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THE LOCUS OF POINTS WITH EQUAL SUM OF RELATIVE DISTANCES TO THREE POINTS

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Abstract: In this paper, we study the problem about relative distance in the relative metric space. By mass point geometry, we get the result that for any given real number $\tau > 4$, the locus of the points P satisfying the condition, $d_{\mathscr{T}}(P,A)+d_{\mathscr{T}}(P,B)+d_{\mathscr{T}}(P,C)=\tau$, is a convex dodecagon or nonagon (where $\mathscr{T} \equiv ABC$ is a triangle formed by the three fixed points A, B, and C), which enriches the field of relative distance.

Keywords: relative distance; plane convex body; dodecagon

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1 Introduction

Let k (≥ 2) be an integer, to find k points on the sphere or in the ball of a Euclidean n-space E^n such that their pairwise distances are as large as possible is a long-standing problem in geometry. Let $\mathcal{C}$ be a plane convex body. Many authors considered this problem in the sense of the following notion of $\mathcal{C}$-distance of points in a plane convex body [3]. Some results concerning this kind of distance appeared in [1, 2, 4] and [6–10].

We recall the following definitions. For arbitrary different points $A, B \in E^2$, denote by AB the line-segment connecting the points A and B, by $|AB|$ the Euclidean length of the line-segment AB, by $\overrightarrow{AB}$ the ray starting at the point A and passing through the point B, and by $\overline{AB}$ the straight line passing through the points A and B. Let $\mathcal{C}$ be a plane convex body and let A_1B_1 be a longest chord of $\mathcal{C}$ parallel to AB. The $\mathcal{C}$-distance $d_{\mathcal{C}}(A,B)$ between the points A and B is defined by the ratio of $|AB|$ to $\frac{1}{2}|A_1B_1|$. If there is no confusion about $\mathcal{C}$, we may use the term relative distance between A and B. Observe that for arbitrary points $A, B \in E^2$ the $\mathcal{C}$-distance between A and B is equal to their $[\frac{1}{2}(\mathcal{C}+(-\mathcal{C}))]$-distance. Thus the metric $d_{\mathcal{C}}(A,B)$ is the metric of E^2 whose unit ball is $\frac{1}{2}(\mathcal{C}+(-\mathcal{C}))$.

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In this paper we consider the related problem. That is, let A, B, C be three fixed points in the plane and let $\mathscr{T} := ABC$ be the triangle formed by the points A, B, and C. We prove that, for any given real number $\tau > 4$, the locus of the points P satisfying the condition $d_{\mathscr{T}}(P, A) + d_{\mathscr{T}}(P, B) + d_{\mathscr{T}}(P, C) = \tau$, is a convex dodecagon (or nonagon) (that is, Theorem 2.4).

For simplicity, if two lines $\overline{PQ}$ and $\overline{RS}$ are parallel, we write $\overline{PQ} \| \overline{RS}$. Denote by $A(\mathcal{P})$ the area of the polygon $\mathcal{P}$. For a plane convex body $\mathcal{C}$, a chord PQ of $\mathcal{C}$ is called an affine diameter if there is no longer chord parallel to PQ in $\mathcal{C}$.

2 The Main Results

We first apply mass point geometry [5] to prove the following lemma. A mass point is a pair (α, P), where α is a positive number (the mass) and P is a point in the plane. By the Archimedes principle of the lever, one can have the following addition rule for mass points.

Addition rule: $(\varphi, A) + (\mu, B) = (\varphi + \mu, C)$, where point C is on AB with $|AC| : |CB| = \mu : \varphi$.

Lemma 2.1 Let $T := PAB$ be a triangle. Suppose that $X \in \overrightarrow{PA}$, $Y \in \overrightarrow{PB}$, and $Z \in XY$ with $Z = \lambda \cdot X + (1-\lambda) \cdot Y$, $0 \le \lambda \le 1$. Then $d_T(P, Z) = \lambda \cdot d_T(P, X) + (1-\lambda) \cdot d_T(P, Y)$.

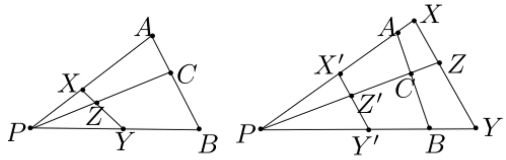

Figure 1

Proof Denote by C the intersection point of the lines $\overline{PZ}$ and $\overline{AB}$. If $X \notin PA$ or $Y \notin PB$, one may take $X' \in PA$, $Y' \in PB$, and $Z' \in PC$ with $\frac{|PX'|}{|PX|} = \frac{|PY'|}{|PY|} = \frac{|PZ'|}{|PZ|}$ (see the right picture in Figure 1), thus $\overline{XY} \| \overline{X'Y'}$ and $Z' = \lambda \cdot X' + (1-\lambda) \cdot Y'$, which implies that $d_T(P, Z) = \lambda \cdot d_T(P, X) + (1-\lambda) \cdot d_T(P, Y)$ if and only if $d_T(P, Z') = \lambda \cdot d_T(P, X') + (1-\lambda) \cdot d_T(P, Y')$. So without loss of generality, we may assume that $X \in PA$ and $Y \in PB$. Let $|PX| = \alpha_1$, $|PY| = \alpha_2$, $|PA| = \beta_1$, and $|PB| = \beta_2$. We assign masses $\alpha_1\varphi$, $\alpha_2\mu$, and $(\beta_1 - \alpha_1)\varphi + (\beta_2 - \alpha_2)\mu$ to points A, B, and P, respectively, where φ and μ satisfy $\beta_1\varphi : \beta_2\mu = \lambda : (1-\lambda)$. Now we apply the addition rule in mass point geometry and prove that Z is the center of the mass system PAB:

$$\begin{aligned} & (\alpha_1\varphi, A) + ((\beta_1 - \alpha_1)\varphi + (\beta_2 - \alpha_2)\mu, P) + (\alpha_2\mu, B) \\ = \ & [(\alpha_1\varphi, A) + ((\beta_1 - \alpha_1)\varphi, P)] + [((\beta_2 - \alpha_2)\mu, P) + (\alpha_2\mu, B)] \\ = \ & (\beta_1\varphi, X) + (\beta_2\mu, Y) = (\beta_1\varphi + \beta_2\mu, Z). \end{aligned}$$

Since Z is the center of the mass system, the mass at C can be obtained by adding the mass points of A and B:

$$(\alpha_1\varphi, A) + (\alpha_2\mu, B) = (\alpha_1\varphi + \alpha_2\mu, C),$$
$$|PZ| : |ZC| = (\alpha_1\varphi + \alpha_2\mu) : [(\beta_1 - \alpha_1)\varphi + (\beta_2 - \alpha_2)\mu].$$

Thus

$$\begin{aligned} d_T(P, Z) &= \frac{2|PZ|}{|PC|} = \frac{2|PZ|}{|PZ| + |ZC|} = 2 \cdot \frac{\alpha_1\varphi + \alpha_2\mu}{\beta_1\varphi + \beta_2\mu} \\ &= \lambda \cdot \frac{2\alpha_1}{\beta_1} + (1 - \lambda) \cdot \frac{2\alpha_2}{\beta_2} = \lambda \cdot d_T(P, X) + (1 - \lambda) \cdot d_T(P, Y), \end{aligned}$$

where the second last equality holds since $\lambda\beta_2\mu = (1 - \lambda)\beta_1\varphi$.

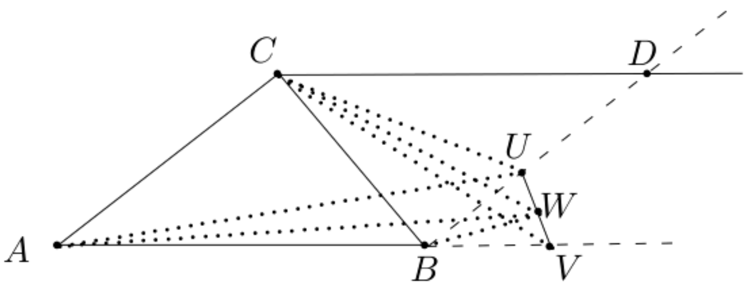

Figure 2

Lemma 2.2 Let $ABDC$ be a parallelogram and let $\mathscr{T} := ABC$ be the triangle formed by the points A, B, and C. Suppose $U \in BD$ and $V \in \overrightarrow{AB}$ with $d_{\mathscr{T}}(U, A) + d_{\mathscr{T}}(U, B) + d_{\mathscr{T}}(U, C) = d_{\mathscr{T}}(V, A) + d_{\mathscr{T}}(V, B) + d_{\mathscr{T}}(V, C) = \tau$ (see Figure 2), then $d_{\mathscr{T}}(W, A) + d_{\mathscr{T}}(W, B) + d_{\mathscr{T}}(W, C) = \tau$ for any point $W \in UV$ with $W = \lambda \cdot U + (1 - \lambda) \cdot V$, $0 \leq \lambda \leq 1$.

Proof Since this lemma satisfies the conditions of Lemma 2.1 (translate some triangle if necessary), we obtain that

$$d_{\mathscr{T}}(A, W) = \lambda \cdot d_{\mathscr{T}}(A, U) + (1 - \lambda) \cdot d_{\mathscr{T}}(A, V),$$
$$d_{\mathscr{T}}(B, W) = \lambda \cdot d_{\mathscr{T}}(B, U) + (1 - \lambda) \cdot d_{\mathscr{T}}(B, V),$$
$$d_{\mathscr{T}}(C, W) = \lambda \cdot d_{\mathscr{T}}(C, U) + (1 - \lambda) \cdot d_{\mathscr{T}}(C, V).$$

Thus

$$\begin{aligned} & d_{\mathscr{T}}(W, A) + d_{\mathscr{T}}(W, B) + d_{\mathscr{T}}(W, C) \\ = \;& \lambda \cdot (d_{\mathscr{T}}(A, U) + d_{\mathscr{T}}(B, U) + d_{\mathscr{T}}(C, U)) + (1 - \lambda) \cdot (d_{\mathscr{T}}(A, V) + d_T(B, V) + d_{\mathscr{T}}(C, V)) \\ = \;& \lambda \cdot \tau + (1 - \lambda) \cdot \tau = \tau. \end{aligned}$$

Theorem 2.3 Let $\mathscr{T} := ABC$ be a triangle. If a point P lies in the interior of $\mathscr{T}$, then $d_{\mathscr{T}}(P, A) + d_{\mathscr{T}}(P, B) + d_{\mathscr{T}}(P, C) = 4$.

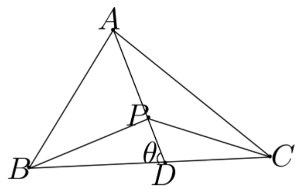

Figure 3

Proof Denote by D the intersection point of the lines $\overline{AP}$ and $\overline{BC}$, and denote by θ the angle formed by the lines $\overline{AD}$ and $\overline{BC}$, as shown in Figure 3. Then we have

$$d_{\mathscr{T}}(P,A)=\frac{2|PA|}{|AD|}=2\cdot(1-\frac{|PD|}{|AD|})=2\cdot(1-\frac{|PD|\cdot|BC|\cdot\sin(\theta)/2}{|AD|\cdot|BC|\cdot\sin(\theta)/2})=2\cdot(1-\frac{A(PBC)}{A(ABC)}).$$

Similarly, we get $d_{\mathscr{T}}(P,B)=2\cdot(1-\frac{A(PAC)}{A(ABC)})$, and $d_{\mathscr{T}}(P,C)=2\cdot(1-\frac{A(PAB)}{A(ABC)})$. Hence

$$\begin{aligned} d_{\mathscr{T}}(P,A)+d_{\mathscr{T}}(P,B)+d_{\mathscr{T}}(P,C) &= 2\cdot(3-\frac{A(PBC)+A(PAC)+A(PAB)}{A(ABC)}) \\ &= 2\cdot(3-1)=4. \end{aligned}$$

Theorem 2.4 Let $\mathscr{T}:=ABC$ be a triangle. Then for any real number $\tau>4$, the locus of the points P satisfying the condition $d_{\mathscr{T}}(P,A)+d_{\mathscr{T}}(P,B)+d_{\mathscr{T}}(P,C)=\tau$, is either a convex dodecagon (when $\tau\neq 8$) or a nonagon (when $\tau=8$).

Proof The proof follows from the following steps.

Step 1 Draw a line-segment CA_1 with $CA_1\|AB$, and draw a line-segment BA_1 with $BA_1\|AC$ (see the left in Figure 4). Let P be an arbitrary point in the triangle A_1BC, and let D the be the intersection point of AP and BC. Then $d_{\mathscr{T}}(P,A)=\frac{2|PA|}{|AD|}=2\cdot\frac{A(ABC)+A(PBC)}{A(ABC)}$. Similarly, one can have

$$d_{\mathscr{T}}(P,B)=2\cdot\frac{A(PAB)}{A(ABC)},$$
$$d_{\mathscr{T}}(P,C)=2\cdot\frac{A(PAC)}{A(ABC)}.$$

Thus

$$d_{\mathscr{T}}(P,A)+d_{\mathscr{T}}(P,B)+d_{\mathscr{T}}(P,C)=4\cdot\frac{A(ABC)+A(PBC)}{A(ABC)}=4+4\cdot\frac{A(PBC)}{A(ABC)}.$$

So in this case the locus of points P must be a line-segment XY parallel to BC.

Step 2 Let P belong to the unbounded angular region MCN bounded by the lines $\overline{AC}$ and $\overline{BC}$, see the right in Figure 4. Then we get

$$d_{\mathscr{T}}(P,A)=2\cdot\frac{A(PAB)}{A(ABC)},$$
$$d_{\mathscr{T}}(P,B)=2\cdot\frac{A(PBA)}{A(ABC)},$$
$$d_{\mathscr{T}}(P,C)=2\cdot\frac{A(PAB)-A(ABC)}{A(ABC)}.$$

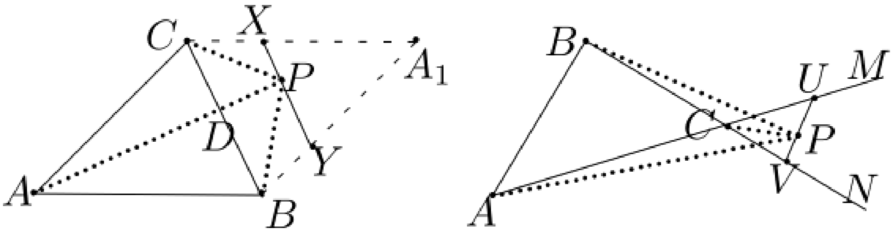

Figure 4

And thus

$$d_{\mathscr{T}}(P,A)+d_{\mathscr{T}}(P,B)+d_{\mathscr{T}}(P,C)=6\cdot\frac{A(PAB)}{A(ABC)}-2.$$

So in this case the locus of points P must be a line-segment UV parallel to AB.

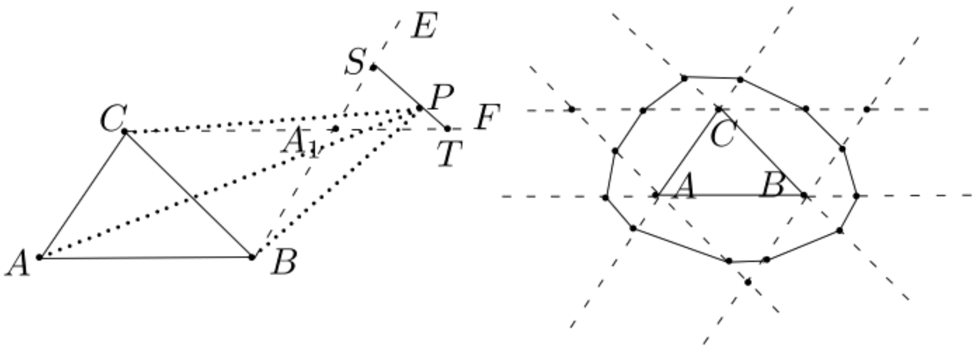

Figure 5

Step 3 Take lines $\overline{BE}$ and $\overline{CF}$ such that $\overline{BE}\|\overline{AC}$, $\overline{CF}\|\overline{AB}$, respectively. Denote by A_1 the intersection point of the lines $\overline{BE}$ and $\overline{CF}$ (see the left in Figure 5). Let P lie in the angular region EA_1F. Then

$$\begin{aligned}
d_{\mathscr{T}}(P,A)&=2\cdot\frac{A(ABC)+A(PBC)}{A(ABC)},\\
d_{\mathscr{T}}(P,B)&=2\cdot\frac{A(PBC)}{A(ABC)},\\
d_{\mathscr{T}}(P,C)&=2\cdot\frac{A(PCB)}{A(ABC)}.
\end{aligned}$$

And thus

$$d_{\mathscr{T}}(P,A)+d_{\mathscr{T}}(P,B)+d_{\mathscr{T}}(P,C)=2+6\cdot\frac{A(PBC)}{A(ABC)}.$$

So in this case the locus of points P must be a line-segment ST parallel to BC.

By symmetry and by Lemma 2.2, from the three steps above we conclude that the locus of the points P satisfying the condition, $d_{\mathscr{T}}(P,A)+d_{\mathscr{T}}(P,B)+d_{\mathscr{T}}(P,C)=\tau$, is a convex dodecagon (when $\tau\neq 8$), (see the right in Figure 5) or a nonagon (when $\tau=8$). The proof is completed.

We now generalize the result of Theorem 2.3 as follows.

Theorem 2.5 Let $\mathcal{Q} := ABCD$ be any convex quadrangle. If a point P lies in the interior of $\mathcal{Q}$, then

$$4 \le d_{\mathcal{Q}}(P,A) + d_{\mathcal{Q}}(P,B) + d_{\mathcal{Q}}(P,C) + d_{\mathcal{Q}}(P,D) \le 6.$$

Proof Denote by O the intersection point of the line-segments AC and BD. The point P must be in at least one of the four triangles OAB, OBC, OCD, and ODA. We suppose without loss of generality that P lies in the interior of OAB (see Figure 6). Since $d_{\mathcal{Q}}(A,C) = d_{\mathcal{Q}}(B,D) = 2$, by the triangle inequality, we have

$$d_{\mathcal{Q}}(P,A) + d_{\mathcal{Q}}(P,C) \ge d_{\mathcal{Q}}(A,C) = 2,$$
$$d_{\mathcal{Q}}(P,B) + d_{\mathcal{Q}}(P,D) \ge d_{\mathcal{Q}}(B,D) = 2.$$

So

$$d_{\mathcal{Q}}(P,A) + d_{\mathcal{Q}}(P,B) + d_{\mathcal{Q}}(P,C) + d_{\mathcal{Q}}(P,D) \ge 4.$$

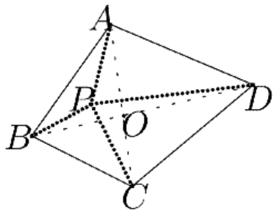

Figure 6

Let $\mathscr{T} := ABC$. Since P lies in the interior of T, by Theorem 2.3 we have $d_{\mathscr{T}}(P,A) + d_{\mathscr{T}}(P,B) + d_{\mathscr{T}}(P,C) = 4$. Since $\mathscr{T} \subset \mathcal{Q}$, we get $d_{\mathcal{Q}}(P,A) + d_{\mathcal{Q}}(P,B) + d_{\mathcal{Q}}(P,C) \le 4$. From $d_{\mathcal{Q}}(P,D) \le 2$, we conclude that

$$d_{\mathcal{Q}}(P,A) + d_{\mathcal{Q}}(P,B) + d_{\mathcal{Q}}(P,C) + d_{\mathcal{Q}}(P,D) \le 6.$$

We also have the following proposition.

Corollary 2.6 Let $\mathcal{S} := ABCD$ be a unit square. Then the locus of the points P satisfying the condition,

$$d_{\mathcal{S}}(P,A) + d_{\mathcal{S}}(P,B) + d_{\mathcal{S}}(P,C) + d_{\mathcal{S}}(P,D) = \tau, 4 \le \tau \le 6$$

is also a square.

Proof We take a Cartesian coordinate system such that the coordinates of the points A, B, C, and D are $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, respectively. Denote by E the intersection point of the line-segments AC and BD. Let $P = (x,y)$ and let P lie in the triangle EBC (see the left in Figure 7). Then

$$d_{\mathcal{S}}(P,A) = \frac{x}{\frac{1}{2}\cdot 1} = 2x, d_{\mathcal{S}}(P,B) = \frac{y}{\frac{1}{2}\cdot 1} = 2y,$$
$$d_{\mathcal{S}}(P,C) = \frac{1-y}{\frac{1}{2}\cdot 1} = 2-2y, d_{\mathcal{S}}(P,D) = \frac{x}{\frac{1}{2}\cdot 1} = 2x.$$

So

$$d_{\mathcal{S}}(P, A) + d_{\mathcal{S}}(P, B) + d_{\mathcal{S}}(P, C) + d_{\mathcal{S}}(P, D) = 4x + 2.$$

Thus we obtain that $4x + 2 = \tau$, that is, $x = \frac{\tau - 2}{4}$.

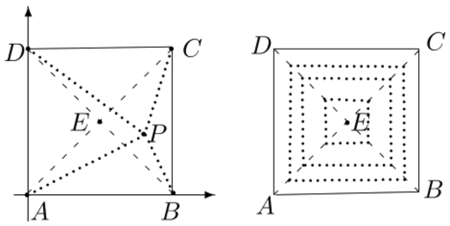

Figure 7

Similarly, when $P \in ECD$, we have $y = \frac{\tau-2}{4}$. When $P \in EAB$, we get

$$d_{\mathcal{S}}(P, A) = \frac{x}{\frac{1}{2} \cdot 1} = 2x, d_{\mathcal{S}}(P, B) = \frac{1-x}{\frac{1}{2} \cdot 1} = 2 - 2x,$$
$$d_{\mathcal{S}}(P, C) = \frac{1-y}{\frac{1}{2} \cdot 1} = 2 - 2y, d_{\mathcal{S}}(P, D) = \frac{1-y}{\frac{1}{2} \cdot 1} = 2 - 2y.$$

So

$$d_{\mathcal{S}}(P, A) + d_{\mathcal{S}}(P, B) + d_{\mathcal{S}}(P, C) + d_{\mathcal{S}}(P, D) = 6 - 4y.$$

Thus $6 - 4y = \tau$, that is, $y = \frac{6-\tau}{4}$. Similarly, when $P \in EAD$, we have $x = \frac{6-\tau}{4}$. It is clear that if P lies in the boundary of $\mathcal{S}$, then

$$d_{\mathcal{S}}(P, A) + d_{\mathcal{S}}(P, B) + d_{\mathcal{S}}(P, C) + d_{\mathcal{S}}(P, D) = 6,$$

and if $P = E$, then

$$d_{\mathcal{S}}(P, A) + d_{\mathcal{S}}(P, B) + d_{\mathcal{S}}(P, C) + d_{\mathcal{S}}(P, D) = 4.$$

Then from the discussions above we conclude that the locus of the points P satisfying the condition,

$$d_{\mathcal{S}}(P, A) + d_{\mathcal{S}}(P, B) + d_{\mathcal{S}}(P, C) + d_{\mathcal{S}}(P, D) = \tau, 4 < \tau \leq 6$$

is a square, whose center is the same as that of $\mathcal{S}$ (see the right in Figure 7). The proof is completed.

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到三定点相对距离的和等于定数的点的轨迹

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摘要: 本文研究了相对测度空间中的距离问题. 利用质点几何的理论方法获得如下结果: 对任意给定的实数, 满足条件$d_{\mathscr{T}}(P,A)+d_{\mathscr{T}}(P,B)+d_{\mathscr{T}}(P,C)=\tau$ 的点P的轨迹是凸十二边形或九边形 (其中$\mathscr{T}:=ABC$ 是由给定的不同三点A, B, C构成的三角形), 所得结果丰富了相对距离研究领域的内容.

关键词: 相对距离; 平面凸体; 十二边形

MR(2010)**主题分类号:** 52A10; 52A38 **中图分类号:** O157.3