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GROUND STATE SOLUTIONS FOR NONLINEAR DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract: In this paper, we study the existence of ground state solutions for nonlinear second order difference equations with periodic coefficients. Using the critical point theory in combination with the Nehari manifold approach, the existence of ground state solutions is established. Under a more general super-quadratic condition than the classical Ambrosetti-Rabinowitz condition, the results considerably generalize some existing ones. Finally, an example is also presented to demonstrate our results.

Keywords: nonlinear difference equations; Nehari manifold; ground state solutions; critical point theory

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1 Introduction

In this paper, we consider the following nonlinear second order difference equation

$$\begin{cases} \Delta[a(k)\Delta u(k-1)] - b(k)u(k) + f(k,u(k)) = 0, & k \in \mathbb{Z}, \\ u(k) \to 0, & |k| \to \infty, \end{cases} \tag{1.1}$$

where $a(k)$, $b(k)$ and $f(k,u)$ are T-periodic in k, $f(k,u) : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a continuous function on u. The forward difference operator Δ is defined by

$$\Delta u(k) = u(k+1) - u(k) \quad \text{for all } k \in \mathbb{Z},$$

where $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all integers and real numbers, respectively.

The solutions of (1.1) are referred to as homoclinic solutions of the equation

$$\Delta[a(k)\Delta u(k-1)] - b(k)u(k) + f(k,u(k)) = 0, \quad k \in \mathbb{Z}. \tag{1.2}$$

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In the theory of differential equations, homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If a system has the smoothly connected homoclinic orbits, then it can not stand the perturbation, its perturbed system probably produces chaotic. Therefore, it is of practical importance and mathematical significance to study the existence of homoclinic solutions.

Difference equations represent the discrete counterpart of ordinary differential equations. The classical methods are used in difference equations, such as numerical analysis, fixed point methods, linear and nonlinear operator theory, see [1–4]. In recent years, the existence and multiplicity of homoclinic solutions for difference equations have been studied in many papers by variational methods, see [5–12] and the reference therein.

Assume the following conditions hold:

(A) $a(k) > 0$ and $a(k+T) = a(k)$ for all $k \in \mathbb{Z}$.

(B) $b(k) > 0$ and $b(k+T) = b(k)$ for all $k \in \mathbb{Z}$.

(H1) $f \in C(\mathbb{Z} \times \mathbb{R}, \mathbb{R})$, and there exist $C > 0$ and $p \in (2, \infty)$ such that

$$|f(k,u)| \le C(1 + |u|^{p-1}) \text{ for all } k \in \mathbb{Z}, u \in \mathbb{R}.$$

(H2) $\lim\limits_{|u|\to 0} f(k,u)/u = 0$ uniformly for $k \in \mathbb{Z}$.

(H3) $\lim\limits_{|u|\to\infty} F(k,u)/|u|^2 = +\infty$ uniformly for $k \in \mathbb{Z}$, where $F(k,u)$ is the primitive function of $f(k,u)$, i.e.,

$$F(k,u) = \int_0^u f(k,s)ds.$$

(H4) $u \mapsto f(k,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.

Remark 1.1 (H2) implies that $u(k) \equiv 0$ is a trivial solution of (1.1).

Our main result is following.

Theorem 1.1 Suppose that conditions (A), (B) and (H1)–(H4) are satisfied. Then equation (1.1) has at least a nontrivial ground state solution, i.e., solution corresponding to the least energy of the action functional of (1.1).

Remark 1.2 In [6], the authors also considered (1.1) and assumed that (H2) and the following classical Ambrosetti-Rabinowitz superlinear condition (see [16, 17]): there exists a constant $\mu > 2$ such that

$$0 < \mu F(k,u) \le f(k,u)u, \quad u \ne 0, \tag{1.3}$$

(1.3) implies that for each $k \in \mathbb{Z}$, there exists a constant $C > 0$ such that

$$F(k,u) \ge C|u|^\mu \text{ for } |u| \ge 1. \tag{1.4}$$

This implies (H3) holds. There exists a superlinear function, such as

$$f(k,u) = u\ln(1+|u|)$$

does not satisfy (1.3). However, it satisfies the conditions (H1)–(H4). So our conditions are weaker than conditions presented in [6]. And in our paper, we do not need periodic approximation technique to obtain homoclinic solutions. Furthermore, the existence of ground state solutions can be obtained.

Remark 1.3 In [13], the authors considered the following difference equation

$$\Delta(\varphi_p(\Delta u(n-1)) - q(n)\varphi_p(u_n) + f(n, u_{n+M}, u_n, u_{n-M}) = 0, \quad n \in \mathbb{Z}. \tag{1.5}$$

Let $p = 0$ and $M = 0$, (1.2) is the special case of (1.5).

The following hypotheses with $p = 0$ and $M = 0$ are satisfied in [13]:

(F1) $F \in C(\mathbb{Z} \times \mathbb{R}, \mathbb{R})$ with $F(n+T, u) = F(n, u)$ and it satisfies $\frac{\partial F}{\partial u} = f(n, u)$;

(F2) there exist positive constants ϱ and $a < \frac{q}{4}(\frac{\kappa_1}{\kappa_2})^2$ such that $|F(n, u)| \le a|u|^2$ for all $n \in \mathbb{Z}$ and $|u| \le \varrho$;

(F3) there exist constants ρ, $c > \frac{1}{4}(\frac{\kappa_1}{\kappa_2})^2(4 + \overline{q})$ and b such that $F(n, u) \ge c|u|^2 + b$ for all $n \in \mathbb{Z}$ and $|u| \ge \rho$;

(F4) $fu - 2F > 0$ for all $n \in \mathbb{Z}$ and $|u| \ne 0$;

(F5) $fu - 2F \to \infty$, as $|u| \to \infty$.

Note that (H2)–(H4) imply that (F2)–(F4). A nontrivial homoclinic orbit of (1.5) is obtained by Mountain Pass lemma in combination with periodic approximations in [13]. However, in our paper, we employ the Nehari manifold approach instead of periodic approximation technique to obtain the ground state solutions. Furthermore, we show that the functional is coercive on Nehari manifold (Lemma 3.2), which is weaker than P.S. condition in [13].

The rest of this paper is organized as follows. In Section 2, we establish the variational framework associated with (1.1), and transfer the existence of solutions of boundary value problem (1.1) into the existence of critical points of the corresponding functional. By employing the critical point theory, we give proofs of the main results in Section 3. Finally, we give a simple example to demonstrate our results.

2 Variational Framework

In this section, we firstly establish the corresponding variational setting associated with (1.1). Let

$$X = \{u = \{u(k)\} : u(k) \in \mathbb{R}, k \in \mathbb{Z}\}$$

be the set of all real sequences

$$u = \{u(k)\}_{k\in\mathbb{Z}} = (\cdots, u(-k), u(-k+1), \cdots, u(-1), u(0), u(1), \cdots, u(k), \cdots).$$

Then X is a vector space with $au + bv = \{au(k) + bu(k)\}$ for $u, v \in X$, $a, b \in \mathbb{R}$.

Define the space

$$E := \{\, u \in X : \sum_{k\in\mathbb{Z}} \left[a(k)(\Delta u(k-1))^2 + b(k)(u(k))^2\right] < +\infty\}.$$

Then E is a Banach space equipped with the corresponding norm

$$\|u\|^2 = \sum_{k\in\mathbb{Z}} \left[a(k)(\Delta u(k-1))^2 + b(k)(u(k))^2\right], \quad \forall u \in E.$$

For $1 \le p < +\infty$, denote

$$l^p = \{u = \{u(k)\} \in X : \sum_{k\in\mathbb{Z}} |u(k)|^p < \infty\}$$

equipped with the norm

$$\|u\|_{l^p} = \left(\sum_{k\in\mathbb{Z}} |u(k)|^p\right)^{\frac{1}{p}},$$

$|\cdot|$ is the usual absolute value in $\mathbb{R}$. Then the following embedding between l^p spaces holds

$$l^q \subset l^p, \|u\|_{l^p} \le \|u\|_{l^q}, 1 \le q \le p \le \infty. \tag{2.1}$$

Now we consider the variational functional J defined on E by

$$\begin{aligned} J(u) &= \sum_{k\in\mathbb{Z}} \left[\frac{1}{2}a(k)(\Delta u(k-1))^2 + \frac{1}{2}b(k)(u(k))^2 - F(k,u(k))\right] \\ &= \frac{1}{2}\|u\|^2 - \sum_{k\in\mathbb{Z}} F(k,u(k)). \end{aligned}$$

Then $J \in C^1(E,\mathbb{R})$, and for all $u, v \in E$, we have

$$(J'(u), v) = \sum_{k\in\mathbb{Z}} \left[a(k)\Delta u(k-1)\Delta v(k-1) + b(k)u(k)v(k) - f(k,u(k))v(k)\right]. \tag{2.2}$$

Then we easily get the variational formulation for (1.1).

Lemma 2.1 Every critical point $u \in E$ of J is a solution of (1.1).

Proof We assume that $u \in E$ is a critical point of J, then $J'(u) = 0$. According to (2.2), this is equivalent to

$$\sum_{k\in\mathbb{Z}} \left[a(k)\Delta u(k-1)\Delta v(k-1) + b(k)u(k)v(k) - f(k,u(k))v(k)\right] = 0, \quad \forall v \in E. \tag{2.3}$$

For any $h \in \mathbb{Z}$, we define $e_h \in E$ by putting $e_h(k) = \delta_{hk}$ for all $k \in \mathbb{Z}$, where $\delta_{hk} = 1$ if $h = k$; $\delta_{hk} = 0$ if $h \ne k$. If we apply (2.3) with $v = e_h$, then

$$-\Delta[a(k)\Delta u(k-1)] + b(k)u(k) - f(k,u(k)) = 0,$$

i.e., u is a solution of (1.1). The proof is completed.

Therefore $u \ne 0$ is a critical point of J on E if and only if u is a solution of (1.1) with $u(\pm\infty) = 0$, that is to say, u is a homoclinic solution emanating from 0. Thus, we have reduced the problem of finding homoclinic solutions of (1.1) to that of seeking critical points of the functional J on E.

3 Proof of Main Results

Now, we consider the Nehari manifold $\mathcal{N} = \{u \in E \setminus \{0\} : J'(u)u = 0\}$, and let $c = \inf\limits_{u\in\mathcal{N}} J(u)$. By the definition of $\mathcal{N}$, we know $\mathcal{N}$ contains all nontrivial critical points of J.

Lemma 3.1 Assume that (A), (B) and (H2)–(H4) are satisfied, then $\mathcal{N}$ is homeomorphic to the unit sphere S in E, where $S = \{u \in E : \|u\| = 1\}$.

Proof Let $\varphi(u) = \sum\limits_{k\in\mathbb{Z}} F(k, u(k))$. By (H2), we have

$$\varphi'(u) = o(\|u\|) \ \text{ as } \ u \to 0. \tag{3.1}$$

Let $U \subset E \setminus \{0\}$ be a weakly compact subset, we know that

$$\varphi(su)/s^2 \to \infty \text{ uniformly for } u \text{ on } U \text{ as } s \to \infty. \tag{3.2}$$

In fact, let $\{u_n\} \subset U$. It needs to show that

$$\text{if } s_n \to \infty, \quad \varphi(s_n u_n)/(s_n)^2 \to \infty$$

as $n \to \infty$. Passing to a subsequence if necessary, $u_n \rightharpoonup u \in E \setminus \{0\}$ and $u_n(k) \to u(k)$ for every k, as $n \to \infty$.

Note that from (H2) and (H4), it is easy to get that

$$F(k, u) > 0 \ \text{ for all } u \neq 0. \tag{3.3}$$

Since $|s_n u_n(k)| \to \infty$ and $u_n \neq 0$, by (H3) and (3.3), we have

$$\frac{\varphi(s_n u_n)}{(s_n)^2} = \sum_{k\in\mathbb{Z}} \frac{F(k, s_n u_n(k))}{(s_n u_n(k))^2}(u_n(k))^2 \to \infty \ \text{ as } n \to \infty.$$

Thus we obtain (3.2) holds.

From (H4), for all $u \neq 0$ and $s > 0$, we have

$$s \mapsto \varphi'(su)u/s \ \text{ is strictly increasing.} \tag{3.4}$$

Let $h(s) := J(sw),\ s > 0$. Then

$$h'(s) = J'(sw)w = s(\|w\|^2 - s^{-1}\varphi(sw)w)$$

from (3.1)–(3.4), then there exists a unique s_w, such that, when $0 < s < s_w$, $h'(s) > 0$; and when $s > s_w$, $h'(s) < 0$. Therefore $h'(s_w) = J'(s_w w)w = 0$ and $s_w w \in \mathcal{N}$.

Therefore s_w is a unique maximum of $h(s)$, and we can define the mapping $\hat{m} : E \setminus \{0\} \to \mathcal{N}$ by setting $\hat{m}(w) := s_w w$. Then the mapping $\hat{m}$ is continuous. Indeed, suppose $w_n \to w \neq 0$. Since $\hat{m}(tu) = \hat{m}(u)$ for each $t > 0$, we may assume $w_n \in S$ for all n. Write $\hat{m}(w_n) = s_{w_n} w_n$. Then $\{s_{w_n}\}$ is bounded. If not, $s_{w_n} \to \infty$ as $n \to \infty$.

Note that by (H4), for all $u \neq 0$,

$$\begin{aligned}\frac{1}{2}f(k,u)u - F(k,u) &= \frac{1}{2}f(k,u)u - \int_0^u f(k,s)ds \\ &> \frac{1}{2}f(k,u)u - \frac{f(k,u)}{u}\int_0^u sds = 0.\end{aligned}$$

So for all $u \in \mathcal{N}$, we have

$$J(u) = J(u) - \frac{1}{2}J'(u)u = \sum_{k\in\mathbb{Z}}\left(\frac{1}{2}f(k,u(k))u(k) - F(k,u(k))\right) > 0. \tag{3.5}$$

By (H3), we have

$$0 < \frac{J(s_{wn}w)}{(s_{wn})^2} = \frac{1}{2}\|w\|^2 - \sum_{k\in\mathbb{Z}}\frac{F(k,s_{wn}w(k))}{(s_{wn}w(k))^2}(w(k))^2 \to -\infty \quad \text{as } n\to\infty,$$

which is a contradiction. Therefore, $s_{w_n} \to s > 0$ after passing to a subsequence if needed. Since $\mathcal{N}$ is closed and $\hat{m}(w_n) = s_{w_n}w_n \to sw, sw \in \mathcal{N}$. Hence $sw = s_w w = \hat{m}(w)$ by the uniqueness of s_w.

Therefore we define a mapping $m : S \to \mathcal{N}$ by setting $m := \hat{m}|_S$, then m is a homeomorphism between S and $\mathcal{N}$.

We also consider the functional $\hat{\Psi} : E\setminus\{0\} \to \mathbb{R}$ and $\Psi : S \to \mathbb{R}$ by

$$\hat{\Psi}(w) := J(\hat{m}(w)) \quad \text{and} \quad \Psi(w) := \hat{\Psi}|_S.$$

Then we have $\hat{\Psi} \in C^1(E\setminus\{0\},\mathbb{R})$ and

$$\hat{\Psi}'(w)z = \frac{\|\hat{m}(w)\|}{\|w\|}J'(\hat{m}(w))z \quad \text{for all } w,z\in E, w\neq 0.$$

In fact, let $w \in E\setminus\{0\}$ and $z \in E$. By Lemma 3.1 and the mean value theorem, we obtain

$$\begin{aligned}\hat{\Psi}(w+tz) - \hat{\Psi}(w) &= J(s_{w+tz}(w+tz)) - J(s_w w) \\ &\leq J(s_{w+tz}(w+tz)) - J(s_{w+tz}(w)) \\ &= J'(s_{w+tz}(w+\tau_t tz))s_{w+tz}tz,\end{aligned}$$

where $|t|$ is small enough and $\tau_t \in (0,1)$. Similarly,

$$\begin{aligned}\hat{\Psi}(w+tz) - \hat{\Psi}(w) &= J(s_{w+tz}(w+tz)) - J(s_w w) \\ &\geq J(s_w(w+tz)) - J(s_w(w)) \\ &= J'(s_w(w+\eta_t tz))s_w tz,\end{aligned}$$

where $\eta_t \in (0,1)$. Combining these two inequalities and the continuity of function $w \mapsto s_w$, we have

$$\lim_{t\to 0}\frac{\hat{\Psi}(w+tz) - \hat{\Psi}(w)}{t} = s_w J'(s_w w)z = \frac{\|\hat{m}(w)\|}{\|w\|}J'(\hat{m}(w))z.$$

Hence the Gâteaux derivative of $\hat{\Psi}$ is bounded linear in z and continuous in w. It follows that $\hat{\Psi}$ is a class of C^1 (see [15], Proposition1.3).

Since $w \in S$, $m(w) = \hat{m}(w)$, so we have $\Psi \in C^1(S, \mathbb{R})$ and

$$\Psi'(w)z = \|m(w)\| J'(m(w))z \quad \text{for all } z \in T_w(S) = \{v \in E : (w, v) = 0\}.$$

Lemma 3.2 Assume that (A), (B) and (H1)–(H3) are satisfied, for $u \in \mathcal{N}$ then $J(u) \to \infty$ as $\|u\| \to \infty$.

Proof By way of contradiction, we assume that there exists a sequence $\{u_n\} \subset \mathcal{N}$ such that $J(u_n) \leq d$, as $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, and then there exists a subsequence, still denoted by v_n, such that $v_n \rightharpoonup v$, and therefore $v_n(k) \to v(k)$ for every k, as $n \to \infty$.

First we can prove that there exist $\delta > 0$ and $k_j \in \mathbb{Z}$ such that

$$|v_n(k_j)| \geq \delta. \tag{3.6}$$

In fact, if not, then $v_n \to 0$ in l^∞ as $n \to \infty$. For $q > 2$,

$$\|v_n\|_{l^q}^q \leq \|v_n\|_{l^\infty}^{q-2} \|v_n\|_{l^2}^2,$$

so we have $v_n \to 0$ in all l^q, $q > 2$.

Note that from (H1) and (H2), we have for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f(k, u)| \leq \varepsilon |u| + c_\varepsilon |u|^{p-1} \quad \text{and} \quad |F(k, u)| \leq \varepsilon |u|^2 + c_\varepsilon |u|^p. \tag{3.7}$$

Then for each $s > 0$, we have

$$\sum_{k \in \mathbb{Z}} F(k, s v_n(k)) \leq \varepsilon s^2 \|v_n\|_{l^2}^2 + c_\varepsilon s^p \|v_n\|_{l^p}^p,$$

which implies that $\sum\limits_{k \in \mathbb{Z}} F(k, s v_n(k)) \to 0$ as $n \to \infty$. So

$$d \geq J(u_n) \geq J(s v_n) = \frac{s^2}{2} \|v^{(k)}\|^2 - \sum_{k \in \mathbb{Z}} F(k, s v_n(k)) \to \frac{s^2}{2}, \tag{3.8}$$

as $n \to \infty$. This is a contradiction if $s > \sqrt{2d}$.

By periodicity of coefficients, we know J and $\mathcal{N}$ are both invariant under T-translation. Making such shifts, we can assume that $1 \leq k_j \leq T - 1$ in (3.6). Moreover, passing to a subsequence, we can assume that $k_j = k_0$ is independent of j.

Next we can extract a subsequence, still denoted by $\{v_n\}$, such that $v_n(k) \to v(k)$ for all $k \in \mathbb{Z}$. Specially, for $k = k_0$, inequality (3.6) shows that $|v(k_0)| \geq \delta$, so $v \neq 0$. Since $|u_n(k)| \to \infty$ as $n \to \infty$, it follows again from (H3) that

$$0 \leq \frac{J(u_n)}{\|u_n\|^2} = \frac{1}{2} - \sum_{k \in \mathbb{Z}} \frac{F(k, u_n(k))}{(u_n(k))^2} (v_n(k))^2 \to -\infty \quad \text{as } n \to \infty,$$

a contradiction again.

From above, we have the following lemma, which is important in this paper.

Lemma 3.3 $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{m(w_n)\}$ is a Palais-Smale sequence for J.

Proof Let $\{w_n\}$ be a Palais-Smale sequence for Ψ, and let $u_n = m(w_n) \in \mathcal{N}$. Since for every $w_n \in S$ we have an orthogonal splitting $E = T_{w_n}S \oplus \mathbb{R}w_n$, we have

$$\|\Psi'(w_n)\| = \sup_{\substack{z\in T_{wn}S \\ \|z\|=1}} \Psi'(w_n)z = \|m(w_n)\| \sup_{\substack{z\in T_{wn}S \\ \|z\|=1}} J'(m(w_n))z = \|u_n\| \sup_{\substack{z\in T_{wn}S \\ \|z\|=1}} J'(u_n)z.$$

Then

$$\begin{aligned}\|\Psi'(w_n)\| &\leq \|u_n\|\|J'(u_n)\| = \|u_n\| \sup_{\substack{z\in T_{wn}S,\ t\in\mathbb{R} \\ z+tw\neq 0}} \frac{J'(u_n)(z+tw)}{\|z+tw\|} \\ &\leq \|u_n\| \sup_{z\in T_{wn}S\setminus\{0\}} \frac{J'(u_n)(z)}{\|z\|} = \|\Psi'(w_n)\|.\end{aligned}$$

Therefore

$$\|\Psi'(w_n)\| = \|u_n\|\|J'(u_n)\|. \tag{3.9}$$

By (3.5), for $u_n \in \mathcal{N}$, $J(u_n) > 0$, so there exists a constant $c_0 > 0$ such that $J(u_n) > c_0$. And since $c_0 \leq J(u_n) = \frac{1}{2}\|u_n\|^2 - I(u_n) \leq \frac{1}{2}\|u_n\|^2$, $\|u_n\| \geq \sqrt{2c_0}$. Together with Lemma3.2, $\sqrt{2c_0} \leq \|u_n\| \leq \sup_n \|u_n\| < \infty$. Hence $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{u_n\}$ is a Palais-Smale sequence for J.

Now, we give the detailed proof of Theorem 1.1.

Proof From(3.9), $\Psi'(w) = 0$ if and only if $J'(m(w)) = 0$. So w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of J. Moreover, the corresponding values of Ψ and J coincide and $\inf_S \Psi = \inf_{\mathcal{N}} J$.

Let $u_0 \in \mathcal{N}$ such that $J(u_0) = c$, then $m^{-1}(u_0) \in S$ is a minimizer of Ψ and therefore a critical point of Ψ, so u_0 is a critical point of J. It needs to show that there exists a minimizer $u \in \mathcal{N}$ of $J|_{\mathcal{N}}$.

Let $\{w_n\} \subset S$ be a minimizing sequence for Ψ. By Ekeland's variational principle we may assume $\Psi(w_n) \to c$, $\Psi'(w_n) \to 0$ as $n \to \infty$, hence $J(u_n) \to c$, $J'(u_n) \to 0$ as $n \to \infty$, where $u_n := m(w_n) \in \mathcal{N}$.

It follows from Lemma 3.2 that $\{u_n\}$ is bounded in $\mathcal{N}$, then there exists a subsequence, still denoted by the same notation, such that u_n weakly converges to some $u \in E$. We know that there exist $\delta > 0$ and $k_j \in \mathbb{Z}$ such that

$$|u_n(k_j)| \geq \delta. \tag{3.10}$$

If not, then $u_n \to 0$ in l^∞ as $n \to \infty$. Note that, for $q > 2$,

$$\|u_n\|_{l^q}^q \leq \|u_n\|_{l^\infty}^{q-2}\|u_n\|_{l^2}^2,$$

then $u_n \to 0$ in all l^q, $q > 2$. By (3.7), we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} f(k,u_n(k))u_n(k) &\le \varepsilon\sum_{k\in\mathbb{Z}}|u_n(k)|\cdot|u_n(k)| + c_\varepsilon\sum_{k\in\mathbb{Z}}|u_n(k)|^{p-1}\cdot|u_n(k)|\\ &\le \varepsilon\|u_n\|_{l^2}\cdot\|u_n\|_{l^2} + c_\varepsilon\|u_n\|_{l^p}^{p-1}\cdot\|u_n\|_{l^p}\\ &\le \varepsilon\|u_n\|_{l^2}\cdot\|u_n\| + c_\varepsilon\|u_n\|_{l^p}^{p-1}\cdot\|u_n\|,\end{aligned}$$

which implies that $\sum\limits_{k\in\mathbb{Z}} f(k,u_n(k))u_n(k) = o(\|u_n\|)$ as $n\to\infty$. Therefore

$$o(\|u_n\|) = (J'(u_n),u_n) = \|u_n\|^2 - \sum_{k\in\mathbb{Z}} f(k,u_n(k))u_n(k) = \|u_n\|^2 - o(\|u_n\|).$$

So $\|u_n\|^2 \to 0$, as $n\to\infty$, which contradicts with $u_n\in\mathcal{N}$.

Since J and J' are both invariant under T-translation. Making such shifts, we assume that $1\le k_j\le T-1$ in (3.10). Moreover passing to a subsequence, we assume that $k_j = k_0$ is independent of j. Extracting a subsequence, still denoted by $\{u_n\}$, we have $u_n \rightharpoonup u$ and $u_n(k)\to u(k)$ for all $k\in\mathbb{Z}$. Specially, for $k=k_0$, inequality (3.10) shows that $|u(k_0)|\ge\delta$, so $u\ne 0$. Hence $u\in\mathcal{N}$.

Now, we prove that $J(u)=c$. By Fatou's lemma,

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J(u_n) - \frac{1}{2}J'(u_n)u_n\right) = \lim_{n\to\infty}\sum_{k\in\mathbb{Z}}\left(\frac{1}{2}f(k,u_n(k))u_n(k) - F(k,u_n(k))\right)\\ &\ge \sum_{k\in\mathbb{Z}}\left(\frac{1}{2}f(k,u(k))u(k) - F(k,u(k))\right) = J(u) - \frac{1}{2}J'(u)u = J(u)\ge c.\end{aligned}$$

So $J(u)=c$. The proof of Theorem 1.1 is completed.

Example 1 Consider the difference equation

$$\begin{aligned}&\Delta[|\sin k|\Delta u(k-1)] - |\cos k|u(k)\\ &+\left[c(\alpha+2)u(k)|u(k)|^\alpha + d(\beta+2)u(k)|u(k)|^\beta\right](\phi(k)+M) = 0,\end{aligned}\tag{3.11}$$

where $c>0, d>0, \alpha\ge\beta>0, M>0, \phi(k)$ is a bounded continuous π-periodic function and $|\phi(k)|<M$, $k\in\mathbb{Z}$. Let $a(k)=|\sin k|$, $b(k)=|\cos k|$,

$$f(k,u) = \left[c(\alpha+2)u|u|^\alpha + d(\beta+2)u|u|^\beta\right](\phi(k)+M)$$

and

$$F(k,u) = \int_0^u f(k,s)ds = \left[c|u|^{\alpha+2} + d|u|^{\beta+2}\right](\phi(k)+M).$$

It is easy to show that all the assumptions of Theorem 1.1 are satisfied. Therefore, equation (3.11) has at least one homoclinic solution.

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一类周期非线性差分方程的基态解

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摘要: 本文研究了一类二阶周期非线性差分方程基态解的存在性问题. 利用临界点理论结合Nehari流形方法, 获得了此类方程基态解的存在性. 在比经典AR条件更一般的超二次条件下, 本文结论推广了已有的结果, 并举例说明此类方程解的存在性.

关键词: 非线性差分方程; Nehari 流形; 基态解; 临界点理论

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