Vol. 36 ( 2016 )
No. 6

数 学 杂 志
J. of Math. (PRC)

# CHEN-RICCI INEQUALITIES FOR SUBMANIFOLDS OF GENERALIZED COMPLEX SPACE FORMS WITH SEMI-SYMMETRIC METRIC CONNECTIONS

HE Guo-qing

(*School of Mathematics and Computer Science, Anhui Normal University, Wuhu 241000, China*)

**Abstract:** In this paper, we study Chen-Ricci inequalities for submanifolds of generalized complex space forms endowed with a semi-symmetric metric connection. By using algebraic techniques, we establish Chen-Ricci inequalities between the mean curvature associated with a semi-symmetric metric connection and certain intrinsic invariants involving the Ricci curvature and $k$-Ricci curvature of submanifolds, which generalize some of Mihai and Özgür's results.

**Keywords:** Chen-Ricci inequality; $k$-Ricci curvature; generalized complex space form; semi-symmetric metric connection

**2010 MR Subject Classification:** 53C40

**Document code:** A **Article ID:** 0255-7797(2016)06-1133-09

## 1 Introduction

Since the celebrated theory of Nash [1] of isometric immersion of a Riemannian manifold into a suitable Euclidean space gave very important and effective motivation to view each Riemannian manifold as a submanifold in a Euclidean space, the problem of discovering simple sharp relationships between intrinsic and extrinsic invariants of a Riemannian submanifold becomes one of the most fundamental problems in submanifold theory. The main extrinsic invariant of a submanifold is the squared mean curvature and the main intrinsic invariants of a manifold include the Ricci curvature and the scalar curvature. There were also many other important modern intrinsic invariants of (sub)manifolds introduced by Chen such as $k$-Ricci curvature (see [2–4]).

In 1999, Chen [5] proved a basic inequality involving the Ricci curvature and the squared mean curvature of submanifolds in a real space form $R^m(C)$. This inequality is now called Chen-Ricci inequality [6]. In [5], Chen also defined the $k$-Ricci curvature of a $k$-plane section of $T_xM^n$, $x \in M$, where $M^n$ is a submanifold of the real space form $R^{n+p}(C)$. And he proved a basic inequality involving the $k$-Ricci curvature and the squared mean curvature of the submanifold $M^n$. These inequalities described relationships between the intrinsic

* **Received date:** 2014-09-13 **Accepted date:** 2015-11-09

**Foundation item:** Supported by the Foundation for Excellent Young Talents of Higher Education of Anhui Province (2011SQRL021ZD).

**Biography:** He Guoqing (1979–), female, born at Chaohu, Anhui, lecturer, major in differential geometry.

invariants and the extrinsic invariants of a Riemannian submanifold and drew attentions of many people. Similar inequalities are studied for different submanifolds in various ambient manifolds (see [7–10]).

On the other hand, Hayden [11] introduced a notion of a semi-symmetric connection on a Riemannian manifold. Yano [12] studied Riemannaian manifolds endowed with a semi-symmetric connection. Nakao [13] studied submanifolds of Riemannian manifolds with a semi-symmetric metric connection. Recently, Mihai and Özgür [14, 15] studied Chen inequalities for submanifolds of real space forms admitting a semi-symmetric metric connection and Chen inequalities for submanifolds of complex space forms and Sasakian space forms with a semi-symmetric metric connection, respectively. Motivated by studies of the above authors, in this paper we establish Chen-Ricci inequalities for submanifolds in generalized complex forms with a semi-symmetric metric connection.

## 2 Preliminaries

Let $N^{n+p}$be an $(n+p)$-dimensional Riemannian manifold with Riemannian metric $g$ and a linear connection $\overline{\nabla}$ on $N^{n+p}$. If the torsion tensor $\overline{T}$ of $\overline{\nabla}$, defined by

$$\overline{T}(\overline{X},\overline{Y})=\overline{\nabla}_{\overline{X}}\overline{Y}-\overline{\nabla}_{\overline{Y}}\overline{X}-[\overline{X},\overline{Y}]$$

for any vector fields $\overline{X}$ and $\overline{Y}$ on $N^{n+p}$, satisfies

$$\overline{T}(\overline{X},\overline{Y})=\phi(\overline{Y})\overline{X}-\phi(\overline{X})\overline{Y}$$

for a 1-form $\phi$, then the connection $\overline{\nabla}$ is called a semi-symmetric connection. Furthermore, if $\overline{\nabla}$ satisfies $\overline{\nabla}g=0$, then $\overline{\nabla}$ is called a semi-symmetric metric connection. Let $\overline{\nabla}'$ denote the Levi-Civita connection with respect to the Riemannian metric $g$. In [12] Yano gave a semi-symmetric metric connection $\overline{\nabla}$ which can be written as

$$\overline{\nabla}_{\overline{X}}\overline{Y}=\overline{\nabla}'_{\overline{X}}\overline{Y}+\phi(\overline{Y})\overline{X}-g(\overline{X},\overline{Y})U \tag{2.1}$$

for any vector field $\overline{X}$, $\overline{Y}$ on $N^{n+p}$, where U is a vector field given by $g(U,\overline{X})=\phi(\overline{X})$.

Let $M^n$ be an $n$-dimensional submanifold of $N^{n+p}$ with a semi-symmetric metric connection $\overline{\nabla}$ and the Levi-Civita connection $\overline{\nabla}'$. On the submanifold $M^n$ we consider the induced semi-symmetric metric connection denoted by $\nabla$ and the induced Levi-Civita connection denoted by $\nabla'$. The Gauss formulas with respect to $\nabla$ and $\nabla'$, respectively, can be written as

$$\overline{\nabla}_XY=\nabla_XY+h(X,Y),\quad \overline{\nabla}'_XY=\nabla'_XY+h'(X,Y) \tag{2.2}$$

for any vector fields $X,Y$ on $M^n$, where $h'$ is the second fundamental form of $M^n$ in $N^{n+p}$ and $h$ is a (0,2)-tensor on $M^n$. According to formula (7) in [13], $h$ is also symmetric.

Let $\overline{R}$ be the curvature tensor of $N^{n+p}$ with respect to $\overline{\nabla}$ and $\overline{R}'$ be the curvature tensor of $N^{n+p}$ with respect to $\overline{\nabla}'$. We also denote by $R$ and $R'$ the curvature tensor of $\nabla$ and

$\nabla'$, respectively, on $M^n$. From [13], we know the curvature tensor $\overline{R}$ with respect to the semi-symmetric metric $\overline{\nabla}$ on $N^{n+p}$ can be written as

$$\begin{aligned}\overline{R}(X,Y,Z,W) =&\overline{R}'(X,Y,Z,W)-\alpha(Y,Z)g(X,W)+\alpha(X,Z)g(Y,W)\\&-\alpha(X,W)g(Y,Z)+\alpha(Y,W)g(X,Z)\end{aligned} \tag{2.3}$$

for any vector fields $X,Y,Z,W$on $M^n$, where $\alpha$ is a (0,2)-tensor field defined by

$$\alpha(X,Y)=(\overline{\nabla}'_X\phi)Y-\phi(X)\phi(Y)+\frac{1}{2}\phi(U)g(X,Y).$$

Denote by $\lambda$ the trace of $\alpha$. The Gauss equation for the submanifold $M^n$ in $N^{n+p}$ is

$$\overline{R}'(X,Y,Z,W)=R'(X,Y,Z,W)+g(h'(X,Z),h'(Y,W))-g(h'(X,W),h'(Y,Z)) \tag{2.4}$$

for any vector fields $X,Y,Z,W$on $M^n$. In [13], the Gauss equation with respect to the semi-symmetric metric connection is

$$\overline{R}(X,Y,Z,W)=R(X,Y,Z,W)+g(h(X,Z),h(Y,W))-g(h(X,W),h(Y,Z)). \tag{2.5}$$

In $N^{n+p}$ we can choose a local orthonormal frame $\{e_1,\cdots,e_n,e_{n+1},\cdots,e_{n+p}\}$ such that restricting to $M^n$, $e_1,\cdots,e_n$ are tangent to $M^n$. Setting $h^r_{ij}=g(h(e_i,e_j),e_r)$, then the squared length of $h$ is

$$||h||^2=\sum_{i,j=1}^{n}g(h(e_i,e_j),h(e_i,e_j))=\sum_{r=n+1}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2.$$

The mean curvature vector of $M^n$ associated to $\overline{\nabla}$ is $H=\frac{1}{n}\sum\limits_{i=1}^{n}h(e_i,e_i)$ and the mean curvature vector of $M^n$ associated to $\overline{\nabla}'$ is $H'=\frac{1}{n}\sum\limits_{i=1}^{n}h'(e_i,e_i)$.

Let $\pi\subset T_xM^n$ be a 2-plane section for any $x\in M^n$ and $K(\pi)$ be the sectional curvature of $\pi$ associated to the induced semi-symmetric metric connection $\nabla$. The scalar curvature $\tau$ at $x$ with respect to $\nabla$ is defined by

$$\tau(x)=\sum_{1\le i<j\le n}K(e_i\wedge e_j). \tag{2.6}$$

The following lemmas will be used in the paper.

**Lemma 2.1** (see [13]) If U is a tangent vector field on $M^n$, we have $H=H'$, $h=h'$.

**Lemma 2.2** (see [13]) Let $M^n$ be an $n$-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $N^{n+p}$ with the semi-symmetric metric connection $\overline{\nabla}$. Then

(i) $M^n$ is totally geodesic with respect to the Levi-Civita connection and with respect to the semi-symmetric metric connection if and only if $U$ is tangent to $M^n$.

(ii) $M^n$ is totally umbilical with respect to the Levi-Civita connection if and only if $M^n$ is totally umbilical with respect to the semi-symmetric metric connection.

**Lemma 2.3** (see [10]) Let $f(x_1, x_2, \cdots, x_n)$ be a function on $R^n$ defined by

$$f(x_1, x_2, \cdots, x_n) = x_1 \sum_{i=2}^{n} x_i.$$

If $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, then we have

$$f(x_1, x_2, \cdots, x_n) \leq \varepsilon^2$$

with the equality holding if and only if $x_1 = x_2 + x_n + \cdots + x_n = \varepsilon$.

A $2m$-dimensional almost Hermitian manifold $(N, J, g)$ is said to be a generalized complex space form (see [16, 17]) if there exists two functions $F_1$ and $F_2$ on $N$ such that

$$\begin{aligned}\overline{R}'(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) =& F_1[g(\overline{Y}, \overline{Z})g(\overline{X}, \overline{W}) - g(\overline{X}, \overline{Z})g(\overline{Y}, \overline{W})] + F_2[g(\overline{X}, J\overline{Z})g(J\overline{Y}, \overline{W}) \\ & - g(\overline{Y}, J\overline{Z})g(J\overline{X}, \overline{W}) + 2g(\overline{X}, J\overline{Y})g(J\overline{Z}, \overline{W})]\end{aligned} \tag{2.7}$$

for any vector fields $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ on $N$, where $\overline{R}'$ is the curvature tensor of $N$ with respect to the Levi-Civita connection $\overline{\nabla}'$. In such a case, we will write $N(F_1, F_2)$. If $N(F_1, F_2)$ is a generalized complex space form with a semi-symmetric metric connection $\overline{\nabla}$, using (2.3) and (2.7), the curvature tensor $\overline{R}$ with respect to the semi-symmetric metric connection $\overline{\nabla}$ of $N(F_1, F_2)$ can be written as

$$\begin{aligned}\overline{R}(X, Y, Z, W) =& F_1[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + F_2[g(X, JZ)g(JY, W) \\ & - g(Y, JZ)g(JX, W) + 2g(X, JY)g(JZ, W)] - \alpha(Y, Z)g(X, W) \\ & + \alpha(X, Z)g(Y, W) - \alpha(X, W)g(Y, Z) + \alpha(Y, W)g(X, Z)\end{aligned} \tag{2.8}$$

for $X, Y, Z, W$ on $M$, where $M$ is a submanifold of $N$.

Let $M$ be an $n$-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$. We set $JX = PX + FX$ for any vector field $X$ tangent to $M$, where $PX$ and $FX$ are tangential and normal components of $JX$, respectively.

## 3 Chen-Ricci Inequality

In this section, we establish a sharp relation between the Ricci curvature along the direction of an unit tangent vector $X$ and the mean curvature $||H||$ with respect to the semi-symmetric metric connect $\overline{\nabla}$.

**Theorem 3.1** Let $M^n$, $n \geq 2$, be an $n$-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ endowed with the semi-symmetric metric connection $\overline{\nabla}$. For each unit vector $X \in T_xM$, we have

(1)

$$\mathrm{Ric}(X) \leq (n-1)F_1 + 3F_2||PX||^2 - (n-2)\alpha(X, X) - \lambda + \frac{n^2}{4}||H||^2. \tag{3.1}$$

(2) If $H(x) = 0$, then a unit tangent vector $X$ at $x$ satisfies the equality case of (3.1) if and only if $X \in N(x) = \{X \in T_xM : h(X, Y) = 0, \ \forall Y \in T_xM\}$.

(3) The equality of inequality (3.1) holds identically for all unit tangent vectors at $x$ if and only if in the case of $n \neq 2$, $h_{ij}^r = 0$, $i, j = 1, 2 \cdots, n$; $r = n+1, \cdots, 2m$ , or in the case of $n = 2$, $h_{11}^r = h_{22}^r$, $h_{12}^r = h_{21}^r = 0$, $r = 3, \cdots, 2m$.

**Proof** (1) Let $X \in T_xM$ be an unit tangent vector at $x$. We choose an orthonormal basis $e_1, \cdots, e_n, e_{n+1} \cdots, e_{2m}$ such that $e_1, \cdots, e_n$ are tangent to $M$ at $x$ and $e_1 = X$.

When we set $X = W = e_i$, $Y = Z = e_j$, $i, j = 1, \cdots, n$, $i \neq j$ in (2.5) and (2.8), we have

$$R_{ijji} = F_1 + 3F_2 g^2(Je_i, e_j) - \alpha(e_i, e_i) - \alpha(e_j, e_j) + \sum_{r=n+1}^{2m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned}
\mathrm{Ric}(X) =& \sum_{j=2}^{n} R_{1jj1} = (n-1)F_1 + \sum_{j=2}^{n} 3F_2 g^2(JX, e_j) \\
&- (n-1)\alpha(X, X) - \sum_{j=2}^{n} \alpha(e_j, e_j) + \sum_{r=n+1}^{2m} \sum_{i=2}^{n} [h_{11}^r h_{ii}^r - (h_{1i}^r)^2] \\
\leq& (n-1)F_1 + 3F_2 ||PX||^2 - (n-2)\alpha(X, X) - \lambda + \sum_{r=n+1}^{2m} \sum_{i=2}^{n} h_{11}^r h_{ii}^r.
\end{aligned} \tag{3.3}$$

We consider the maximum of the function

$$f_r(h_{11}^r, \cdots, h_{nn}^r) = \sum_{i=2}^{n} h_{11}^r h_{ii}^r$$

under the condition $h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = k^r$, where $k^r$ is a real constant.

From Lemma 2.3 we know the solution $(h_{11}^r, \cdots, h_{nn}^r)$ of this problem must satisfy

$$h_{11}^r = \sum_{i=2}^{r} h_{ii}^r = \frac{k^r}{2}. \tag{3.4}$$

So it follows that

$$f_r \leq \frac{(k^r)^2}{4} = \frac{1}{4}(\sum_{i=1}^{n} h_{ii}^r)^2. \tag{3.5}$$

From (3.3) and (3.5) we have

$$\begin{aligned}
\mathrm{Ric}(X) &\leq (n-1)F_1 + 3F_2 ||PX||^2 - (n-2)\alpha(X, X) - \lambda + \sum_{r=n+1}^{2m} \frac{1}{4}(\sum_{i=1}^{n} h_{ii}^r)^2 \\
&= (n-1)F_1 + 3F_2 ||PX||^2 - (n-2)\alpha(X, X) - \lambda + \frac{n^2}{4}||H||^2.
\end{aligned}$$

(2) For the unit vector $X$ at $x$, if the equality case of inequality (3.1) holds, using (3.3), (3.4) and (3.5) we have

$$h_{1i}^r = 0, i \neq 1, \forall r; \tag{3.6}$$

$$h_{11}^r + h_{22}^r + \cdots + h_{nn}^r - 2h_{11}^r = 0, \forall r. \tag{3.7}$$

From $H(x) = 0$, we have $h_{11}^r = 0$, then $h_{1j}^r = 0$, $\forall j, r$. So we get X$\in N(x) = \{X \in T_xM : h(X, Y) = 0, \ \forall Y \in T_xM\}$.

The converse is obvious.

(3) For all unit vector $X$ at $x$, the equality case of inequality (3.1) holds. Let $X = e_i$, $i = 1, 2 \cdots n$, as in (2), we have

$$h_{ij}^r = 0, \ \ i \neq j, \ \ \forall r;$$
$$h_{11}^r + h_{22}^r + \cdots + h_{nn}^r - 2h_{ii}^r = 0, \ \ \forall i = 1, \cdots, n; \ \ r = n+1, \cdots, 2m.$$

We can distinguish two cases:

(a) in the case of $n \neq 2$, we have $h_{ij}^r = 0$, $i, j = 1, 2, \cdots, n$, $r = n+1, \cdots, 2m$.

(b) in the case of $n = 2$, we have $h_{11}^r = h_{22}^r$, $h_{12}^r = h_{21}^r = 0$, $r = 3, \cdots, 2m$.

The converse is trivial.

**Corollary 3.2** If the equality case of inequality (3.1) holds for all unit tangent vector X of $M^n$, then we have

(1) the equality case of inequality (3.1) holds for all unit tangent vector X of $M^n$ if and only if $M^n$ is a totally umbilical submanifold;

(2) if $U$ is a tangent field on $M^n$ and $n \geq 3$, $M^n$ is a totally geodesic submanifold.

**Proof** (1) For $n = 2$, from Theorem 3.1 we know the equality case of inequality (3.1) holds for all unit tangent vector $X$ of $M^2$ if and only if $M^2$ is a totally umbilical submanifold with respect to the semi-symmetric metric connection. Then from Lemma 2.2, $M^2$ is a totally umbilical submanifold with respect to the Levi-Civita connection.

For $n \geq 3$, from Theorem 3.1 we know the equality case of inequality (3.1) holds for all unit tangent vector $X$ of $M^n$ if and only $h_{ij}^r = 0$, $\forall i, j, r$. According to formula (7) from [13], we have $h_{ij}'^r = h_{ij}^r + k^r g_{ij}$, where $k^r$ are real-valued functions on $M$. Thus we have $h_{ij}'^r = k^r g_{ij}$. So $M^n$ is a totally umbilical submanifold.

(2) If $U$ is a tangent vector field on $M^n$, from Lemma 2.1 we have $h' = h$. For $n \geq 3$, from Theorem 3.1 the equality case of inequality (3.1) holds for all unit tangent vector $X$ of $M^n$ if and only if $h_{ij}^r = 0$, $\forall i, j, r$. Thus we have $h_{ij}'^r = 0$, $\forall i, j, r$. So $M^n$ is a totally geodesic submanifold.

## 4 $k$-Ricci Curvature

In this section, we establish a sharp relation between the $k$-Ricci curvature and the mean curvature $||H||$ with respect to the semi-symmetric metric connect $\overline{\nabla}$.

Let $L$ be a $k$-plane section of $T_xM^n$, $x \in M^n$, and $X$ be a unit vector in $L$. We choose an orthonormal frame $e_1, \cdots, e_k$ of $L$ such that $e_1 = X$. In [5] the $k$-Ricci curvature of $L$ at $X$ is defined by

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{4.1}$$

The scalar curvature of a $k$-plane section $L$ is given by

$$\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}. \tag{4.2}$$

For an integer $k$, $2 \leq k \leq n$, the Riemannian invariant $\Theta_k$ on $M^n$ at $x \in M^n$ defined by

$$\Theta_k(x) = \frac{1}{k-1} \inf\{\mathrm{Ric}_L(X) : L, X\}, \tag{4.3}$$

where $L$ runs over all $k$-plane sections in $T_xM$ and $X$ runs over all unit vectors in $L$. From (2.6), (4.1) and (4.2) for any $k$-plane section $L_{i_1\cdots i_k}$ spanned by $\{e_{i_1}, \cdots, e_{i_k}\}$, it follows that

$$\tau(L_{i_1\cdots i_k}) = \frac{1}{2} \sum_{i\in\{i_1,\cdots i_k\}} \mathrm{Ric}_{i_1,\cdots i_k}(e_i) \tag{4.4}$$

and

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1\leq i_1<\cdots<i_k\leq n} \tau(L_{i_1\cdots i_k}). \tag{4.5}$$

From (4.3), (4.4) and (4.5), we have

$$\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(x). \tag{4.6}$$

**Theorem 4.1** Let $M^n$, $n \geq 3$, be an $n$-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ endowed with a semi-symmetric connection $\overline{\nabla}$. Then we have

$$||H||^2 \geq \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - F_1 - \frac{3F_2}{n(n-1)}\|P\|^2.$$

**Proof** For $x \in M^n$, let $\{e_1, \cdots, e_n\}$ and $\{e_{n+1}, \cdots, e_{2m}\}$ be an orthonormal basis of $T_x^M$ and $T_x^\perp M$, respectively, where $e_{n+1}$ is parallel to the mean curvature vector $H$.

From (3.2), we have

$$R_{ijji} = F_1 + 3F_2g^2(Je_i, e_j) - \alpha(e_i, e_i) - \alpha(e_j, e_j) + \sum_{r=n+1}^{2m}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{4.7}$$

Setting $||P||^2 = \sum\limits_{i,j=1}^{n} g^2(Je_i, e_j)$. From (2.6), it follows that

$$2\tau(x) = n(n-1)F_1 + 3F_2||P||^2 - 2(n-1)\lambda + n^2||H||^2 - ||h||^2. \tag{4.8}$$

Then equation (4.8) can be also written as

$$n^2||H||^2 = 2\tau + ||h||^2 + 2(n-1)\lambda - n(n-1)F_1 - 3F_2||P||^2. \tag{4.9}$$

We choose an orthonormal basis $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{2m}\}$ such that $e_1, \cdots, e_n$ diagonalize the shape operator $Ae_{n+1}$, i.e.,

$$Ae_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

and $Ae_r = (h^r_{ij}),\ \ i,j = 1\cdots\cdots n;\ \ r = n+2,\cdots,2m,\ \ \mathrm{trace}Ae_r = 0$. So (4.9) turns into

$$n^2||H||^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h^r_{ij})^2 + 2(n-1)\lambda - n(n-1)F_1 - 3F_2||P||^2. \quad (4.10)$$

On the other hand, we get

$$(n||H||)^2 = (\sum_{i=1}^{n} a_i)^2 \le n\sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \ge n||H||^2. \quad (4.11)$$

From (4.10) and (4.11), it follows that

$$n^2||H||^2 \ge 2\tau + n||H||^2 + 2(n-1)\lambda - n(n-1)F_1 - 3F_2||P||^2,$$

which means

$$||H||^2 \ge \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - F_1 - \frac{3F_2}{n(n-1)}||P||^2. \quad (4.12)$$

Using Theorem 4.1 and (4.6) we can obtain the following theorem.

**Theorem 4.2** Let $M^n,\ \ n \ge 3$, be an $n$-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ endowed with a semi-symmetric connection $\overline{\nabla}$. Then for any integer $k,\ \ 2 \le k \le n$, and for any point $x \in M$, we have

$$||H||^2(x) \ge \Theta_k(x) + \frac{2}{n}\lambda - F_1 - \frac{3F_2}{n(n-1)}||P||^2.$$

**Proof** Let $\{e_1,\cdots,e_n\}$ be an orthonormal basis of $T_xM^n$ at $x \in M^n$. The $k$-plane section spanned by $e_{i_1},\cdots,e_{i_k}$ is denoted by $L_{i_1\cdots i_k}$.

Then from (4.6) and (4.12), we have

$$||H||^2(x) \ge \Theta_k(x) + \frac{2}{n}\lambda - F_1 - \frac{3F_2}{n(n-1)}||P||^2.$$

**Remark 4.3** For $F_1 = F_2 = C$ ($C$ is constant) in Theorem 3.1, we obtain a Chen-Ricci inequality for submanifolds of complex space forms with a semi-symmetric metric connection.

For $F_1 = F_2 = C$ ($C$ is constant) in Theorem 4.1 and Theorem 4.2, the results can be found in [15].

## References

[1] Nash J F. The imbedding problem for Riemannian manifolds[J]. Ann. Math., 1956, 63: 20–63.

[2] Chen B Y. Some pinching and classification theorems for minimal submanifolds[J]. Arch. Math. (Basel), 1993, 60(6): 568–578.

[3] Chen B Y. Strings of Riemannian invariants, inequalities, ideal immersions and their applications[J]. Third Pacific Rim Geom. Conf. (Seoul), 1996: 7–60.

[4] Chen B Y. Riemannian submanifolds[M]. North-Holland, Amsterdam: Handbook Diff. Geom., 2000, 1: 187–418.

[5] Chen B Y. Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions[J]. Glasg. Math. J., 1999, 41(1): 33–41.

[6] Tripathi M M. Chen-Ricci inequality for submanifolds of contact metric manifolds[J]. J. Adv. Math. Studies, 2008, 1(1-2): 111–135.

[7] Chen B Y. On Ricci curvature of isotropic and Lagrangian submanifolds in complex space forms[J]. Arch. Math. (Basel), 2000, 74: 154–160.

[8] Matsumoto K, Mihai I, Oiaga A. Ricci curvature of submanifolds in complex space forms[J]. Rev. Roumaine Math. Pures Appl., 2001, 46: 775–782.

[9] Mihai I. Ricci curvature of submanifolds in Sasakian space forms[J]. J. Aust. Math. Soc., 2002, 72(2): 247–256.

[10] Zhang Pan, Zhang Liang, Song Wei-dong. Some remarks on geometric inequalities for submanifolds of a riemannian manifold of quasi-constant curvature[J]. J. Math., 2016, 36(3): 445–457.

[11] Haydan H A. Subspaces of a space with torsion[J]. Proc. London Math. Soc., 1932, 34: 27–50.

[12] Yano K. On semi-symmetric metric connection[J]. Rev. Roumaine Math. Pures Appl., 1970, 15: 1579–1586.

[13] Nakao Z. Submanifolds of a Riemanian with semi-symmetric metric connections[J]. Proc. Amer. Math. Soc., 1976, 54: 261–266.

[14] Mihai A, Özgür C. Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection[J]. Taiwanese J. Math., 2010, 14 (4): 1465–1477.

[15] Mihai A, Özgür C. Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connections[J]. Rocky Mountain J. Math., 2011, 5(41): 1653–1673.

[16] Tricerri F, Vanhecke L. Curvature tensors on almost Hermitian manifolds[J]. Trans. Amer. Math. Soc., 1981, 267(2): 365–397.

[17] Vanhecke L. Almost Hermitian manifolds with $J$-invariant Riemann curvature tensor[J]. Rend. Sem. Mat. Univ. Politec. Torino, 1975, 34: 487–498.

# 容有半对称度量联络的广义复空间中子流形上的 Chen-Ricci不等式

何国庆

(安徽师范大学数学计算机科学学院, 安徽 芜湖 241000)

**摘 要**: 本文研究了容有半对称度量联络的广义复空间中的子流形上的Chen-Ricci不等式. 利用代数技巧, 建立了子流形上的Chen-Ricci不等式. 这些不等式给出了子流形的外在几何量－关于半对称联络的平均曲率与内在几何量－Ricci曲率及$k$-Ricci曲率之间的关系, 推广了Mihai 和Özgür的一些结果.

**关键词**: Chen-Ricci不等式; $k$-Ricci曲率; 广义复空间; 半对称度量联络

MR(2010)**主题分类号**: 53C40 **中图分类号**: O186.12