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STABILITY AND HOPF BIFURCATION OF A PREDATOR-PREY BIOLOGICAL ECONOMIC SYSTEM

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Abstract: In this paper, we mainly study the Hopf-bifurcation and the stability of differential-algebraic biological economic system with predator harvesting. By using the method of stability theory and Hopf bifurcation theorem dynamical systems and differential algebraic system, we find some related conclusions about stability and Hopf-bifurcation. We have improved the ratio-dependent predator-prey system, take economic effect μ as the bifurcation parameter and make a numerical simulation by using Matlab at last, so the conclusions are made more practical.

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1 Introduction

According to the lack of biological resources on the earth, more and more people increasingly realized the importance of the modelling and research of biological system. The predator-prey was one of the most popular models that many researchers [1–8] studied and acquired some valuable characters of dynamic behavior. For example, the stability of equilibrium, Hopf bifurcation, flip bifurcation, limit cycle and other relevant conducts. At the same time, the development and utilization of biological resources and artificial arrest was researched commonly in the fields of fishery, wildlife and forestry management by some experts [9–11]. Most of them choose differential equations and difference equations to research biological models. It is well known that economic profit become more and more important and take a fundamental gradually situation in social development. In recent years, biological economic systems were researched by many authors [12–16], who describe the system by differential-algebraic equations or differential-difference-algebraic equations.

Basic analysis model which applied by differential-algebraic equations and differential-difference-algebraic equations are familiar at present. However, there still exist some disadvantages in many systems such as harvesting function. In this paper, the main research is

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the stability and Hopf bifurcation of a biological-algebraic biological economic system, which is changed in some details and meaningful.

Our basic model is based on the following ratio-dependent predator-prey system with harvest

$$\begin{cases} \dot{u} = u(r_1 - \epsilon v), \\ \dot{v} = v(r_2 - \theta \frac{v}{u}) - \alpha v E^*, \end{cases} \tag{1.1}$$

where u and v represent the predator density and prey density at time t, respectively, ϵ, θ and α are all positive constants, and r_1 and r_2 stand for the densities of predator and prey populations, and E represents harvesting effort. $\alpha E v$ denotes that the harvests for predator population are proportional to their densities at time t.

In 1954, Gordon [17] studied the effect of the harvest effort on ecosystem form an economic perspective and proposed the following economic principle:

$$\text{Net Economic Revenue } (NER) = \text{Total Revenue } (TR) - \text{Total Cost } (TC).$$

Associated with system (1.1), an algebraic equation which considers the economic profit m of the harvest effort on predator can be established as follows

$$E(t)(pv - q) = m,$$

where $E(t)$ represents the harvest effort, p denotes harvesting reward per unit harvesting effort for unit weight, c represents harvesting cost per unit harvesting effort. Combining the economic theory of fishery resources, we can establish a differential algebraic biological economic system

$$\begin{cases} \dot{u} = u(r_1 - \epsilon v), \\ \dot{v} = v(r_2 - \theta \frac{v}{u}) - \alpha v E^*, \\ 0 = E^*(pv - q) - m. \end{cases} \tag{1.2}$$

Nevertheless, the capture effect to predator is not always shown in the liner in nature based on many factors that can affect the predation such as the ability of search, illness and death. Therefore, the harvesting function of system (1.2) is modified as follows

$$\begin{cases} \dot{u} = u(r_1 - \epsilon v), \\ \dot{v} = v(r_2 - \theta \frac{v}{u}) - \alpha E^* \frac{v}{1+\gamma v}, \\ 0 = E^*(p\frac{v}{1+\gamma v} - q) - m. \end{cases} \tag{1.3}$$

To simplify system (1.2), we use these dimensionless variables

$$x = \frac{\epsilon}{\theta} u, \ \ y = \epsilon v, \ \ E = \alpha E^*, \ \ \beta = \frac{\gamma}{\epsilon}, \ \ c = \frac{\epsilon q}{p}, \ \ \mu = \frac{\alpha \epsilon m}{p}$$

and then obtain the following system

$$\begin{cases} \dot{x} = x(r_1 - y), \\ \dot{y} = y(r_2 - \frac{y}{x}) - E\frac{y}{1+\beta y}, \\ 0 = E(\frac{y}{1+\beta y} - q) - \mu. \end{cases} \tag{1.4}$$

For simplicity, let

$$f(Z,E,\mu)=\begin{pmatrix} f_1(Z,E,\mu) \\ f_2(Z,E,\mu) \end{pmatrix}=\begin{pmatrix} x(r_1-y) \\ y(r_2-\frac{y}{x})-E\frac{y}{1+\beta y} \end{pmatrix},$$
$$g(Z,E,\mu)=E(\frac{y}{1+\beta y}-q)-\mu,$$

where $Z=(x,y)^T$, μ is a bifurcation parameter, which will be defined in the follows.

In this paper, we discuss the effects of the economic profit on the dynamics of system (1.4) in the region $R_+^3=\{(x,y,E)|x>0,y>0,E>0\}$.

Next, the paper will be organized as follows. In Section 2, the stability of the positive equilibrium point is discussed by corresponding characteristic equation of system (2.2). In Section 3, we provide Hopf bifurcation analysis of system (1.4). In Section 4, we use numerical simulations to illustrate the effectiveness of result. Then give a brief conclusion in Section 5.

2 Local Stability Analysis of System (1.4)

It is obvious that there exists an equilibrium in R_+^3 if only if this point $\chi_0:=(x_0,y_0,E_0)^T$ is a real solution of the equations

$$\begin{cases} x(r_1-y)=0, \\ y(r_2-\frac{y}{x})-E\frac{y}{1+\beta y}=0, \\ E(\frac{y}{1+\beta y}-q)-\mu=0. \end{cases} \tag{2.1}$$

By the calculation, we get

$$\chi_0=(x_0,y_0,E_0)=(\frac{r_1G_0}{r_2G_0-\mu},r_1,\frac{\mu(1+\beta r_1)}{G_0}),$$

where

$$G=G(y)=y-c(1+\beta y),\quad G_0=G(y=r_1)=r_1-c(1+\beta r_1).$$

According to this analysis procedure, this essay only concentrate on the interior equilibrium of system (1.4). Based on the ecology meaningful of the interior equilibrium, the predator and the harvest effort to predator are all exist that it is the key point to the study. Thus, a simple assumption that the inequality $0<\mu<r_2G_0$ holds in this paper. Following, we use the linear transformation $\chi^T=QM^T$, where

$$M=(X,Y,\bar{E}),\quad Q=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\dfrac{E_0}{y_0(1+\beta y_0)-c(1+\beta y_0)^2} & 1 \end{pmatrix}.$$

Then we obtain $D_\chi g(\chi)Q=(0,0,\frac{y_0}{1+\beta y_0}-c)$, $M=(x,y,\frac{\mu}{G_0^2}y+E)$, where

$$\bar{E}=\frac{\mu}{G_0^2}y+E=\frac{E_0}{y_0(1+\beta y_0)-c(1+\beta y_0)^2}y+E.$$

Next, let $E = \bar{E} - \frac{\mu}{G_0^2} y$. Thus we transform system (1.4) into

$$\begin{cases} \dot{X} = X(r_1 - Y), \\ \dot{Y} = Y[r_2 - \frac{Y}{X} + (\frac{\mu}{G_0^2} Y - \bar{E})\frac{1}{1+\beta Y}], \\ 0 = (\bar{E} - \frac{\mu}{G_0^2} Y)(\frac{Y}{1+\beta Y} - c) - \mu. \end{cases} \tag{2.2}$$

From Section 1, we obtain

$$f(M,\mu) = \begin{pmatrix} f_1(M,\mu) \\ f_2(M,\mu) \end{pmatrix} = \begin{pmatrix} X(r_1 - Y) \\ Y[r_2 - \frac{Y}{X} + (\frac{\mu}{G_0^2} Y - \bar{E})\frac{1}{1+\beta Y}] \end{pmatrix},$$
$$g(M,\mu) = (\bar{E} - \frac{\mu}{G_0^2} Y)(\frac{Y}{1+\beta Y} - c) - \mu.$$

For system (2.2), we consider the local parametric ψ, which defined as follows

$$(X, Y, \bar{E}^T) = \psi(\bar{Z}) = M_0^T + U_0 \bar{Z} + V_0 h(\bar{Z}),$$

where

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{Z} = (y_1, y_2)^T, \quad M_0 = (X_0, Y_0, \bar{E}_0),$$

$h : R^2 \to R^3$ is a smooth mapping. Then we can obtain the parametric system (2.2) as follows:

$$\begin{cases} \dot{y_1} = f_1(\mu, \psi(\mu, M)), \\ \dot{y_2} = f_2(\mu, \psi(\mu, M)). \end{cases} \tag{2.3}$$

More details about the definition can be found in [18]. Based on system (2.3), we can get Jacobian matrix $E(M_0)$, which takes the form of

$$\begin{aligned} E(M_0) &= \begin{pmatrix} D_M f_1(M_0) \\ D_M f_2(M_0) \end{pmatrix} \begin{pmatrix} D_M G_1(M_0) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_1 \end{pmatrix} \\ &= \begin{pmatrix} D_X f_1(M_0) & D_Y f_1(M_0) \\ D_X f_2(M_0) & D_Y f_2(M_0) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\frac{r_1 G_0}{r_2 G_0 - \mu} \\ (r_2 - \frac{\mu}{G_0})^2 & -r_2 \frac{\mu}{G_0^2}(2G_0 + c) \end{pmatrix}. \end{aligned}$$

Then the following theorem summarizes the stability of the positive equilibrium point of system (1.4).

Theorem 2.1 For system (2.2)

(i) If $(r_2 - \mu\frac{2G_0+c}{G_0^2})^2 \geqslant 4r_1\frac{r_2 G_0 - \mu}{G_0}$ and $\mu < \min\left\{\frac{r_2 G_0^2}{2G_0+c}, r_2 G_0\right\}$, the positive equilibrium point of system (1.4) is asymptotically stable; otherwise when $\frac{r_2 G_0^2}{2G_0+c} < \mu < r_2 G_0$, the positive equilibrium point of system (1.4) is unstable.

(ii) If $(r_2-\mu\frac{2G_0+c}{G_0^2})^2<4r_1\frac{r_2G_0-\mu}{G_0}$ and $\mu<\min\left\{\frac{r_2G_0^2}{2G_0+c},r_2G_0\right\}$, the positive equilibrium point of system (1.4) is a sink; otherwise when $\frac{r_2G_0^2}{2G_0+c}<\mu<r_2G_0$, the positive equilibrium point of system (1.4) is a source.

Proof First, the characteristic equation of the matrix $E(M_0)$ can be written as

$$\lambda^2+(r_2-\mu\frac{2G_0+c}{G_0^2})\lambda+r_1\frac{r_2G_0-\mu}{G_0}=0. \tag{2.4}$$

Now donate Δ by

$$\Delta=(r_2-\mu\frac{2G_0+c}{G_0^2})^2-4r_1\frac{r_2G_0-\mu}{G_0}.$$

If $\Delta\geq 0$ and $\mu<\min\left\{\frac{r_2G_0^2}{2G_0+c},r_2G_0\right\}$, eq. (2.4) has two negative real roots; when $\Delta\geq 0$ and $\frac{r_2G_0^2}{2G_0+c}<\mu<r_2G_0$, eq. (2.4) has two positive real roots. We can obtain part (i) of the theorem by the Routh-Hurwitz criteria, part (ii) can be similar proofed. Thus, we complete the proof of Theorem 2.1.

Remark 1 The local stability of χ_0 is equivalent to the local stability of M_0.

Remark 2 When the roots of eq. (2.4) exist zero real parts, system (1.4) will occur bifurcation, which will be discussed in Section 3.

3 Hopf Bifurcation Analysis of the Positive Equilibrium

In this section, we discuss the Hopf bifurcation from the equilibrium point χ_0 by choosing μ as the bifurcation parameter. Based on the Hopf bifurcation theorem in [19], we need find some sufficient conditions.

According to the definition of Δ, we obtain

$$J_\pm=\frac{(2r_2G_0+r_2c-2r_1G_0)G_0^2}{(2G_0+c)^2}\pm\sqrt{\frac{(2r_2G_0+r_2c-2r_1G_0)^2G_0^4}{(2G_0+c)^4}+B}=A\pm\sqrt{A^2+B},$$

where

$$A=\frac{(2r_2G_0+r_2c-2r_1G_0)G_0^2}{(2G_0+c)^2},$$

$$B=(4r_1r_2-r_2^2)\cdot\frac{G_0^4}{(2G_0+c)^2},$$

here, we assume that $A^2+B\geqslant 0$ in this paper.

Thus, for eq. (2.4), if $B>0$ and $0<\mu<\min\{r_2G_0,J_+\}$. Eq. (2.4) has one pair of imaginary roots. When $B>0,A>0,J_-<r_2G_0$ and $J_-<\mu<\min\{r_2G_0,J_+\}$, eq. (2.4) has one pair of imaginary roots.

In the case of meet the above conditions, we can get the roots as follows:

$$\begin{aligned}\lambda_{1,2}&=-\frac{1}{2}(r_2-\mu\frac{2G_0+c}{G_0^2})\pm\sqrt{r_1\frac{r_2G_0-\mu}{G_0}-\frac{1}{4}(r_2-\mu\frac{2G_0+c}{G_0^2})^2}\\&=\alpha(\mu)\pm i\omega(\mu),\end{aligned}$$

where

$$\alpha(\mu) = -\frac{1}{2}(r_2 - \mu\frac{2G_0 + c}{G_0^2}),$$
$$i\omega(\mu) = \sqrt{r_1\frac{r_2G_0 - \mu}{G_0} - \alpha^2(\mu)}.$$

By calculating, we obtain

$$\mu_0 = \frac{r_2G_0^2}{2G_0 + c}, \quad \alpha'(\mu_0) = \frac{2G_0 + c}{G_0^2} > 0,$$
$$\omega_0 = \omega(\mu_0) = \sqrt{r_1r_2(\frac{1}{G_0} - \frac{G_0}{2G_0 + c})}. \tag{3.1}$$

Eq. (3.1) indicates that eq. (2.2) occurs Hopf bifurcation at μ_0.

In order to calculate the Hopf bifurcation, we need to lead the normal form of system (2.2) as follows

$$\begin{aligned}\dot{y_1} &= \alpha(\mu)y_1 - \omega(\mu)y_2 + \frac{1}{2}a_{11}^1y_1^2 + a_{12}^1y_1y_2 + \frac{1}{2}a_{22}^1y_2^2 + \frac{1}{6}a_{111}^1y_1^3\\ &\quad + \frac{1}{2}a_{112}^1y_1^2y_2 + \frac{1}{2}a_{122}^1y_1y_2^2 + \frac{1}{6}a_{222}^1y_2^3 + o(|\bar{Z}|^4),\\ \dot{y_2} &= \omega(\mu)y_1 + \alpha(\mu)y_2 + \frac{1}{2}a_{11}^2y_1^2 + a_{12}^2y_1y_2 + \frac{1}{2}a_{22}^2y_2^2 + \frac{1}{6}a_{111}^2y_1^3\\ &\quad + \frac{1}{2}a_{112}^2y_1^2y_2 + \frac{1}{2}a_{122}^2y_1y_2^2 + \frac{1}{6}a_{222}^2y_2^3 + o(|\bar{Z}|^4).\end{aligned} \tag{3.2}$$

From eq. (2.3), we have

$$f_1(M,\mu) = X(r_1 - Y),$$
$$f_2(M,\mu) = Y[r_2 - \frac{Y}{X} + (\frac{\mu}{G_0^2}Y - \bar{E})\frac{1}{1 + \beta Y}],$$
$$g(M,\mu) = (\bar{E} - \frac{\mu}{G_0^2}Y)(\frac{Y}{1 + \beta Y} - c) - \mu.$$

Then we can easily obtain

$$D_Nf_1(M,\mu) = (r_1 - Y, -X, 0),$$
$$D_Nf_2(M,\mu) = (\frac{Y^2}{X^2}, r_2 - \frac{2Y}{X} + F, \frac{-Y}{1 + \beta Y}),$$
$$D_Ng(M,\mu) = (0, \frac{\mu c}{G_0^2} - F, \frac{Y}{1 + \beta Y} - c),$$

where

$$F = \frac{\mu}{G_0^2}\cdot\frac{2Y + \beta Y^2}{(1 + \beta Y)^2} - \frac{\bar{E}}{(1 + \beta Y)^2}$$

and

$$\begin{aligned}
D_\psi(\bar{Z},\mu) &= \begin{pmatrix} D_N g(M,\mu) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ \frac{1+\beta Y}{G_0} & 0 & \frac{1+\beta Y}{Y-c(1+\beta Y)}(F-\frac{\mu c}{G_0^2}) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{1+\beta Y}{Y-c(1+\beta Y)}(F-\frac{\mu c}{G_0^2}) \end{pmatrix}.
\end{aligned}$$

Then we get

$$\begin{aligned}
&f_{1y_1}(M,\mu) = D_N f_1(M,\mu) D_{y_1}\psi(\bar{Z},\mu) = r_1 - Y, \\
&f_{1y_2}(M,\mu) = D_N f_1(M,\mu) D_{y_2}\psi(\bar{Z},\mu) = -X, \\
&f_{2y_1}(M,\mu) = D_N f_2(M,\mu) D_{y_1}\psi(\bar{Z},\mu) = \frac{Y^2}{X^2}, \\
&f_{2y_2}(M,\mu) = D_N f_2(M,\mu) D_{y_2}\psi(\bar{Z},\mu) = \frac{-Y}{Y-c(1+\beta Y)}(F-\frac{\mu c}{G_0^2}) + r_2 - \frac{2Y}{X} + F.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&D_N f_{1y_1}(M,\mu) = (0,-1,0), \\
&D_N f_{1y_2}(M,\mu) = (-1,0,0), \\
&D_N f_{2y_1}(M,\mu) = (-\frac{2Y^2}{X^3}, \frac{2Y}{X^2}, 0), \\
&D_N f_{2y_2}(M,\mu) = (\frac{2Y}{X^2}, \cdots, \frac{c}{[Y-c(1+\beta Y)](1+\beta Y)}).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&f_{1y_1y_1}(M,\mu) = D_N f_{1y_1}(M,\mu) D_{y_1}\psi(\bar{Z},\mu) = 0, \\
&f_{1y_1y_2}(M,\mu) = D_N f_{1y_1}(M,\mu) D_{y_2}\psi(\bar{Z},\mu) = -1, \\
&f_{1y_2y_2}(M,\mu) = D_N f_{1y_2}(M,\mu) D_{y_2}\psi(\bar{Z},\mu) = 0, \\
&f_{2y_1y_1}(M,\mu) = D_N f_{2y_1}(M,\mu) D_{y_1}\psi(\bar{Z},\mu) = -\frac{2Y^2}{X^3}, \\
&f_{2y_1y_2}(M,\mu) = D_N f_{2y_1}(M,\mu) D_{y_2}\psi(\bar{Z},\mu) = \frac{2Y}{X^2}, \\
&f_{2y_2y_2}(M,\mu) = D_N f_{2y_2}(M,\mu) D_{y_2}\psi(\bar{Z},\mu) = \frac{2c}{G_0^2}(F+\beta F-\frac{\mu c}{G_0^2}-\frac{\mu}{G_0^2}) - \frac{2}{X}.
\end{aligned}$$

Substituting M_0, μ_0 into above, we have

$$
\begin{aligned}
&f_{1y_1}(M_0,\mu_0)=0, \quad f_{1y_2}(M_0,\mu_0)=-X_0, \quad f_{2y_1}(M_0,\mu_0)=\frac{Y_0^2}{X_0^2},\\
&f_{1y_1y_2}(M_0,\mu_0)=-1, \quad f_{2y_1y_1}(M_0,\mu_0)=-\frac{2Y_0^2}{X_0^3}, \quad f_{2y_1y_2}(M_0,\mu_0)=\frac{2Y_0}{X_0^2},\\
&f_{2y_2y_2}(M_0,\mu_0)=\frac{2c\mu}{G_0^4}(\beta c-1)+\frac{2}{X_0}.
\end{aligned}
$$

Now, we get

$$
\begin{aligned}
&D_N f_{2y_1y_1}(M,\mu)=(\frac{6Y^2}{X^4},-\frac{4Y}{X^3},0),\\
&D_N f_{2y_1y_2}(M,\mu)=(-\frac{4Y}{X^3},\frac{2}{X^2},0),\\
&D_N f_{2y_2y_1}(M,\mu)=\frac{2}{X^2},\frac{G_0}{1+\beta Y}\cdot(\frac{2\mu}{G_0^2}-F(1-c\beta)),-\frac{c(1+\beta)}{G_0^2(1+\beta Y)^2}).
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
f_{2y_1y_1y_1}&=D_N f_{2y_1y_1}(M,\mu)D_{y_1\psi}(\bar{Z},\mu)=\frac{6Y^2}{X^4},\\
f_{2y_1y_1y_2}&=D_N f_{2y_1y_1}(M,\mu)D_{y_2\psi}(\bar{Z},\mu)=-\frac{4Y}{X^3},\\
f_{2y_2y_2y_1}&=D_N f_{2y_2y_2}(M,\mu)D_{y_1\psi}(\bar{Z},\mu)=\frac{2}{X^2},\\
f_{2y_2y_2y_2}&=D_N f_{2y_2y_2}(M,\mu)D_{y_2\psi}(\bar{Z},\mu)=\frac{1-\beta c}{G_0^2}(\frac{2\mu G_0}{1+\beta Y_0}-\mu c).
\end{aligned}
$$

Thus we have eq. (3.3)

$$
\begin{aligned}
\dot{y_1}&=-X_0y_2-y_1y_2,\\
\dot{y_2}&=\frac{Y_0^2}{X_0^2}y_1-\frac{Y_0^2}{X_0^3}y_1^2+\frac{2Y_0}{X_0^2}y_1y_2+[\frac{c\mu}{G_0^4}(\beta c-1)+\frac{1}{X_0}]y_2^2+\frac{Y_0^2}{X_0^4}y_1^3-\frac{2Y}{X^3}y_1^2y_2+\frac{1}{X^2}y_1y_2^2\\
&\quad+\frac{1}{6}\cdot\frac{1-\beta c}{G_0^2}(\frac{2\mu G_0}{1+\beta Y_0}-\mu c)y_2^3+o(|\bar{Z}|^4).
\end{aligned} \tag{3.3}
$$

Comparing with the normal form (3.2), we chosse the nonsingular matrix

$$
N=\begin{pmatrix} X_0\sqrt{X_0} & 0\\ 0 & Y_0\end{pmatrix},
$$

then we use the linear transformation $H=N\bar{Z}$, noticing $\omega_0=\frac{Y_0}{\sqrt{X_0}}$, we derive the normal form as follows

$$
\begin{aligned}
\dot{u_1}&=-\omega_0u_2-Y_0u_1u_2,\\
\dot{u_2}&=\omega_0u_1-Y_0u_1^2+\frac{2Y_0}{\sqrt{X_0}}u_1u_2+[\frac{c\mu}{G_0^4}(\beta c-1)+\frac{1}{X_0}]Y_0u_2^2+Y_0\sqrt{X_0}u_1^3-2Y_0u_1^2u_2\\
&\quad+\frac{Y_0}{\sqrt{X_0}}u_1u_2^2+\frac{1}{6}\cdot\frac{1-\beta c}{G_0^2}(\frac{2\mu G_0}{1+\beta Y_0}-\mu c)Y_0^2u_2^3+o(|\bar{Z}|^4),
\end{aligned} \tag{3.4}
$$

where $H = (u_1, u_2)^T$. Then

$$
\begin{aligned}
&a_{11}^1 = a_{22}^1 = 0,\\
&a_{12}^1 = -Y_0 \qquad a_{11}^2 = -Y_0, \qquad a_{12}^2 = \frac{2Y_0}{\sqrt{X_0}},\\
&a_{22}^2 = [\frac{c\mu}{G_0^4}(\beta c - 1) + \frac{1}{X_0}]Y_0,\\
&a_{111}^1 = a_{122}^1 = 0,\\
&a_{111}^2 = Y_0\sqrt{X_0}, \qquad a_{112}^2 = -2Y_0, \qquad a_{122}^2 = \frac{Y_0}{\sqrt{X_0}},\\
&a_{222}^2 = \frac{1}{6}\cdot\frac{1-\beta c}{G_0^2}(\frac{2\mu G_0}{1+\beta Y_0} - \mu c)Y_0^2.
\end{aligned}
$$

According to the Hopf bifurcation theorem in [19], now we only need to calculate the value of a

$$
\begin{aligned}
16a &= [a_{12}^1(a_{11}^1 + a_{22}^1) - a_{12}^2(a_{11}^2 + a_{22}^2) - a_{11}^1 a_{11}^2 + a_{22}^1 a_{22}^2]/\omega + (a_{111}^1 + a_{122}^1 + a_{112}^2 + a_{222}^2)\\
&= (-a_{12}^2 a_{11}^2 - a_{12}^2 a_{22}^2)/\omega + a_{112}^2 + a_{222}^2\\
&= -\frac{2Y_0}{X_0} + 2Y_0\frac{c\mu}{G_0^4}(1-\beta + \frac{1}{6}\cdot\frac{1-\beta c}{G_0^2}(\frac{2\mu G_0}{1+\beta Y_0} - \mu c)Y_0^2.
\end{aligned}
$$

Next, there are two cases should be discussed. That is $a > 0$ and $a < 0$. Based on the Hopf bifurcation theorem in [19], we obtain Theorem 3.1.

Theorem 3.1 For the system (2.2), there exist an $\varepsilon > 0$ and two small enough neighborhoods P_1 and P_2 of $\chi_0(\mu)$, where $P_1 \subset P_2$.

(i) If

$$2Y_0\frac{c\mu}{G_0^4}(1-\beta c) + \frac{1}{6}\cdot\frac{1-\beta c}{G_0^2}(\frac{2\mu G_0}{1+\beta Y_0} - \mu c)Y_0^2 > \frac{2Y_0}{X_0},$$

then

(1) when $\mu_0 < \mu < \mu_0 + \varepsilon$, $\chi_0(\mu)$ is unstable, and repels all the points in P_2;

(2) when $\mu_0 - \varepsilon < \mu < \mu_0$, there exist at least one periodic solution in $\bar{P}_1$, which is the closure of P_1, one of them repel all the points in $\bar{P}_1\backslash\{\chi_0(\mu)\}$, and also have another periodic solution (may be the same that) repels all the points in $P_2\backslash\bar{P}_1$,and $\chi_0(\mu)$ is locally asymptotically stable.

(ii) If

$$2Y_0\frac{c\mu}{G_0^4}(1-\beta c) + \frac{1}{6}\cdot\frac{1-\beta c}{G_0^2}(\frac{2\mu G_0}{1+\beta Y_0} - \mu c)Y_0^2 < \frac{2Y_0}{X_0},$$

then

(1) when $\mu_0 - \varepsilon < \mu < \mu_0, \chi_0(\mu)$ is locally asymptotically stable,and repels all the points in P_2;

(2) when $\mu_0 < \mu < \mu_0 + \varepsilon$, there exist at least one periodic solution in $\bar{P}_1$, one of them repel all the points in $\bar{P}_1\backslash\{\chi_0(\mu)\}$, and also have another periodic solution (may be the same that) repels all the points in $P_2\backslash\bar{P}_1$, and $\chi_0(\mu)$ is unstable.

Proof Theorem 3.1 can be similarly proved as the Hopf bifurcation theorem in [19], so we omit the process here.

4 Numerical Simulations

In this section, we give a numerical example of system (1.4) with the parameters $r_1 = 3, r_2 = 1, c = 1, \beta = 0.195$, then system (1.4) becomes

$$\begin{cases} \dot{x} = x(3-y), \\ \dot{y} = y(1-\frac{y}{x}) - E\frac{y}{1+0.195y}, \\ 0 = E(\frac{y}{1+0.195y} - q) - \mu. \end{cases} \tag{4.1}$$

By simple computing, the only positive equilibrium point of above system is

$$\chi(\mu_0) = (4.7578, 3, 0.5856),$$

and the Hopf bifurcation value $\mu_0 = \frac{r_2 G_0^2}{2G_0+c} = \frac{2.0002225}{3.83}$.

Therefore, by Theorem 3.1, we can easily show that the positive equilibrium point $\chi_0(\mu)$ of system (4.1) is locally asymptitically stable when $\mu = 0.505 < \mu_0$ as is illustrated by computer simulations in Fig. 1; periodic solutions occur from $\chi_0(\mu)$ when $\mu = 0.5195 < \mu_0$ as is illustrated in Fig. 2; the positive equilibrium point $\chi_0(\mu)$ of system (4.1) is unstable when $\mu = 0.535 > \mu_0$ as is illustrated in Fig. 3.

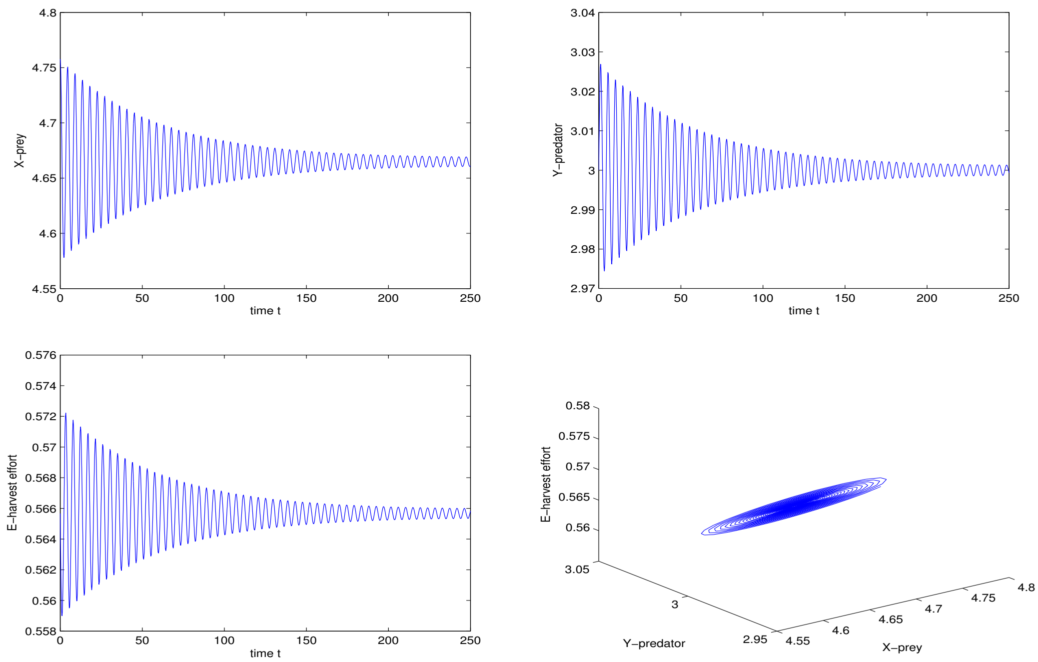

Figure 1: When $\mu = 0.505 < \mu_0$, that show the positive equilibrium point $\chi_0(\mu)$ is locally asymptotically stable.

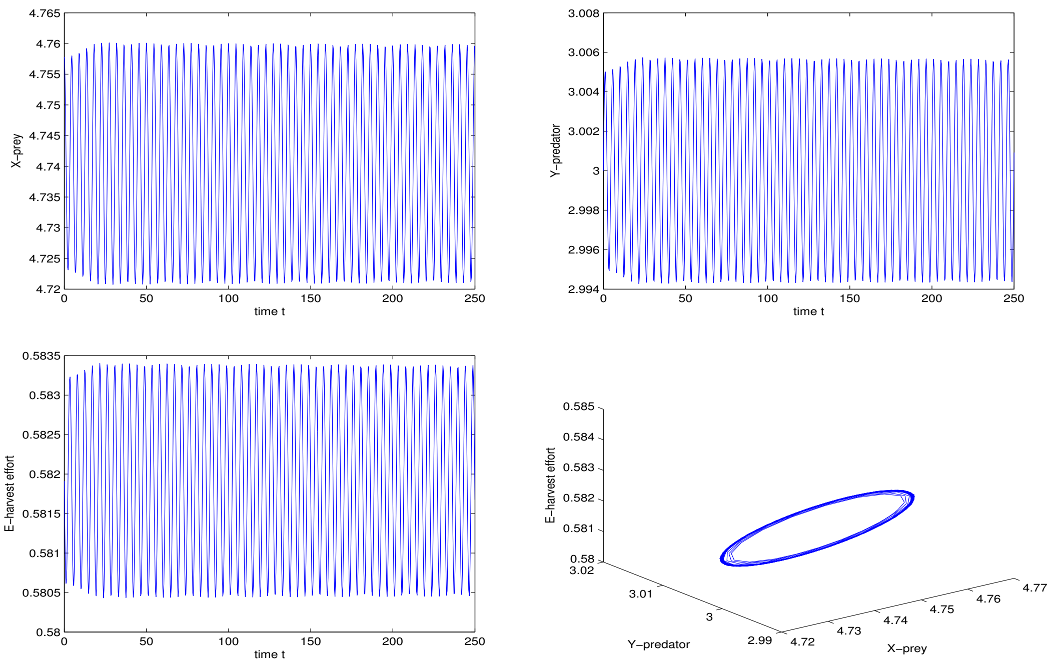

Figure 2: Periodic solutions bifurcating from $\chi_0(\mu)$ when$\mu = 0.5195 < \mu_0$.

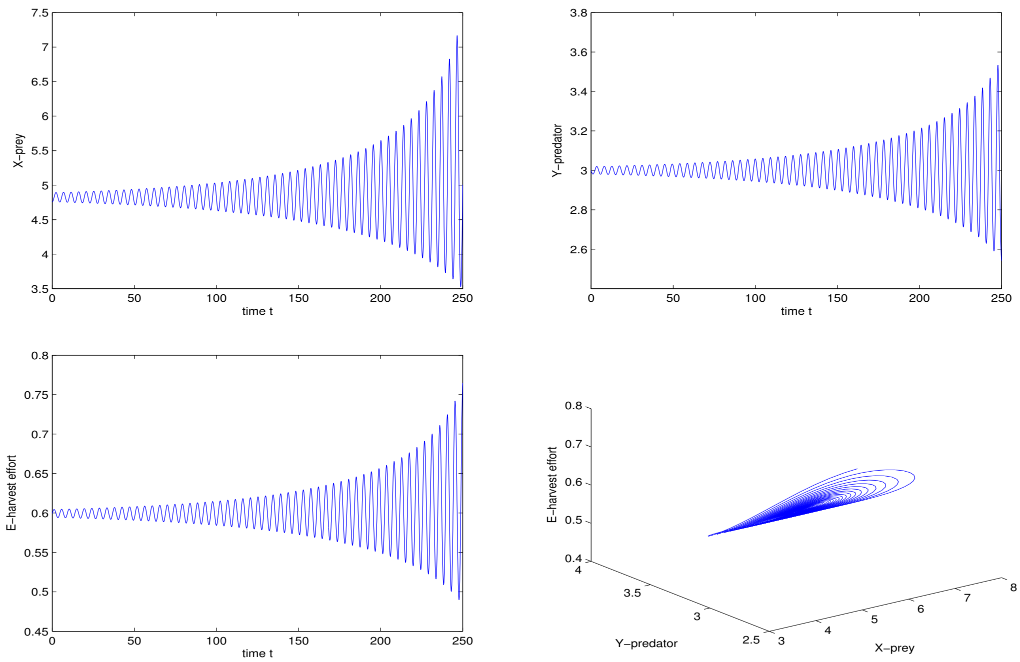

Figure 3: When $\mu = 0.535 > \mu_0$, that show the positive equilibrium point $\chi_0(\mu)$ is unstable.

5 Conclusions

Based on the above inference and calculation, we find that economic effect will influence the stability of differential-algebraic biological economic system. For instance, according to those statistics and graphs, if people fix the economic index at a high level, over the bifurcation value of Hopf-bifurcation, the system will become unstable that means people have destroyed the economic balance even led to the extinction of ecologic species. Therefore, with an aim to realize the harmonious sustainable development co-existence between man and nature, we should not seek economic effect blindly and control it within a certain limit, such as less than bifurcation value.

In addition, we can make some improvements in our model. For example, we do not consider the influence of time delays and double harvesting that is, human harvesting will harvest predator and prey at the same time. So it is necessary for us to go on with our research in these aspects in the future.

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一类捕食食饵微分经济系统的稳定性与Hopf分支

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摘要: 本文主要研究了一个带有对捕食者进行捕获的微分代数经济系统的稳定性和Hopf分支问题. 利用了动力系统和微分代数系统中的稳定性理论和分支理论的方法, 得到了稳定性和Hopf分支稳定性的相关结论. 本文对Ratio-Dependent捕食食饵模型进行了一定程度的完善, 并且选取经济效益μ 为分支参数进行研究, 最后利用Matlab 进行数值模拟, 这样使得到的结论更符合现实意义.

关键词: 稳定性; 经济系统; Hopf分支; 捕获

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