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ENDOMORPHISM ALGEBRAS IN THE YETTER-DRINFEL'D MODULE CATEGORY OVER A REGULAR MULTIPLIER HOPF ALGEBRA

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Abstract: Endomorphism algebras in Yetter-Drinfel'd module category over a regular multiplier Hopf algebra are studied in this paper. By the tools of multiplier Hopf algebra and Homological algebra theories, we get that two endomorphism algebras are isomorphic in the Yetter-Drinfel'd module category, which generalizes the results of Panaite et al. in Hopf algebra case.

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1 Introduction

Multiplier Hopf algebra, introduced by Van Daele [1], can be naturally considered as a generalization of Hopf algebra when the underlying algebra is no longer assumed to have a unit. Yetter-Drinfel'd module category, as an important content in Hopf algebras theory, was also studied by Van Daele and his collaborators. All the objects they discussed are (non-degenerate) algebras (see [2]).

However, in the well-known Hopf algebras case, the objects of Yetter-Drinfel'd module category are only vector spaces satisfying some certain conditions. So in [3], the authors gave a new category structure for regular multiplier Hopf algebra A: (α,β)-Yetter-Drinfel'd module category ${}_A\mathcal{YD}^A(\alpha,\beta)$, in which the objects were vector spaces, generalizing the former notions.

In this paper, we focus our work on (α,β)-Yetter-Drinfel'd module, mainly consider some algebras in Yetter-Drinfel'd modules category and get some isomorphisms.

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The paper is organized in the following way. In Section 2, we recall some notions which we will use in the following, such as multiplier Hopf algebras, modules and complete modules for a multiplier Hopf algebras, comodules and (α,β)-Yetter-Drinfel'd modules.

In Section 3, we consider algebras in ${}_A\mathcal{YD}^A(\alpha,\beta)$. Let $\alpha,\beta\in \mathrm{Aut}(A)$, and $M\in {}_A\mathcal{YD}^A(\alpha,\beta)$ be finite dimensional. Consider the object $M'\in {}_A\mathcal{YD}^A(\alpha\beta^{-1}\alpha,\alpha)$, coinciding with M as left A-modules, and having a right A-comodule structure given by

$$\Gamma(m)(1\otimes a') = m_{\langle 0\rangle}\otimes m_{\langle 1\rangle}a' = m_{(0)}\otimes \alpha\beta^{-1}(m_{(1)})a',$$

then $\overline{\mathrm{End}(M)}\cong \mathrm{End}(M')^{op}$ as algebras in ${}_A\mathcal{YD}^A$. Let ${}^{\diamond}M$ be the dual vector space M^* with certain left A-module and right A-comodule structures, then $\mathrm{End}(M)^{op}\cong \mathrm{End}({}^{\diamond}M)$.

2 Preliminaries

Throughout this paper, all spaces we considered are over a fixed field k. We consider A as an algebra with a nondegenerate product, it is possible to construct the multiplier algebra $M(A)$. $M(A)$ is an algebra with identity such that A sits in $M(A)$ as an essential two-sided ideal, it can be also characterized as the largest algebra with identity containing A as an essential ideal. More details about the concept of the multiplier algebra of an algebra, we refer to [1].

An algebra morphism (or homomorphism) $\Delta: A\longrightarrow M(A\otimes A)$ is called a comultiplication on A if $T_1^A(a\otimes b)=\Delta(a)(1\otimes b)$ and $T_2^A(a\otimes b)=(a\otimes 1)\Delta(b)$ are elements of $A\otimes A$ for all $a,b\in A$ and if Δ is coassociative in the sense that the linear mappings $T_1^A, T_2^A: A\otimes A\longrightarrow A\otimes A$ obey the relation

$$(T_2^A\otimes id)\circ(id\otimes T_1^A)=(id\otimes T_1^A)\circ(T_2^A\otimes id),$$

where id denotes the identity map.

A pair (A,Δ) of an algebra A with nondegenerate product and a comultiplication Δ on A is called a multiplier Hopf algebra if T_1^A and T_2^A are bijective (see [1]), (A,Δ) is regular if and only if the antipode of (A,Δ^{cop}) is bijective.

Let (A,Δ,ε,S) be a regular multiplier Hopf algebra and M a vector space. Then M is called a (left-right) (α,β)-Yetter-Drinfel'd module over regular multiplier Hopf algebra A, if

(1) $(M,\cdot)$ is a left unital A-module, i.e., $A\cdot M=M$.

(2) (M,Γ) is a (right) A-comodule, where $\Gamma: M\to M_0(M\otimes A)$ denotes the right coaction of A on M, $M_0(M\otimes A)$ denote the completed module.

(3) Γ and $\cdot$ satisfy the following compatible conditions

$$(a\cdot v)_{(0)}\otimes(a\cdot v)_{(1)}a' = a_{(2)}\cdot v_{(0)}\otimes\beta(a_{(3)})v_{(1)}\alpha(S^{-1}(a_{(1)}))a'. \tag{2.1}$$

By the definition of Yetter-Drinfel'd modules, we can define (left-right) Yetter-Drinfel'd module category ${}_A\mathcal{YD}^A(\alpha,\beta)$. The other three Yetter-Drinfel'd module categories are similar (more details see [3–5]).

${}_A\mathcal{YD}^A(id, id) = {}_A\mathcal{YD}^A$, the left-right Yetter-Drinfel'd module category.

3 Endomorphism Algebras

Let A be a regular multiplier Hopf algebra, in this section, we mainly consider (left-right) Yetter-Drinfel'd module category ${}_A\mathcal{YD}^A$ over regular multiplier Hopf algebra A.

Definition 3.1 Let A be a multiplier Hopf algebra and C a unital algebra. C is called a left A-module algebra, if

(1) $(C, \cdot)$ is a left unital A-module,

(2) the module action satisfies

$$a \cdot (cc') = (a_{(1)} \cdot c)(a_{(2)} \cdot c'), \quad a \cdot 1_C \;=\; \varepsilon(a)1_C,$$

C is called right A-comodule algebra, if

(1) (C, ρ) is a right A-comodule,

(2) the comodule structure map ρ satisfies: for all $a \in A$,

$$\begin{aligned} \rho(cc')(1 \otimes a) &= c_{(0)}c'_{(0)} \otimes c_{(1)}c'_{(1)}a, \\ \rho(1_C) &= 1_C \otimes 1_A, \quad 1_A \text{ is the unit of } M(A). \end{aligned}$$

Let C be a unital associative algebra in ${}_A\mathcal{YD}^A$. That means C is an object in ${}_A\mathcal{YD}^A$, and the multiplication $C \otimes C \to C$ and a unit map $\iota : k \to C$ satisfying associativity and unit axioms.

Proposition 3.2 C is a unital algebra in ${}_A\mathcal{YD}^A$ if and only if C is an object in ${}_A\mathcal{YD}^A$ and C is a left A-module algebra and a right A^{op}-comodule algebra.

We denote by C^{op} the usual opposite algebra, with the multiplication $c \bullet c' = c'c$ for all $c, c' \in C$, and by $\overline{C}$ the A-opposite algebra, which means C as an object in ${}_A\mathcal{YD}^A$, but with the multiplication $c * c' = c'_{(0)}(c'_{(1)} \cdot c)$ for all $c, c' \in C$, i.e., the opposite of C in the category ${}_A\mathcal{YD}^A$.

Proposition 3.3 By above notation, if C is an algebra in ${}_A\mathcal{YD}^A$, then $\overline{C}$ is an algebra in ${}_A\mathcal{YD}^A$.

Proof For $c, c' \in \overline{C}$ and any $a \in A$,

$$\begin{aligned} a \cdot (c * c') &= a \cdot (c'_{(0)}(c'_{(1)} \cdot c)) = (a_{(1)} \cdot c'_{(0)})(a_{(2)} \cdot (c'_{(1)} \cdot c)) \\ &= (a_{(1)} \cdot c'_{(0)})(a_{(2)}c'_{(1)} \cdot c), \\ (a_{(1)} \cdot c) * (a_{(2)} \cdot c') &= (a_{(2)} \cdot c')_{(0)}((a_{(2)} \cdot c')_{(1)} \cdot (a_{(1)} \cdot c)) \\ &= (a_{(2)(2)} \cdot c'_{(0)})((a_{(2)(3)}c'_{(1)}S^{-1}(a_{(2)(1)})) \cdot (a_{(1)} \cdot c)) \\ &= (a_{(1)} \cdot c'_{(0)})(a_{(2)}c'_{(1)} \cdot c), \\ \rho(c * c')(1 \otimes a) &= \rho(c'_{(0)}(c'_{(1)} \cdot c))(1 \otimes a) = c'_{(0)(0)}(c'_{(1)} \cdot c)_{(0)} \otimes (c'_{(1)} \cdot c))_{(1)}c'_{(0)(1)}a \\ &= c'_{(0)(0)}(c'_{(1)(2)} \cdot c_{(0)}) \otimes c'_{(1)(3)}c_{(1)}S^{-1}(c'_{(1)(1)})c'_{(0)(1)}a \\ &= c'_{(0)}(c'_{(1)} \cdot c_{(0)}) \otimes c'_{(2)}c_{(1)}a \\ &= c_{(0)} * c'_{(0)} \otimes c'_{(1)}c_{(1)}a. \end{aligned}$$

Proposition 3.4 If C, D are algebras in ${}_A\mathcal{YD}^A$, then $C \otimes D$ is also an algebra in ${}_A\mathcal{YD}^A$ with the following structures

$$\begin{aligned} a\cdot(c\otimes d) &= a_{(1)}\cdot c\otimes a_{(2)}\cdot d,\\ \rho(c\otimes d)(1\otimes a') &= (c_{(0)}\otimes d_{(0)})\otimes d_{(1)}c_{(1)}a' \text{ for all } a'\in A,\\ (c\otimes d)(c'\otimes d') &= cc'_{(0)}\otimes(c'_{(1)}\cdot d)d'. \end{aligned}$$

Proof It is obvious. Indeed, this algebra structure on $C \otimes D$ given above is just the braided tensor product of C and D in the braided tensor category ${}_A\mathcal{YD}^A$.

We now introduce the endomorphism algebras associated to (α, β)-Yetter-Drinfel'd modules.

Proposition 3.5 Let $\alpha, \beta \in \mathrm{Aut}(A)$ and $M \in {}_A\mathcal{YD}^A(\alpha, \beta)$ be finite dimensional. Then

(1) $\mathrm{End}(M)$ is an algebra in ${}_A\mathcal{YD}^A$ with structures

$$\begin{aligned} &(a\cdot u)(m)=\alpha^{-1}(a_{(1)})\cdot u(\alpha^{-1}S(a_{(2)})\cdot m),\\ &u_{(0)}(m)\otimes a'u_{(1)}=u(m_{(0)})_{(0)}\otimes a'S^{-1}(m_{(1)})u(m_{(0)})_{(1)} \end{aligned}$$

for all $a, a' \in A$, $u \in \mathrm{End}(M)$ and $m \in M$.

(2) $\mathrm{End}(M)^{op}$ is an algebra in ${}_A\mathcal{YD}^A$ with structures

$$\begin{aligned} &(a\cdot u)(m)=\beta^{-1}(a_{(2)})\cdot u(\beta^{-1}S^{-1}(a_{(1)})\cdot m),\\ &u_{(0)}(m)\otimes u_{(1)}a'=u(m_{(0)})_{(0)}\otimes u(m_{(0)})_{(1)}S(m_{(1)})a' \end{aligned}$$

for all $a, a' \in A$, $u \in \mathrm{End}(M)^{op}$ and $m \in M$.

Proof We only prove (1) here, (2) is similar. For (1), we first show that $\mathrm{End}(M)$ is an object in ${}_A\mathcal{YD}^A$. In the following, we show the main process: the compatible condition of ${}_A\mathcal{YD}^A$, i.e.,

$$(a\cdot u)_{(0)}(m)\otimes a'(a\cdot u)_{(1)}=(a_{(2)}\cdot u_{(0)})(m)\otimes a'(a_{(3)}u_{(1)}S^{-1}(a_{(1)})).$$

It holds, since

$$\begin{aligned} &(a\cdot u)_{(0)}(m)\otimes a'(a\cdot u)_{(1)}\\ =\;&(a\cdot u)(m_{(0)})_{(0)}\otimes a'S^{-1}(m_{(1)})(a\cdot u)(m_{(0)})_{(1)}\\ =\;&(\alpha^{-1}(a_{(1)})\cdot u(\alpha^{-1}S(a_{(2)})\cdot m_{(0)}))_{(0)}\\ &\otimes a'S^{-1}(m_{(1)})(\alpha^{-1}(a_{(1)})\cdot u(\alpha^{-1}S(a_{(2)})\cdot m_{(0)}))_{(1)}\\ =\;&\alpha^{-1}(a_{(1)(2)})\cdot u(\alpha^{-1}S(a_{(2)})\cdot m_{(0)})_{(0)}\\ &\otimes a'S^{-1}(m_{(1)})\beta(\alpha^{-1}(a_{(1)(3)}))u(\alpha^{-1}S(a_{(2)})\cdot m_{(0)})_{(1)}\alpha S^{-1}(\alpha^{-1}(a_{(1)(1)}))\\ =\;&\alpha^{-1}(a_{(2)})\cdot u(\alpha^{-1}S(a_{(4)})\cdot m_{(0)})_{(0)}\\ &\otimes a'S^{-1}(m_{(1)})\beta\alpha^{-1}(a_{(3)})u(\alpha^{-1}S(a_{(2)})\cdot m_{(0)})_{(1)}S^{-1}(a_{(1)}) \end{aligned}$$

and

$$
\begin{aligned}
&(a_{(2)}\cdot u_{(0)})(m)\otimes a'(a_{(3)}u_{(1)}S^{-1}(a_{(1)}))\\
=\ &\alpha^{-1}(a_{(2)(1)})\cdot u_{(0)}(\alpha^{-1}S(a_{(2)(2)})\cdot m)\otimes a'(a_{(3)}u_{(1)}S^{-1}(a_{(1)}))\\
=\ &\alpha^{-1}(a_{(2)(1)})\cdot u((\alpha^{-1}S(a_{(2)(2)})\cdot m)_{(0)})_{(0)}\\
&\otimes a'(a_{(3)}S^{-1}((\alpha^{-1}S(a_{(2)(2)})\cdot m)_{(1)})u((\alpha^{-1}S(a_{(2)(2)})\cdot m)_{(0)})_{(1)}S^{-1}(a_{(1)}))\\
=\ &\alpha^{-1}(a_{(2)(1)})\cdot u(\alpha^{-1}S(a_{(2)(2)(2)})\cdot m_{(0)})_{(0)}\\
&\otimes a'(a_{(3)}S^{-1}(\beta\alpha^{-1}S(a_{(2)(2)(1)})m_{(1)}a_{(2)(2)(3)})u(\alpha^{-1}S(a_{(2)(2)(2)})\cdot m_{(0)})_{(1)}S^{-1}(a_{(1)}))\\
=\ &\alpha^{-1}(a_{(2)})\cdot u(\alpha^{-1}S(a_{(4)})\cdot m_{(0)})_{(0)}\\
&\otimes a'S^{-1}(m_{(1)})\beta\alpha^{-1}(a_{(3)})u(\alpha^{-1}S(a_{(2)})\cdot m_{(0)})_{(1)}S^{-1}(a_{(1)}).
\end{aligned}
$$

Then we need to show that the product defined in (1) is A-linear and A-colinear.

$$
\begin{aligned}
&((a_{(1)}\cdot u)(a_{(2)}\cdot u'))(m)\\
=\ &(a_{(1)}\cdot u)\alpha^{-1}(a_{(2)(1)})\cdot u'(\alpha^{-1}S(a_{(2)(2)})\cdot m)\\
=\ &\alpha^{-1}(a_{(1)(1)})\cdot u(\alpha^{-1}S(a_{(1)(2)})\cdot\alpha^{-1}(a_{(2)(1)})\cdot u'(\alpha^{-1}S(a_{(2)(2)})\cdot m))\\
=\ &\alpha^{-1}(a_{(1)})\cdot u(u'(\alpha^{-1}S(a_{(2)})\cdot m))\\
=\ &\alpha^{-1}(a_{(1)})\cdot(uu')(\alpha^{-1}S(a_{(2)})\cdot m))\\
=\ &(a\cdot(uu'))(m)
\end{aligned}
$$

and

$$
\begin{aligned}
&u_{(0)}u'_{(0)}(m)\otimes a'u'_{(1)}u_{(1)}\\
=\ &u_{(0)}(u'(m_{(0)})_{(0)})\otimes a'S^{-1}(m_{(1)})u'(m_{(0)})_{(1)}u_{(1)}\\
=\ &u(u'(m_{(0)})_{(0)(0)})_{(0)}\otimes a'S^{-1}(m_{(1)})u'(m_{(0)})_{(1)}S^{-1}(u'(m_{(0)})_{(0)(1)})u(u'(m_{(0)})_{(0)(0)})_{(1)}\\
=\ &u(u'(m_{(0)}))_{(0)}\otimes a'S^{-1}(m_{(1)})u(u'(m_{(0)}))_{(1)}\\
=\ &(uu')(m_{(0)})_{(0)}\otimes a'S^{-1}(m_{(1)})(uu')(m_{(0)})_{(1)}\\
=\ &(uu')_{(0)}(m)\otimes a'(uu')_{(1)}.
\end{aligned}
$$

It is easy to get $a\cdot id=\varepsilon(a)id$ and $\rho(id)=id\otimes 1$, where id is the unit in $\mathrm{End}(M)$. This completes the proof.

Remark here that

$$
\begin{aligned}
u_{(0)}(m)\otimes a'u_{(1)}&=u(m_{(0)})_{(0)}\otimes a'S^{-1}(m_{(1)})u(m_{(0)})_{(1)},\\
u(m)_{(0)}\otimes a'u(m)_{(1)}&=u_{(0)}(m_{(0)})\otimes a'm_{(1)}u_{(1)}
\end{aligned}
$$

are equivalent.

Proposition 3.6 Let $\alpha,\beta\in\mathrm{Aut}(A)$, and $M\in{}_A\mathcal{YD}^A(\alpha,\beta)$. Define a new object M' as follows: M' coincides with M as left A-modules, and has a right A-comodule structure given by

$$\begin{aligned}\Gamma(m)(1\otimes a') &= m_{\langle 0\rangle}\otimes m_{\langle 1\rangle}a'\\ &:= m_{(0)}\otimes \alpha\beta^{-1}(m_{(1)})a' = m_{(0)}\otimes \alpha\beta^{-1}(m_{(1)}\beta\alpha^{-1}(a'))\end{aligned}$$

for all $a'\in A$ and $m\in M$, where

$$m_{(0)}\otimes m_{(1)}\beta\alpha^{-1}(a') = \rho(m)(1\otimes \beta\alpha^{-1}(a')),$$

and ρ is the comodule structure of M. Then

$$M'\in {}_A\mathcal{YD}^A(\alpha\beta^{-1}\alpha,\alpha).$$

Proof We can get the conclusion by direct computation.

$$\begin{aligned}(a\cdot m)_{<0>}\otimes (a\cdot m)_{<1>}a' &= (a\cdot m)_{(0)}\otimes \alpha\beta^{-1}((a\cdot m)_{(1)})a'\\ &= a_{(2)}\cdot m_{(0)}\otimes \alpha\beta^{-1}(\beta(a_{(3)})m_{(1)}\alpha S^{-1}(a_{(1)}))a'\\ &= a_{(2)}\cdot m_{(0)}\otimes \alpha(a_{(3)})\alpha\beta^{-1}(m_{(1)})\alpha\beta^{-1}\alpha S^{-1}(a_{(1)}))a'\\ &= a_{(2)}\cdot m_{<0>}\otimes \alpha(a_{(3)})m_{<1>}\alpha\beta^{-1}\alpha S^{-1}(a_{(1)}))a',\end{aligned}$$

this implies $M'\in {}_A\mathcal{YD}^A(\alpha\beta^{-1}\alpha,\alpha)$.

Theorem 3.7 Let $\alpha,\beta\in \mathrm{Aut}(A)$, and $M\in {}_A\mathcal{YD}^A(\alpha,\beta)$ be finite dimensional. Consider the object $M'\in {}_A\mathcal{YD}^A(\alpha\beta^{-1}\alpha,\alpha)$ as above. Define the map

$$\begin{aligned}&\tau:\overline{\mathrm{End}(M)}\to \mathrm{End}(M')^{op},\\ &\tau(u)(m) = u_{(0)}(\alpha^{-1}(u_{(1)})\cdot m)\end{aligned}$$

for all $u\in \mathrm{End}(M)$ and $m\in M'$. Then τ is an isomorphism of algebras in ${}_A\mathcal{YD}^A$.

Proof Similar to the proof of Proposition 4.10 in [6].

First, τ is a homomorphism, since for $u,v\in\mathrm{End}(M)$,

$$\begin{aligned}\tau(u*v)(m) &= \tau(v_{(0)}(v_{(1)}\cdot u))(m)\\ &= (v_{(0)}(v_{(1)}\cdot u))_{(0)}(\alpha^{-1}((v_{(0)}(v_{(1)}\cdot u))_{(1)})\cdot m)\\ &= (v_{(0)(0)}(v_{(1)}\cdot u)_{(0)})(\alpha^{-1}((v_{(1)}\cdot u)_{(1)}v_{(0)(1)})\cdot m)\\ &= v_{(0)(0)}(v_{(1)(2)}\cdot u_{(0)})(\alpha^{-1}(v_{(1)(3)}u_{(1)}S^{-1}(v_{(1)(1)})v_{(0)(1)})\cdot m)\\ &= v_{(0)}(v_{(1)}\cdot u_{(0)})(\alpha^{-1}(v_{(2)}u_{(1)})\cdot m)\\ &= v_{(0)}(\alpha^{-1}(v_{(1)(1)})\cdot u_{(0)}(\alpha^{-1}S^{-1}(v_{(1)(2)})\alpha^{-1}(v_{(2)}u_{(1)})\cdot m))\\ &= v_{(0)}(\alpha^{-1}(v_{(1)})\cdot u_{(0)}(\alpha^{-1}(u_{(1)})\cdot m))\\ &= \tau(v)(u_{(0)}(\alpha^{-1}(u_{(1)})\cdot m))\\ &= \tau(v)(\tau(u)(m))\\ &= (\tau(u)\bullet\tau(v))(m).\end{aligned}$$

Second, τ is A-linear, since

$$\begin{aligned}\tau(a\cdot u)(m) &= (a\cdot u)_{(0)}(\alpha^{-1}((a\cdot u)_{(1)})\cdot m)\\ &= (a_{(2)}\cdot u_{(0)})(\alpha^{-1}(a_{(3)}u_{(1)}S^{-1}(a_{(1)}))\cdot m)\\ &= \alpha^{-1}(a_{(2)(1)})\cdot u_{(0)}(\alpha^{-1}S(a_{(2)(2)})\cdot(\alpha^{-1}(a_{(3)}u_{(1)}S^{-1}(a_{(1)}))\cdot m))\\ &= \alpha^{-1}(a_{(2)})\cdot u_{(0)}(\alpha^{-1}(u_{(1)}S^{-1}(a_{(1)}))\cdot m)\\ &= \alpha^{-1}(a_{(2)})\cdot u_{(0)}(\alpha^{-1}(u_{(1)})\alpha^{-1}S^{-1}(a_{(1)}))\cdot m)\\ &= \alpha^{-1}(a_{(2)})\cdot \tau(u)(\alpha^{-1}S^{-1}(a_{(1)}))\cdot m)\\ &= (a\cdot\tau(u))(m).\end{aligned}$$

Third, τ is A-colinear. To prove this, we have to show that $\rho\tau=(\tau\otimes l)\rho$, where ρ is the A-comodule structure of $End(M')^{op}$. Denote $\rho(v)(1\otimes a)=v^{(0)}\otimes v^{(1)}a$, we have to prove

$$\tau(u)^{(0)}(m)\otimes\tau(u)^{(1)}a=\tau(u_{(0)})(m)\otimes u_{(1)}a$$

for all $a\in A$,

$$\begin{aligned}&\tau(u)^{(0)}(m)\otimes\tau(u)^{(1)}a\\ =\ &\tau(u)(m_{<0>})_{<0>}\otimes\tau(u)(m_{<0>})_{<1>}S(m_{<1>})a\\ =\ &\tau(u)(m_{(0)})_{(0)}\otimes\alpha\beta^{-1}(\tau(u)(m_{(0)})_{(1)})S(\alpha\beta^{-1}(m_{(1)}))a\\ =\ &\tau(u)(m_{(0)})_{(0)}\otimes\alpha\beta^{-1}(\tau(u)(m_{(0)})_{(1)}S(m_{(1)}))a\\ =\ &(u_{(0)}(\alpha^{-1}(u_{(1)})\cdot m_{(0)}))_{(0)}\otimes\alpha\beta^{-1}((u_{(0)}(\alpha^{-1}(u_{(1)})\cdot m_{(0)}))_{(1)}S(m_{(1)}))a\\ =\ &u_{(0)(0)}((\alpha^{-1}(u_{(1)})\cdot m_{(0)})_{(0)})\otimes\alpha\beta^{-1}((\alpha^{-1}(u_{(1)})\cdot m_{(0)})_{(1)}u_{(0)(1)}S(m_{(1)}))a\\ =\ &u_{(0)(0)}(\alpha^{-1}(u_{(1)(2)})\cdot m_{(0)(0)})\\ &\otimes\alpha\beta^{-1}((\beta(\alpha^{-1}(u_{(1)(3)}))m_{(0)(1)}\alpha(S^{-1}(\alpha^{-1}(u_{(1)(1)}))))u_{(0)(1)}S(m_{(1)}))a\\ =\ &u_{(0)(0)}(\alpha^{-1}(u_{(0)(1)})\cdot m)\otimes u_{(1)}a\\ =\ &\tau(u_{(0)})(m)\otimes u_{(1)}a.\end{aligned}$$

Finally, we will show that τ is bijective, we define

$$\begin{aligned}&\tau^{-1}:\mathrm{End}(M')^{op}\to\overline{\mathrm{End}(M)},\\ &\tau^{-1}(v)(m)=v^{(0)}(\alpha^{-1}(S(v^{(1)}))\cdot m)\end{aligned}$$

for $v\in\mathrm{End}(M')^{op}$. We can check that $\tau\tau^{-1}=\tau^{-1}\tau=id$ and τ^{-1} is A-linear and A-colinear.

This completes the proof.

Remark that $m=\sum\limits_{i=1}^{n}a_i\cdot m_i$, since M is a unital A-module.

$$\begin{aligned}\tau(u)(m) &= u_{(0)}(\alpha^{-1}(u_{(1)})\cdot m)=\sum_{i=1}^{n}u_{(0)}(\alpha^{-1}(u_{(1)})\cdot(a_i\cdot m_i))\\ &= \sum_{i=1}^{n}u_{(0)}(\alpha^{-1}(u_{(1)}\alpha(a_i))\cdot m_i).\end{aligned}$$

The definition of τ is meaningful. Because for finite i, there is an $e \in A$ such that $ea_i = a_i$ for all $i = 1, \cdots, n$. Here

$$\sum_{i=1}^{n} u_{(0)} \otimes u_{(1)}\alpha(a_i)) = \rho(u)(1 \otimes e)(1 \otimes \sum_{i=1}^{n} a_i),$$

where ρ is the right A-comodule structure of $\mathrm{End}(M)$.

From Proposition 3.5 and the notion ${}^{\diamond}M$ defined in Section 3 of [5], we can get the following results:

Proposition 3.8 Let $\alpha, \beta \in \mathrm{Aut}(A)$, and $M \in {}_A\mathcal{YD}^A(\alpha, \beta)$ be finite dimensional. Then $\mathrm{End}(M)^{op} \cong \mathrm{End}({}^{\diamond}M)$ as algebras in ${}_A\mathcal{YD}^A$.

Proof Denote the map $\imath : \mathrm{End}(M)^{op} \longrightarrow \mathrm{End}({}^{\diamond}M)$ by $\imath(u) = u^*$ for $u \in \mathrm{End}(M)^{op}$. It is an algebra isomorphism.

The map $\imath$ is A-linear, the proof is similar as in Proposition 4.11 in [6]. Then we need to show $\imath$ is A-colinear. Indeed, by Proposition 3.5 and the structures of ${}^{\diamond}M$, we can compute as follows: for all $u \in \mathrm{End}(M)^{op}$, $f \in {}^{\diamond}M$, $m \in M$, and $a \in A$,

$$\begin{aligned}
(\rho(\imath(u))(1 \otimes a))(f)(m) &= \imath(u)_{(0)}(f)(m) \otimes \imath(u)_{(0)}a \\
&= \imath(u)(f_{(0)})_{(0)}(m) \otimes S^{-1}(f_{(1)})\imath(u)(f_{(0)})_{(1)}a \\
&= \imath(u)(f_{(0)})(m_{(0)}) \otimes S^{-1}(f_{(1)})S(m_{(1)})a \\
&= f_{(0)}(u(m_{(0)})) \otimes S^{-1}(f_{(1)})S(m_{(1)})a \\
&= f(u(m_{(0)})_{(0)}) \otimes S^{-1}(S(u(m_{(0)})_{(1)}))S(m_{(1)})a \\
&= f(u(m_{(0)})_{(0)}) \otimes (u(m_{(0)})_{(1)})S(m_{(1)})a
\end{aligned}$$

and

$$\begin{aligned}
((\imath \otimes 1)\rho(u)(1 \otimes a))(f)(m) &= \imath(u_{(0)})(f)(m) \otimes u_{(1)}a \\
&= f(u_{(0)}(m)) \otimes u_{(1)}a \\
&= f(u(m_{(0)})_{(0)}) \otimes (u(m_{(0)})_{(1)})S(m_{(1)})a.
\end{aligned}$$

From all above, we use the adapted Sweedler notation, it seems that the definitions and proofs are similar as in the (weak) Hopf algebra case (see, e.g. [7]). However, we should notice the 'cover' technique introduced in [8].

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正则乘子Hopf代数上Yetter-Drinfel'd模范畴中的自同构代数

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摘要: 本文研究了正则乘子Hopf代数上Yetter-Drinfel'd模范畴中自同构代数的问题. 利用乘子Hopf代数以及同调代数理论中的方法, 获得了Yetter-Drinfel'd模范畴中两个自同构代数是同构的结果, 推广了Panaite等人在Hopf代数中的结果.

关键词: 乘子Hopf代数; Yetter-Drinfel'd模; Yetter-Drinfel'd模范畴

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