha)}{\Psi} \circ \bar{x}|_{\overset{(\alpha)}{M}} : \overset{(\alpha)}{M} \to \mathbb{S}_1^{m+1}, \quad \overset{(\alpha)}{M} = \{p \in M; x(p) \in U_\alpha\}, \quad \alpha = 1, 2. \tag{2.21}$$
Then $M^m = \overset{(1)}{M} \bigcup \overset{(2)}{M}$, and the following lemma is clearly true by a direct computation:

Lemma 2.1 The conformal position vector $\overset{(1)}{Y}$ of $\overset{(1)}{x}$ is nothing but $Y|_{\overset{(1)}{M}}$, while the conformal position vector $\overset{(2)}{Y}$ of $\overset{(2)}{x}$ is given by
$$\overset{(2)}{Y} = \mathbb{T}(Y|_{\overset{(2)}{M}}), \text{ where } \mathbb{T} = \left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{0} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & I_{m+1} \end{array}\right). \tag{2.22}$$

Corollary 2.2 The basic conformal invariants g, Φ, A, B of $\bar{x}$ coincide accordingly with those of each of $\overset{(1)}{x}$ and $\overset{(2)}{x}$ on where $\overset{(1)}{x}$ or $\overset{(2)}{x}$ is defined, respectively.

Therefore $\overset{(1)}{\Psi}$ and $\overset{(2)}{\Psi}$ can be viewed as two non-homogenous coordinate maps preserving the conformal invariants of the regular space-like hypersurfaces.

Corollary 2.3 $\overset{(1)}{x}$ and $\overset{(2)}{x}$ are conformal equivalent to each other on $\overset{(1)}{M} \cap \overset{(2)}{M}$.

On the other hand, all the regular space-like hypersurfaces in the three Lorentzian space forms can be viewed as ones in $\mathbb{Q}_1^{m+1}$ via the conformal embeddings σ_1, σ_0 and σ_{-1} defined in (1.1). Now, using $\overset{(1)}{\Psi}$ and $\overset{(2)}{\Psi}$, one can shift the conformal geometry of regular space-like hypersurfaces in $\mathbb{Q}_1^{m+1}$ to that of regular space-like hypersurfaces in the de Sitter space $\mathbb{S}_1^{m+1}$. It follows that, in a sense, the conformal geometry of regular space-like hypersurfaces can also be unified as that of the corresponding hypersurfaces in the de Sitter space. Concisely, we can achieve this simply by introducing the following four conformal maps

$$\overset{(1)}{\sigma} = \overset{(1)}{\Psi} \circ \sigma_0 : \overset{(1)}{\mathbb{R}}{}_1^{m+1} \to \mathbb{S}_1^{m+1}, \quad u \mapsto \left(\frac{2u}{1+\langle u,u\rangle}, \frac{1-\langle u,u\rangle}{1+\langle u,u\rangle} \right), \tag{2.23}$$

$$\overset{(2)}{\sigma} = \overset{(2)}{\Psi} \circ \sigma_0 : \overset{(2)}{\mathbb{R}}{}_1^{m+1} \to \mathbb{S}_1^{m+1}, \quad u \mapsto \left(\frac{1+\langle u,u\rangle}{2u_1}, \frac{u_2}{u_1}, \frac{1-\langle u,u\rangle}{2u_1} \right), \tag{2.24}$$

$$\overset{(1)}{\tau} = \overset{(1)}{\Psi} \circ \sigma_{-1} : \overset{(1)}{\mathbb{H}}{}_1^{m+1} \to \mathbb{S}_1^{m+1}, \quad y \mapsto \left(\frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{1}{y_1} \right), \tag{2.25}$$

$$\overset{(2)}{\tau} = \overset{(2)}{\Psi} \circ \sigma_{-1} : \overset{(2)}{\mathbb{H}}{}_1^{m+1} \to \mathbb{S}_1^{m+1}, \quad y \mapsto \left(\frac{y_1}{y_2}, \frac{y_3}{y_2}, \frac{1}{y_2} \right), \tag{2.26}$$

where

$$\overset{(1)}{\mathbb{R}}{}_1^{m+1} = \{u \in \mathbb{R}_1^{m+1}; \quad 1+\langle u,u\rangle \neq 0\}, \tag{2.27}$$

$$\overset{(2)}{\mathbb{R}}{}_1^{m+1} = \{u = (u_1, u_2) \in \mathbb{R}_1^{m+1}; u_1 \neq 0\}, \tag{2.28}$$

$$\overset{(1)}{\mathbb{H}}{}_1^{m+1} = \{y = (y_1, y_2, y_3) \in \mathbb{H}_1^{m+1}; y_1 \neq 0\}, \tag{2.29}$$

$$\overset{(2)}{\mathbb{H}}{}_1^{m+1} = \{y = (y_1, y_2, y_3) \in \mathbb{H}_1^{m+1}; y_2 \neq 0\}. \tag{2.30}$$

The following theorem will be used later in this paper.

Theorem 2.4 [2] Two hypersurfaces $x : M^m \to \mathbb{S}_1^{m+1}$ and $\tilde{x} : \tilde{M}^m \to \mathbb{S}_1^{m+1}$ $(m \geq 3)$ are conformal equivalent if and only if there exists a diffeomorphism $f : M \to \tilde{M}$ which preserves the conformal metric and the conformal second fundamental form.

3 Examples

Before proving the main theorem, we first present some regular space-like hypersurfaces in $\mathbb{S}_1^{m+1}$ with parallel Blaschke tensors.

Example 3.1 (see [1, 4]) Let $\mathbb{R}^+$ be the half line of positive real numbers. For any two given natural numbers p, q with $p + q < m$ and a real number $a > 1$, consider the hypersurface of warped product embedding

$$u : \mathbb{H}^q\left(-\frac{1}{a^2-1}\right) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{m-p-q-1} \to \mathbb{R}_1^{m+1}$$

defined by

$$u = (tu', tu'', u'''), \text{ where } u' \in \mathbb{H}^q\left(-\frac{1}{a^2-1}\right),\ u'' \in \mathbb{S}^p(a),\ t \in \mathbb{R}^+,\ u''' \in \mathbb{R}^{m-p-q-1}.$$

Then $\bar{x} := \sigma_0 \circ u$ is a regular space-like hypersurface in the conformal space $\mathbb{Q}_1^{m+1}$ with parallel conformal second fundamental form. This hypersurface is denoted as $WP(p,q,a)$ in [1]. By Proposition 3.1 together with its proof in [1], $\bar{x}$ is also of parallel Blaschke tensor. It follows from Corollary 2.2 that the composition map

$$x = \Psi \circ \bar{x} : \mathbb{H}^q\left(-\frac{1}{a^2-1}\right) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{m-p-q-1} \to \mathbb{S}_1^{m+1},$$

where Ψ denotes $\overset{(1)}{\Psi}$ or $\overset{(2)}{\Psi}$, defines a regular space-like hypersurface in $\mathbb{S}_1^{m+1}$ with both parallel conformal second fundamental form and parallel Blaschke tensor. For convenience, we also denote x by the same symbol $WP(p,q,a)$. Note that, by a direct calculation, one easily finds that $WP(p,q,a)$ has exactly three distinct conformal principal curvatures.

The similar example of $WP(p,q,a)$ in Möbius geometry was originally found by [4] and denoted by $CSS(p,q,a)$.

Example 3.2 Given $r > 0$. For any integers m and K satisfying $m \geq 3$ and $2 \leq K \leq m-1$, let $\tilde{y}_1 : M_1^K \to \mathbb{S}_1^{K+1}(r) \subset \mathbb{R}_1^{K+2}$ be a regular and maximal space-like hypersurface with constant scalar curvature

$$\tilde{S}_1 = \frac{mK(K-1)+(m-1)r^2}{mr^2} \tag{3.1}$$

and

$$\tilde{y} = (\tilde{y}_0, \tilde{y}_2) : \mathbb{H}^{m-K}\left(-\frac{1}{r^2}\right) \to \mathbb{R}_1^1 \times \mathbb{R}^{m-K} \equiv \mathbb{R}_1^{m-K+1}$$

be the canonical embedding, where $\tilde{y}_0 > 0$. Set

$$\tilde{M}^m = M_1^K \times \mathbb{H}^{m-K}\left(-\frac{1}{r^2}\right), \quad \tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2). \tag{3.2}$$

Then $\tilde{Y} : \tilde{M}^m \to \mathbb{R}_2^{m+3}$ is an immersion satisfying $\langle \tilde{Y}, \tilde{Y} \rangle_2 = 0$. The induced metric

$$g = \langle d\tilde{Y}, d\tilde{Y} \rangle_2 = -d\tilde{y}_0^2 + \langle d\tilde{y}_1, d\tilde{y}_1 \rangle_1 + d\tilde{y}_2 \cdot d\tilde{y}_2$$

by $\tilde{Y}$ is clearly a Riemannian one, and thus as Riemannian manifolds we have

$$(\tilde{M}^m, g) = (M_1, \langle d\tilde{y}_1, d\tilde{y}_1 \rangle_1) \times \left(\mathbb{H}^{m-K}\left(-\frac{1}{r^2}\right), \langle d\tilde{y}, d\tilde{y} \rangle_1\right). \tag{3.3}$$

Define

$$\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}, \quad \tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}, \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2). \tag{3.4}$$

Then $\tilde{x}^2 = 1$ and thus we have a smooth map $\tilde{x} : M^m \to \mathbb{S}_1^{m+1}$. Clearly,

$$d\tilde{x} = -\frac{d\tilde{y}_0}{\tilde{y}_0^2}(\tilde{y}_1, \tilde{y}_2) + \frac{1}{\tilde{y}_0}(d\tilde{y}_1, d\tilde{y}_2). \tag{3.5}$$

Therefore the induced "metric" $\tilde{g} = d\tilde{x} \cdot d\tilde{x}$ is derived as

$$\begin{aligned}\tilde{g} &= \tilde{y}_0^{-4} d\tilde{y}_0^2(\langle \tilde{y}_1, \tilde{y}_1\rangle_1 + \tilde{y}_2 \cdot \tilde{y}_2) + \tilde{y}_0^{-2}(\langle d\tilde{y}_1, d\tilde{y}_1\rangle_1 + d\tilde{y}_2 \cdot d\tilde{y}_2) \\ &\quad - 2\tilde{y}_0^{-3} d\tilde{y}_0(\langle \tilde{y}_1, d\tilde{y}_1\rangle_1 + \tilde{y}_2 \cdot d\tilde{y}_2) \qquad (3.6)\\ &= \tilde{y}_0^{-2}(d\tilde{y}_0^2 + g + d\tilde{y}_0^2 - 2d\tilde{y}_0^2) \qquad (3.7)\\ &= \tilde{y}_0^{-2} g, \qquad (3.8)\end{aligned}$$

implying that $\tilde{x}$ is a regular space-like hypersurface.

If $\tilde{n}_1$ is the time-like unit normal vector field of $\tilde{y}_1$ in $\mathbb{S}_1^{K+1}(r) \subset \mathbb{R}_1^{K+2}$, then $\tilde{n} = (\tilde{n}_1, 0) \in \mathbb{R}_1^{m+2}$ is a time-like unit normal vector field of $\tilde{x}$. Consequently, by (3.5), the second fundamental form $\tilde{h}$ of $\tilde{x}$ is given by

$$\begin{aligned}\tilde{h} &= \langle d\tilde{n}, d\tilde{x}\rangle_1 = \langle (d\tilde{n}_1, 0), -\tilde{y}_0^{-2} d\tilde{y}_0(\tilde{y}_1, \tilde{y}_2) + \tilde{y}_0^{-1}(d\tilde{y}_1, d\tilde{y}_2)\rangle_1 \\ &= -\tilde{y}_0^{-2} d\tilde{y}_0 \langle d\tilde{n}_1, \tilde{y}_1\rangle_1 + \tilde{y}_0^{-1}\langle d\tilde{n}_1, d\tilde{y}_1\rangle_1 = \tilde{y}_0^{-1} h,\end{aligned} \tag{3.9}$$

where h is the second fundamental form of $\tilde{y}_1 : M_1^K \to \mathbb{S}_1^{K+1}$.

Let $\{E_i\,; 1 \le i \le K\}$ (resp. $\{E_i\,; K+1 \le i \le m\}$) be a local orthonormal frame field on $(M_1, d\tilde{y}_1^2)$ (resp. on $\mathbb{H}^{m-K}(-\frac{1}{r^2})$). Then $\{E_i\,; 1 \le i \le m\}$ gives a local orthonormal frame field on $(\tilde{M}^m, g)$. Put $e_i = \tilde{y}_0 E_i$, $i = 1, \cdots, m$. Then $\{e_i\,; 1 \le i \le m\}$ is a orthonormal frame field along $\tilde{x}$. Thus we obtain

$$\tilde{h}_{ij} = \tilde{h}(e_i, e_j) = \tilde{y}_0^2 \tilde{h}(E_i, E_j) = \begin{cases} \tilde{y}_0 h(E_i, E_j) = \tilde{y}_0 h_{ij}, & \text{when } 1 \le i, j \le K, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Since the mean curvature of $\tilde{y}_1 \equiv 0$ by the maximality of $\tilde{y}_1$, the mean curvature $\tilde{H}$ of $\tilde{x}$ vanishes. Therefore

$$\tilde{\rho}^2 = \frac{m}{m-1}\left(\sum_{i,j} \tilde{h}_{ij}^2 - m|\tilde{H}|^2\right) = \frac{m}{m-1}\tilde{y}_0^2 \sum_{i,j=1}^{K} h_{ij}^2 = \tilde{y}_0^2,$$

where we have used the Gauss equation and (3.1). It follows that $\tilde{x}$ is regular and its conformal factor $\tilde{\rho} = \tilde{y}_0$. Thus $\tilde{Y}$, given in (3.2), is exactly the conformal position vector of $\tilde{x}$, implying the induced metric g by $\tilde{Y}$ is nothing but the conformal metric of $\tilde{x}$. Furthermore, the conformal second fundamental form of $\tilde{x}$ is given by

$$\tilde{B} = \tilde{\rho}^{-1} \sum (\tilde{h}_{ij} - \tilde{H}\delta_{ij})\omega^i\omega^j = \sum_{i,j=1}^{K} h_{ij}\omega^i\omega^j, \tag{3.11}$$

where $\{\omega^i\}$ is the local coframe field on M^m dual to $\{E_i\}$.

On the other hand, by (3.3) and the Gauss equations of $\tilde{y}_1$ and $\tilde{y}$, one finds that the Ricci tensor of g is given as follows

$$R_{ij} =\frac{K-1}{r^2}\delta_{ij} + \sum_{k=1}^{K} h_{ik}h_{kj}, \text{ if } 1 \le i, j \le K, \tag{3.12}$$

$$R_{ij} =-\frac{m-K-1}{r^2}\delta_{ij}, \text{ if } K+1 \le i, j \le m, \tag{3.13}$$

$$R_{ij} =0, \text{ if } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K, \tag{3.14}$$

which implies that the normalized scalar curvature of g is given by

$$\kappa = \frac{m(K(K-1)-(m-K)(m-K-1))+(m-1)r^2}{m^2(m-1)r^2}. \tag{3.15}$$

Thus

$$\frac{1}{2m}(m^2\kappa - 1) = \frac{K(K-1)-(m-K)(m-K-1)}{2(m-1)r^2}. \tag{3.16}$$

Since $m \ge 3$, it follows from (2.15) and (3.11)–(3.16) that the Blaschke tensor of $\tilde{x}$ is given by $A = \sum A_{ij}\omega^i\omega^j$, where

$$A_{ij} =\frac{1}{2r^2}\delta_{ij}, \text{ if } 1 \le i, j \le K, \quad A_{ij} = -\frac{1}{2r^2}\delta_{ij}, \text{ if } K+1 \le i, j \le m, \tag{3.17}$$

$$A_{ij} =0, \text{ if } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K. \tag{3.18}$$

Clearly, A has two distinct eigenvalues $\lambda_1 = -\lambda_2 = \frac{1}{2r^2}$, which are constant. Thus by (3.3), A is parallel.

Example 3.3 Given $r > 0$. For any integers m and K satisfying $m \ge 3$ and $2 \le K \le m-1$, let

$$\tilde{y} : M_1^K \to \mathbb{H}_1^{K+1}\left(-\frac{1}{r^2}\right) \subset \mathbb{R}_2^{K+2}$$

be a regular and maximal space-like hypersurface with constant scalar curvature

$$\tilde{S}_1 = \frac{-mK(K-1)+(m-1)r^2}{mr^2} \tag{3.19}$$

and $\tilde{y}_2 : \mathbb{S}^{m-K}(r) \to \mathbb{R}^{m-K+1}$ be the canonical embedding. Set

$$\tilde{M}^m = M_1^K \times \mathbb{S}^{m-K}(r), \quad \tilde{Y} = (\tilde{y}, \tilde{y}_2), \tag{3.20}$$

then $\langle \tilde{Y}, \tilde{Y} \rangle_2 = 0$. Thus we have an immersion $\tilde{Y} : M^m \to \mathbb{C}^{m+2} \subset \mathbb{R}_2^{m+3}$ with the induced metric $g = \langle d\tilde{Y}, d\tilde{Y} \rangle_2 = \langle d\tilde{y}, d\tilde{y} \rangle_2 + d\tilde{y}_2 \cdot d\tilde{y}_2$, which is certainly positive definite. It follows that, as Riemannian manifolds

$$(\tilde{M}^m, g) = (M_1, \langle d\tilde{y}, d\tilde{y} \rangle_2) \times \left(\mathbb{S}^{m-K}(r), d\tilde{y}_2^2\right). \tag{3.21}$$

If we write $\tilde{y} = (\tilde{y}_0, \tilde{y}_1', \tilde{y}_1'') \in \mathbb{R}_1^1 \times \mathbb{R}_1^1 \times \mathbb{R}^K \equiv \mathbb{R}_2^{K+2}$, then $\tilde{y}_0$ and $\tilde{y}_1'$ can not be zero simultaneously. So without loss of generality, we can assume that $\tilde{y}_0 \neq 0$. In this case, we denote $\varepsilon = \mathrm{Sgn}\,(\tilde{y}_0)$ and write $\tilde{y}_1 := (\tilde{y}_1', \tilde{y}_1'')$. Define

$$\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}, \quad \tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}, \quad \tilde{x} = \varepsilon(\tilde{x}_1, \tilde{x}_2). \tag{3.22}$$

Then $\tilde{x} \in \mathbb{R}_1^{m+2}, \tilde{x}^2 = 1$ and, similar to that in Example 3.2, $\tilde{x} : \tilde{M}^m \to \mathbb{S}_1^{m+1}$ defines a regular space-like hypersurface. In fact, since

$$\varepsilon d\tilde{x} = -\frac{d\tilde{y}_0}{\tilde{y}_0^2}(\tilde{y}_1, \tilde{y}_2) + \frac{1}{\tilde{y}_0}(d\tilde{y}_1, d\tilde{y}_2), \tag{3.23}$$

the induced metric $\tilde{g} = d\tilde{x} \cdot d\tilde{x}$ is related to g by

$$\begin{aligned}\tilde{g} =&\tilde{y}_0^{-4} d\tilde{y}_0^2(\langle \tilde{y}_1, \tilde{y}_1\rangle_1 + \tilde{y}_2 \cdot \tilde{y}_2) + \tilde{y}_0^{-2}(\langle d\tilde{y}_1, d\tilde{y}_1\rangle_1 + d\tilde{y}_2 \cdot d\tilde{y}_2)\\ &- 2\tilde{y}_0^{-3} d\tilde{y}_0(\langle \tilde{y}_1, d\tilde{y}_1\rangle_1 + \tilde{y}_2 \cdot d\tilde{y}_2)\\ =&\tilde{y}_0^{-2}(-d\tilde{y}_0^2 + \langle d\tilde{y}_1, d\tilde{y}_1\rangle_1 + \tilde{y}_2 \cdot d\tilde{y}_2)\\ =&\tilde{y}_0^{-2} g.\end{aligned} \tag{3.24}$$

Suitably choose the time-like unit normal vector field $(\tilde{n}_0, \tilde{n}_1)$ of $\tilde{y}$, define

$$\tilde{n} = (\tilde{n}_1, 0) - \varepsilon \tilde{n}_0 \tilde{x} \in \mathbb{R}_1^{m+2}.$$

Then $\langle \tilde{n}, \tilde{n}\rangle_1 = -1, \langle \tilde{n}, \tilde{x}\rangle_1 = 0, \langle \tilde{n}, d\tilde{x}\rangle_1 = 0$ indicating that $\tilde{n}$ is a time-like unit normal vector field of $\tilde{x}$. The second fundamental form $\tilde{h}$ of $\tilde{x}$ is given by

$$\begin{aligned}\tilde{h} =&\langle d\tilde{n}, d\tilde{x}\rangle_1 = \langle (d\tilde{n}_1, 0) - \varepsilon d\tilde{n}_0 \tilde{x} - \varepsilon \tilde{n}_0 d\tilde{x}, d\tilde{x}\rangle_1\\ =&\langle (d\tilde{n}_1, 0), d\tilde{x}\rangle_1 - \varepsilon d\tilde{n}_0 \langle \tilde{x}, d\tilde{x}\rangle_1 - \varepsilon \tilde{n}_0 \langle d\tilde{x}, d\tilde{x}\rangle_1\\ =&\varepsilon(\tilde{y}_0^{-1}(-d\tilde{n}_0 \cdot d\tilde{y}_0 + \langle d\tilde{n}_1, d\tilde{y}_1\rangle_1) - \tilde{n}_0 \langle d\tilde{x}, d\tilde{x}\rangle_1)\\ =&\varepsilon(\tilde{y}_0^{-1}\langle d(\tilde{n}_0, \tilde{n}_1), d\tilde{y}\rangle_1 - \tilde{n}_0 \langle d\tilde{x}, d\tilde{x}\rangle_1)\\ =&\varepsilon(\tilde{y}_0^{-1} h - \tilde{n}_0 \tilde{y}_0^{-2} g),\end{aligned} \tag{3.25}$$

where h is the second fundamental form of $\tilde{y}$.

Let $\{E_i\,; 1 \leq i \leq K\}$ (resp. $\{E_i\,; K+1 \leq i \leq m\}$) be a local orthonormal frame field on $(M_1, d\tilde{y}^2)$ (resp. on $\mathbb{S}^{m-K}(r)$). Then $\{E_i\,; 1 \leq i \leq m\}$ is a local orthonormal frame field on (M^m, g). Put $e_i = \varepsilon \tilde{y}_0 E_i$, $i = 1, \cdots, m$. Then $\{e_i\,; 1 \leq i \leq m\}$ is a local orthonormal frame field with respect to the metric $\tilde{g} = \langle d\tilde{x}, d\tilde{x}\rangle_1$. Thus

$$\tilde{h}_{ij} = \tilde{h}(e_i, e_j) = \tilde{y}_0^2 \tilde{h}(E_i, E_j) = \begin{cases} \varepsilon(\tilde{y}_0 h_{ij} - \tilde{n}_0 \delta_{ij}), & \text{when } 1 \leq i, j \leq K,\\ -\varepsilon \tilde{n}_0 g(E_i, E_j) = -\varepsilon \tilde{n}_0 \delta_{ij}, & \text{when } K+1 \leq i, j \leq m,\\ 0, & \text{for other} \quad i, j. \end{cases} \tag{3.26}$$

By the maximality of $\tilde{y}_1$, the mean curvature of $\tilde{x}$ is

$$\tilde{H} = \frac{1}{m}\sum \tilde{h}_{ii} = \varepsilon\frac{1}{m}(\tilde{y}_0 KH_1 - K\tilde{n}_0) - \varepsilon\frac{1}{m}(m-K)\tilde{n}_0 = -\varepsilon\tilde{n}_0 \tag{3.27}$$

and

$$|\tilde{h}|^2 = \sum_{i,j=1}^{K} \tilde{y}_0^2 h_{ij}^2 + \tilde{n}_0^2\delta_{ij}^2 - 2\tilde{n}_0\tilde{y}_0 h_{ij}\delta_{ij} + \sum_{i,j=K+1}^{m}(-\tilde{n}_0)^2\delta_{ij}^2 = \tilde{y}_0 h^2 + m\tilde{n}_0^2. \tag{3.28}$$

Therefore, by definition, the conformal factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\tilde{\rho}^2 = \frac{m}{m-1}\left(\sum_{i,j}\tilde{h}_{ij}^2 - m|\tilde{H}|^2\right) = \frac{m}{m-1}\tilde{y}_0^2\sum_{i,j}h_{ij}^2 = \tilde{y}_0^2,$$

where we have used the Gauss equation and (3.19). Hence $\tilde{\rho} = |\tilde{y}_0| = \varepsilon\tilde{y}_0 > 0$ and thus $\tilde{Y} = \tilde{\rho}(1, \tilde{x})$ is the conformal position vector of $\tilde{x}$. Consequently, the conformal metric of $\tilde{x}$ is defined by $\langle d\tilde{Y}, d\tilde{Y}\rangle_2 = g$. Furthermore, the conformal second fundamental form of $\tilde{x}$ is given by

$$\tilde{B} = \tilde{\rho}^{-1}(\tilde{h}_{ij} - \tilde{H}\delta_{ij})\omega^i\omega^j = \sum_{i,j=1}^{K} h_{ij}\omega^i\omega^j, \tag{3.29}$$

where $\{\omega^i\}$ is the local coframe field on M^m dual to $\{E_i\}$.

On the other hand, by (3.21) and the Gauss equations of $\tilde{y}_1$ and $\tilde{y}$, one finds the Ricci tensor of g as follows

$$R_{ij} = -\frac{K-1}{r^2}\delta_{ij} + \sum_{k=1}^{K} h_{ik}h_{kj}, \quad \text{if } 1 \le i, j \le K, \tag{3.30}$$

$$R_{ij} = \frac{m-K-1}{r^2}\delta_{ij}, \quad \text{if } K+1 \le i, j \le m, \tag{3.31}$$

$$R_{ij} = 0, \quad \text{if } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K, \tag{3.32}$$

which implies that the normalized scalar curvature of g is given by

$$\kappa = \frac{m((m-K)(m-K-1) - K(K-1)) + (m-1)r^2}{m^2(m-1)r^2}. \tag{3.33}$$

Thus

$$\frac{1}{2m}(m^2\kappa - 1) = \frac{(m-K)(m-K-1) - K(K-1)}{2(m-1)r^2}. \tag{3.34}$$

Since $m \ge 3$, it follows from (2.15) and (3.29)–(3.34) that the Blaschke tensor of $\tilde{x}$ is given by $A = \sum A_{ij}\omega^i\omega^j$, where

$$A_{ij} = -\frac{1}{2r^2}\delta_{ij}, \quad \text{if } 1 \le i, j \le K, \quad A_{ij} = \frac{1}{2r^2}\delta_{ij}, \quad \text{if } K+1 \le i, j \le m, \tag{3.35}$$

$$A_{ij} = 0, \quad \text{if } 1 \le i \le K,\ K+1 \le j \le m, \text{ or } K+1 \le i \le m,\ 1 \le j \le K, \tag{3.36}$$

which, once again, implies that A is parallel with two distinct eigenvalues $\lambda_1 = -\lambda_2 = -\frac{1}{2r^2}$.

4 Proof of Main Theorem

To make the argument more readable, we divide the proof into several lemmas.

Let $x : M^m \to \mathbb{S}_1^{m+1}$ be a regular space-like hypersurface.

Lemma 4.1 If the Blaschke tensor A is parallel, then the conformal form Φ vanishes identically.

Proof For any given point $p \in M^m$, take an orthonormal frame field $\{E_i\}$ around p with respect to the conformal metric g, such that $B_{ij}(p) = B_i\delta_{ij}$. Then it follows from (2.11) that

$$A_{ijk} - A_{ikj} = B_{ij}\Phi_k - B_{ik}\Phi_j.$$

Since A is parallel, $A_{ijk} = 0$ for any i, j, k. Thus at the given point p, we have

$$B_i(\delta_{ij}\Phi_k(p) - \delta_{ik}\Phi_j(p)) = 0. \tag{4.1}$$

By (2.14), there are different indices i_1, i_2 such that $B_{i_1} \neq 0$ and $B_{i_2} \neq 0$. Then for any indices i, j, we have

$$\delta_{i_1 j}\Phi_i(p) - \delta_{i_1 i}\Phi_j(p) = 0, \quad \delta_{i_2 j}\Phi_i(p) - \delta_{i_2 i}\Phi_j(p) = 0. \tag{4.2}$$

If $i = i_1$, put $j = i_2$; if $i \neq i_1$, put $j = i_1$. Then it follows from (4.2) that $\Phi_i(p) = 0$. By the arbitrariness of i and p, we obtain that $\Phi \equiv 0$.

Remark 4.1 Since A is parallel, then all eigenvalues of the Blaschke tensor A of x are constant on M^m. From the equation

$$0 = \sum A_{ijk}\omega^k = dA_{ij} - A_{kj}\omega_i^k - A_{ik}\omega_j^k,$$

we obtain that

$$\omega_j^i = 0 \quad \text{in case that } A_i \neq A_j. \tag{4.3}$$

Lemma 4.2 If A is parallel, then $B_{ij} = 0$ as long as $A_i \neq A_j$.

Proof Since A is parallel, there exists around each point a local orthonormal frame field $\{E_i\}$ such that

$$A_{ij} = A_i\delta_{ij}. \tag{4.4}$$

It follows from (2.10) and Lemma 4.1 that $\sum B_{ik}A_{kj} - A_{ik}B_{kj} = \Phi_{ij} - \Phi_{ji} = 0$. Then we have $B_{ij}(A_j - A_i) = 0$.

Now, let t be the number of the distinct eigenvalues of A, and $\lambda_1, \cdots, \lambda_t$ denote the distinct eigenvalues of A. Fix a suitably chosen orthonormal frame field $\{E_i\}$ for which the matrix (A_{ij}) can be written as

$$(A_{ij}) = \mathrm{Diag}(\underbrace{\lambda_1, \cdots, \lambda_1}_{k_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{k_2}, \cdots, \underbrace{\lambda_t, \cdots, \lambda_t}_{k_t}), \tag{4.5}$$

or equivalently,

$$A_1 = \cdots = A_{k_1} = \lambda_1, \cdots, A_{m-k_t+1} = \cdots = A_m = \lambda_t. \tag{4.6}$$

Lemma 4.3 Suppose that $t \geq 3$. If, with respect to an orthonormal frame field $\{E_i\}$, (4.5) holds and at a point p, $B_{ij} = B_i\delta_{ij}$, then $B_i = B_j$ in the case that $A_i = A_j$.

Proof By (4.3), for any i, j satisfying $A_i \neq A_j$, we have $\omega^i_j = 0$. Differentiating this equation, we obtain from (2.13) that $0 = R_{ijji} = B^2_{ij} - B_{ii}B_{jj} + A_{ii} - A_{ij}\delta_{ij} + A_{jj} - A_{ij}\delta_{ij}$. Thus at p, it holds that

$$-B_iB_j + A_i + A_j = 0. \tag{4.7}$$

If there exist indices i, j such that $A_i = A_j$ but $B_i \neq B_j$, then for all k satisfying $A_k \neq A_i$, we have

$$-B_iB_k + A_i + A_k = 0, \quad -B_jB_k + A_j + A_k = 0. \tag{4.8}$$

It follows from (4.8) that $(B_i - B_j)B_k = 0$, which implies that $B_k = 0$. Thus by (4.8), we obtain $A_k = -A_i = -A_j$. This implies that $t = 2$, contradicting the assumption.

Corollary 4.4 If $t \geq 3$, then there exists an orthonormal frame field $\{E_i\}$ such that

$$A_{ij} = A_i\delta_{ij}, \quad B_{ij} = B_i\delta_{ij}. \tag{4.9}$$

Furthermore, if (4.5) holds, then

$$(B_{ij}) = \mathrm{Diag}(\underbrace{\mu_1, \cdots, \mu_1}_{\mathrm{k_1}}, \underbrace{\mu_2, \cdots, \mu_2}_{\mathrm{k_2}}, \cdots, \underbrace{\mu_t, \cdots, \mu_t}_{\mathrm{k_t}}), \tag{4.10}$$

that is

$$B_1 = \cdots = B_{k_1} = \mu_1, \cdots, B_{m-k_t+1} = \cdots = B_m = \mu_t. \tag{4.11}$$

Proof Since A is parallel, we can find a local orthonormal frame field $\{E_i\}$, such that (4.5) holds. It then suffices to show that, at any point, the component matrix (B_{ij}) of B with respect to $\{E_i\}$ is diagonal. Note that $k_1, \ldots, k_t$ are the multiplicities of the eigenvalues $\lambda_1, \cdots, \lambda_t$, respectively. By Lemma 4.2, we can write $(B_{ij}) = \mathrm{Diag}(B_{(1)}, \cdots, B_{(t)})$, where $B_{(1)}, \cdots, B_{(t)}$ are square matrices of orders $k_1, \cdots, k_t$, respectively. For any point p, we can choose a suitable orthogonal matrix T of the form $T = \mathrm{Diag}(T_{(1)}, \cdots, T_{(t)})$, with $T_{(1)}, \cdots, T_{(t)}$ being orthogonal matrices of orders $k_1, \cdots, k_t$, such that

$$T \cdot (B_{ij}(p)) \cdot T^{-1} = \mathrm{Diag}(B_1, \cdots, B_m),$$

where $B_1, \cdots, B_m$ are the eigenvalues of tensor B at p. It then follows from Lemma 4.3 that $B_1 = \cdots = B_{k_1} := \mu_1, \cdots, B_{m-k_t+1} = \cdots = B_m := \mu_t$. Hence

$$T_{(1)}B_{(1)}(p)T^{-1}_{(1)} = \mathrm{Diag}(\mu_1, \cdots, \mu_1), \tag{4.12}$$

$$\vdots$$

$$\vdots$$

$$T_{(t)}B_{(t)}(p)T^{-1}_{(t)} = \mathrm{Diag}(\mu_t, \cdots, \mu_t). \tag{4.13}$$

Therefore

$$B_{(1)}(p) = T_{(1)}^{-1} \cdot \mathrm{Diag}(\mu_1, \cdots, \mu_1) \cdot T_{(1)} = \mathrm{Diag}(\mu_1, \cdots, \mu_1). \tag{4.14}$$

In the same way,

$$B_{(2)}(p) = \mathrm{Diag}(\mu_2, \cdots, \mu_2), \cdots, B_{(t)}(p) = \mathrm{Diag}(\mu_t, \cdots, \mu_t). \tag{4.15}$$

Thus

$$(B_{ij}(p)) = \mathrm{Diag}(\mu_1, \cdots, \mu_1, \cdots, \mu_t, \cdots, \mu_t).$$

Lemma 4.5 If $t \geq 3$, then all the conformal principal curvatures $\mu_1, \cdots, \mu_t$ of x are constant, and hence x is conformal isoparametric.

Proof Without loss of generality, we only need to show that μ_1 is constant. To this end, choose a frame field $\{E_i\}$ such that (4.5) and (4.10) hold. Note that, by (4.3), when $1 \leq i \leq k_1$ and $j > k_1$, we have

$$\sum B_{ijk}\omega^k = dB_{ij} - \sum B_{kj}\omega_i^k - \sum B_{ik}\omega_j^k = 0,$$

which implies that $B_{ijk} = 0$.

By Lemma 4.1, $\Phi \equiv 0$. Hence from (2.12) one seen that B_{ijk} is symmetric with respect to i, j, k. It follows that $B_{ijk} = 0$, in case that two indices in i, j, k are less than or equal to k_1 with the other index larger than k_1, or one index in i, j, k is less than or equal to k_1 with the other two indices larger than k_1. In particular, for any i, j satisfying $1 \leq i, j \leq k_1$,

$$\sum_{k=1}^{k_1} B_{ijk}\omega^k = dB_{ij} - \sum B_{kj}\omega_i^k - \sum B_{ik}\omega_j^k = dB_i\delta_{ij} - B_j\omega_i^j - B_i\omega_j^i.$$

Putting $j = i$, one obtains

$$\sum_{k=1}^{k_1} B_{iik}\omega^k = d\mu_1, \tag{4.16}$$

which implies that

$$E_k(\mu_1) = 0, \quad k_1 + 1 \leq k \leq m. \tag{4.17}$$

Similarly,

$$E_i(B_j) = 0, \quad 1 \leq i \leq k_1,\ k_1 + 1 \leq j \leq m. \tag{4.18}$$

On the other hand, we see from (4.7) that

$$-\mu_1 B_j + \lambda_1 + A_j = 0, \quad k_1 + 1 \leq j \leq m \tag{4.19}$$

hold identically. Differentiating (4.19) in the direction of E_k, $1 \leq k \leq k_1$, and using (4.18), we obtain

$$E_k(\mu_1)B_j = 0, \quad 1 \leq k \leq k_1,\ k_1 + 1 \leq j \leq m.$$

By (2.14) there exists some index j such that $k_1 + 1 \leq j \leq m$ and $B_j \neq 0$. Therefore, $E_k(\mu_1) = 0$ for $1 \leq k \leq k_1$. This together with (4.17) implies that μ_1 is a constant.

Corollary 4.6 If $t \geq 3$, then $t = 3$ and B is parallel.

Proof Indeed, the conclusion that B is parallel comes from (4.3), Corollary 4.4 and Lemma 4.5.

If $t > 3$, then there exist at least four indices i_1, i_2, i_3, i_4, such that $A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4}$ are distinct each other. Then it follows from (4.7) that

$$-B_{i_1}B_{i_2} + A_{i_1} + A_{i_2} = 0, \quad -B_{i_3}B_{i_4} + A_{i_3} + A_{i_4} = 0, \tag{4.20}$$

$$-B_{i_1}B_{i_3} + A_{i_1} + A_{i_3} = 0, \quad -B_{i_2}B_{i_4} + A_{i_2} + A_{i_4} = 0. \tag{4.21}$$

Consequently, we obtain $(A_{i_1} - A_{i_4})(A_{i_2} - A_{i_3}) = 0$, a contradiction.

Lemma 4.7 If $t \leq 2$ and B is not parallel, then one of the following cases holds:

(1) $t = 1$ and x is conformal isotropic;

(2) $t = 2$, $\lambda_1 + \lambda_2 = 0$ and $B_i = 0$ either for all $1 \leq i \leq k_1$, or for all $k_1 + 1 \leq i \leq m$.

Proof Note that $\Phi \equiv 0$. Thus x is conformal isotropic if and only if $t = 1$.

If $t = 2$, then for any point $p \in M^m$, we can find an orthonormal frame field $\{E_i\}$ such that (4.9) holds at p.

By (4.3), we see that

$$\omega_j^i = 0, \quad 1 \leq i \leq k_1, \quad k_1 + 1 \leq j \leq m \tag{4.22}$$

hold identically. Taking exterior differentiation of (4.22) and making use of (2.13), we find that, at p

$$-B_iB_j + A_i + A_j = 0, \quad 1 \leq i \leq k_1, \quad k_1 + 1 \leq j \leq m. \tag{4.23}$$

If there exist one pair of indices i_0, j_0 satisfying $1 \leq i_0 \leq k_1$, $k_1 + 1 \leq j_0 \leq m$ such that $B_{i_0} \neq 0$ and $B_{j_0} \neq 0$, then for each index i satisfying $1 \leq i \leq k_1$, we obtain

$$-B_{i_0}B_{j_0} + A_{i_0} + A_{j_0} = 0, \quad -B_iB_{j_0} + A_i + A_{j_0} = 0$$

from which it follows that $(B_i - B_{i_0})B_{j_0} = 0$, or equivalently $B_i = B_{i_0}, \quad 1 \leq i \leq k_1$. Similarly, we obtain $B_j = B_{j_0}, \quad k_1 + 1 \leq j \leq m$. Consequently, (4.10) also holds in the case that $t = 2$. Now, an argument similar to that in the proof of Lemma 4.5 shows that the conformal principal curvatures B_i are all constant. Therefore B is parallel by (4.22), contradicting to the assumption. Thus either $B_i = 0$ for all indices i satisfying $1 \leq i \leq k_1$, or $B_j = 0$ for all indices j satisfying $k_1 + 1 \leq j \leq m$. In both cases we have, by (4.23), $\lambda_1 + \lambda_2 = 0$.

Proof of Theorem 1.2 By Theorem 1.1 and Theorem 1.2, it clearly suffices to consider the case that x neither is conformal isotropic nor has parallel conformal second fundamental form. Hence from those Lemmas proved in this section, we can suppose without loss of generality that

$$t = 2, \quad \lambda_1 = -\lambda_2 = \lambda \neq 0, \quad B_{k_1+1} = \cdots = B_m = 0. \tag{4.24}$$

Since $\sum B_i = 0$ and $\sum B_i^2 = (m-1)/m$, one sees easily that $m \geq 3$. Since A is parallel, the tangent bundle TM^m of M^m has a decomposition $TM^m = V_1 \oplus V_2$, where V_1 and V_2 are the eigenspaces of A corresponding to the eigenvalues $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$, respectively.

Let $\{E_i\,;1 \le i \le k_1\}$ and $\{E_j\,;k_1+1 \le j \le m\}$ be orthonormal frame fields for subbundles V_1 and V_2, respectively. Then $\{E_i\,;1 \le i \le m\}$ is an orthonormal frame field on M^m with respect to the conformal metric g. Then (4.22) implies that both V_1 and V_2 are integrable, and thus Riemannian manifold (M^m, g) can be locally decomposed into a direct product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2), that is, as a Riemannian manifold, locally

$$(M^m, g) = (M_1, g_1) \times (M_2, g_2). \tag{4.25}$$

It follows from (2.13), (4.5), (4.24) and (4.25) that the Riemannian curvature tensors of (M_1, g_1) and (M_2, g_2) have the following components, respectively,

$$R_{ijkl} = 2\lambda(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + (B_{ik}B_{jl} - B_{il}B_{jk}), \quad 1 \le i,j,k,l \le k_1, \tag{4.26}$$

$$R_{ijkl} = -2\lambda(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \quad k_1+1 \le i,j,k,l \le m. \tag{4.27}$$

Thus (M_2, g_2) is of constant sectional curvature -2λ.

Next we consider the following cases separately.

Case (1) $\lambda > 0$. In this case, set $r = (2\lambda)^{-1/2}$. Then (M_2, g_2) can be locally identified with $\mathbb{H}^{m-k_1}\left(-\frac{1}{r^2}\right)$. Let $\tilde{y} = (\tilde{y}_0, \tilde{y}_2) : \mathbb{H}^{m-k_1}\left(-\frac{1}{r^2}\right) \to \mathbb{R}_1^{m-k_1+1}$ be the canonical embedding.

Since $h = \sum\limits_{i,j=1}^{k_1} B_{ij}\omega^i\omega^j$ is a Codazzi tensor on (M_1, g_1), it follows from (4.26) that there exists a maximal immersed hypersurface

$$\tilde{y}_1 : (M_1, g_1) \to \mathbb{S}_1^{k_1+1}(r) \subset \mathbb{R}_1^{k_1+2}, \quad 2 \le k_1 \le m-1,$$

which has h as its second fundamental form. Clearly, $\tilde{y}_1$ has constant scalar curvature

$$S_1 = \frac{mk_1(k_1-1) + (m-1)r^2}{mr^2},$$

and M^m can be locally identified with $\tilde{M}^m = (M_1, g_1) \times \mathbb{H}^{m-k_1}(-\frac{1}{r^2})$.

Define $\tilde{x}_1 = \tilde{y}_1/\tilde{y}_0$, $\tilde{x}_2 = \tilde{y}_2/\tilde{y}_0$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$. Then, by the discussion in Example 3.2, $\tilde{x} : \tilde{M}^m \to \mathbb{S}_1^{m+1}$ yields a regular space-like hypersurface with the given g and B as its conformal metric and conformal second fundamental form, respectively. Therefore, by Theorem 2.4, x is conformal equivalent to $\tilde{x}$.

Case (2) $\lambda < 0$. In this case, set $r = (-2\lambda)^{-1/2}$, then (M_2, g_2) can be locally identified with $\mathbb{S}^{m-k_1}(r)$. Let $\tilde{y}_2 : \mathbb{S}^{m-k_1}(r) \to \mathbb{R}^{m-k_1+1}$ be the canonical embedding.

Since $h = \sum\limits_{i,j=1}^{k_1} B_{ij}\omega^i\omega^j$ is a Codazzi tensor on (M_1, g_1), it follows from (4.26) that there exists a maximal immersed hypersurface

$$\tilde{y} = (\tilde{y}_0, \tilde{y}_1) : (M_1, g_1) \to \mathbb{H}_1^{k_1+1}\left(-\frac{1}{r^2}\right) \subset \mathbb{R}_2^{k_1+2}, \quad 2 \le k_1 \le m-1,$$

which has h as its second fundamental form. Clearly, $\tilde{y}$ has constant scalar curvature

$$S_1 = \frac{-mk_1(k_1-1) + (m-1)r^2}{mr^2},$$

and M^m can be locally identified with $\tilde{M}^m = (M_1, g_1) \times \mathbb{S}^{m-k_1}(r)$.

Assume without loss of generality that $\tilde{y}_0 \neq 0$. Define $\varepsilon = \mathrm{Sgn}(\tilde{y}_0)$ and let $\tilde{x}_1 = \varepsilon\tilde{y}_1/\tilde{y}_0$, $\tilde{x}_2 = \varepsilon\tilde{y}_2/\tilde{y}_0$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$. Then, by the discussion in Example 3.3, $\tilde{x} : \tilde{M}^m \to \mathbb{S}^{m+1}_1$ defines a regular space-like hypersurface with the given g and B as its conformal metric and conformal second fundamental form, respectively. It follows by Theorem 2.4 that x is conformal equivalent to $\tilde{x}$.

References

[1] Nie Changxiong, Wu Chuanxi. Space-like hypersurfaces with parallel conformal second fundamental forms in the conformal space (in Chinese)[J]. Acta Math. Sinica, Chin. Ser., 2008, 51(4): 685–692.

[2] Nie Changxiong, Wu Chuanxi. Regular submanifolds in the conformal space $\mathbb{Q}^n_p$[J]. Chin. Ann. Math., 2012, 33: 695–714.

[3] Nie Changxiong, Li Tongzhu, He Yijun, Wu Chuanxi. Conformal isoparametric hypersurfaces with two distinct conformal principal curvatures in conformal space[J]. Sci. China Math., 2010, 53(4): 953–965.

[4] Hu Zejun, Li Haizhong. Classification of hypersurfaces with parallel Möbius second fundamental form in S^{n+1}[J]. Sci. China Ser. A, 2004, 47: 417–430.

[5] Han Yingbo, Feng Shuxiang. On complete hypersurfaces in hyperbolic space form $H^{n+1}(-1)$[J]. J. Math., 2013, 33(5): 767–772.

[6] Hu Zejun, Li Xingxiao, Zhai Shujie. On the Blaschke isoparametric hypersurfaces in the unit sphere with three distinct Blaschke eigenvalues[J]. Sci. China Math., 2011: 54: 2171–2194.

[7] Li Xingxiao, Zhang Fengyun. A classification of immersed hypersurfaces in spheres with parallel Blaschke tensors[J]. Tohoku Math. J., 2006, 58: 581–597.

[8] Li Xingxiao, Zhang Fengyun. Immersed hypersurfaces in the unit sphere S^{m+1} with constant Blaschke eigenvalues[J]. Acta Math. Sinica, Engl. Ser., 2007, 23(3): 533–548.

[9] Li Xingxiao, Zhang Fengyun. On the Blaschke isoparametric hypersurfaces in the unit sphere[J]. Acta Math. Sinica, Engl. Ser., 2009, 25(4): 657–678.

[10] Wang Changping. Möbius geometry of submanifolds in $\mathbb{S}^n$[J]. Manus. Math., 1998, 96: 517–534.

de Sitter空间$\mathbb{S}^{m+1}_1$中具有平行Blaschke张量的正则类空超曲面

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摘 要: 本文引入两个以de Sitter空间为模型的非齐性坐标来覆盖共形空间$\mathbb{Q}^{m+1}_1$. 利用球面$\mathbb{S}^{m+1}$中超曲面的Möbius 几何的方法, 本文研究了$\mathbb{Q}^{m+1}_1$中正则类空超曲面的共形几何. 作为其结果, 本文对所有具有平行Blaschke张量的正则类空超曲面进行了完全分类.

关键词: 共形形式; 平行Blaschke张量; 共形度量; 共形第二基本形式; 极大超曲面; 常数量曲率

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