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A GENERAL LAW OF PRECISE ASYMPTOTICS FOR MOVING AVERAGE PROCESS UNDER DEPENDENCE

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Abstract: In this paper, the problem of precise asymptotics of complete convergence for moving average processes under dependence is studied. By using the method of approximation of normal distribution and associated inequalities, a general law of precise asymptotics is obtained, which extends the existing results of precise asymptotics in the law of the logarithm and of the iterated logarithm.

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1 Introduction and Main Results

Suppose that $\{\xi_i, -\infty < i < \infty\}$ is a doubly infinite sequence of random variables, and $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers. Let

$$X_k = \sum_{i=-\infty}^{\infty} a_{i+k}\xi_i, \quad k \geq 1 \tag{1.1}$$

be the moving average process based on $\{\xi_i, -\infty < i < \infty\}$. So far, there were detailed studies about the asymptotic behavior of the moving average process $\{X_k, k \geq 1\}$.

Throughout the paper, let N be the standard normal random variable. We denote by C a positive constant which may vary from place to place, $\xrightarrow{d}$ means convergence in distribution, and $\lfloor x \rfloor = \sup\{m : m \leq x, m \in \mathbb{Z}^+\}$. Also we let $\log x = \ln(x \vee e)$ and $\log\log x = \ln(\ln(x \vee e^e))$.

We begin with a brief review of the definition of φ-mixing. Let $\mathbb{F}_k^l$ denote the σ-field

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generated by $X_k, X_{k+1}, \cdots, X_l$ and define

$$\begin{aligned}
&\varphi(\mathbb{F}_1^k, \mathbb{F}_{k+n}^\infty) := \sup\{|\mathbb{P}(B|A) - \mathbb{P}(B)|; A \in \mathbb{F}_1^k, B \in \mathbb{F}_{k+n}^\infty\},\\
&\varrho(\mathbb{F}_1^k, \mathbb{F}_{k+n}^\infty) := \sup\{|\mathbb{C}orr(U,V)|; U \in \mathbb{L}^2(\mathbb{F}_1^k), V \in \mathbb{L}^2(\mathbb{F}_{k+n}^\infty)\},\\
&\varphi(n) := \sup_{k\ge 1} \varphi(\mathbb{F}_1^k, \mathbb{F}_{k+n}^\infty), \qquad \varrho(n) := \sup_{k\ge 1} \varrho(\mathbb{F}_1^k, \mathbb{F}_{k+n}^\infty).
\end{aligned}$$

A sequence $\{X_n\}_{n\ge1}$ of random variables is said to be φ-mixing if $\varphi(n) \to 0$ and ϱ-mixing if $\varrho(n) \to 0$. It is well known that a φ-mixing sequence is ϱ-mixing, since $\varrho(n) \le 2\varphi^{1/2}(n)$.

In the sequel, we suppose $\{\xi_i, -\infty < i < \infty\}$ is a sequence of identically distributed and φ-mixing random variables with zero mean and finite variance with $0 < \sigma^2 = \mathbb{E}\xi_1^2 + 2\sum\limits_{k=2}^{\infty} \mathbb{E}\xi_1\xi_k < \infty$ and $\sum\limits_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$. For the moving average processes $\{X_k, k \ge 1\}$ defined in (1.1), where $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers with $\sum\limits_{i=-\infty}^{\infty} |a_i| < \infty$, we set $S_n = \sum\limits_{k=1}^{n} X_k$ and $\tau = \sigma \cdot \sum\limits_{i=-\infty}^{\infty} a_i$.

Li and Zhang [1] showed the precise rates in the law of the iterated logarithm of the moving average process defined in (1.1) for φ-mixing or negatively associated sequences under conditions above. For any $\delta \ge 0$, if $\mathbb{E}\xi_1^2(\log\log|\xi_1|)^{\delta-1} < \infty$, they proved that

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\delta+2} \sum_{n=1}^{\infty} \frac{(\log\log n)^\delta}{n \log n} \mathbb{P}\{|S_n| \ge \varepsilon\tau\sqrt{2n\log\log n}\} = \frac{1}{(\delta+1)\sqrt{\pi}} \Gamma(\delta + 3/2), \tag{1.2}$$

where $\Gamma(\cdot)$ is a Gamma function.

In this paper, we consider the general law of complete convergence rates of the moving average process $\{X_k, k \ge 1\}$ defined in (1.1) for φ-mixing sequences, and we have the following results.

Theorem 1.1 Suppose that $g(x)$ is a positive and differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞. For $b > 0$, assume that $\phi(x) = g'(x)g^{b-1}(x)$ is monotone nondecreasing or monotone nonincreasing on $[n_0, \infty)$, and if $\phi(x)$ is monotone nondecreasing, we assume that $\lim\limits_{x\to\infty} \phi(x+1)/\phi(x) = 1$. If $\mathbb{E}|\xi_1|^{2+\delta} < \infty$ for some $\delta \ge 0$, then we have

$$\lim_{\varepsilon\searrow 0} \varepsilon^{\frac{b}{s}} \sum_{n=n_0}^{\infty} \phi(n)\mathbb{P}\{|S_n| \ge (\varepsilon + a_n)\sqrt{n} g^s(n)\} = \frac{\tau^{\frac{b}{s}}\mathbb{E}|N|^{\frac{b}{s}}}{b}, \quad 0 < \tfrac{b}{s} < 2 + \delta, \tag{1.3}$$

$$\lim_{\varepsilon\searrow 0} \varepsilon^{\frac{b}{s}} \sum_{n=n_0}^{\infty} \frac{\phi(n)}{\sqrt{n}} \mathbb{E}\{|S_n| - (\varepsilon + a_n)\sqrt{n} g^s(n)\}_+ = \frac{s\tau^{\frac{b}{s}+1}\mathbb{E}|N|^{\frac{b}{s}+1}}{b(b+s)}, \quad 0 < \tfrac{b}{s} < 1 + \delta, \tag{1.4}$$

where $a_n = o(g^{-s}(n))$ as $n \to \infty$.

Theorem 1.2 Suppose that $g(x)$ is a positive and differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞. Assume that $\phi(x) = g'(x)g^{-1}(x)$ is monotone nondecreasing or monotone nonincreasing on $[n_0, \infty)$, and if $\phi(x)$ is monotone nondecreasing, we assume that $\lim\limits_{x\to\infty}\phi(x+1)/\phi(x)=1$. Then for $s>0$, we have

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\frac{g'(n)}{g(n)}\mathbb{P}\{|S_n|\geq(\varepsilon+a_n)\sqrt{n}g^s(n)\}=\frac{1}{s}, \tag{1.5}$$

$$\lim_{\varepsilon\searrow 0}\frac{1}{-\log\varepsilon}\sum_{n=n_0}^{\infty}\frac{g'(n)}{\sqrt{n}g(n)}\mathbb{E}\{|S_n|-(\varepsilon+a_n)\sqrt{n}g^s(n)\}_+=\frac{\tau}{s}\sqrt{\frac{2}{\pi}}, \tag{1.6}$$

where $a_n=o(g^{-s}(n))$ as $n\to\infty$.

Remark 1.1 Applying Lemma 2.3 of [1], Theorems 1.1 and 1.2 are still true when $\{\xi_i,-\infty<i<\infty\}$ is a sequence of identically distributed negatively associated random variables with $\mathbb{E}\xi_1=0$, $\mathbb{E}\xi_1^2<\infty$ and $0<\sigma^2=\mathbb{E}\xi_1^2+2\sum\limits_{k=2}^{\infty}\mathbb{E}\xi_1\xi_k<\infty$.

Remark 1.2 Specially, for $k\geq 1$, if we let $a_{2k}=1$ and $a_i=0,-\infty<i<\infty$ for $i\neq 2k$, that is to say, $X_k=\xi_k$ with $\mathbb{E}\xi_1=0$, $\mathbb{E}\xi_1^2<\infty$ and $0<\sigma^2=\mathbb{E}\xi_1^2+2\sum\limits_{k=2}^{\infty}\mathbb{E}\xi_1\xi_k<\infty$, then for $S_n=\sum\limits_{k=1}^{n}\xi_k$, Theorems 1.1 and 1.2 are still valid for $\tau=\sigma$.

Remark 1.3 The conditions about $\phi(x)$ and $g(x)$ in the above two theorems are mild for many common functions like $g(x)=x^\gamma$, $(\log x)^\gamma$ and $(\log\log x)^\gamma$ with $\gamma>0$, and the corresponding results were obtained by many researchers.

2 Some Lemmas

For proving of our main results, we introduce the following lemmas.

Lemma 2.1 (see [2]) Let $\sum\limits_{i=-\infty}^{\infty}a_i$ be an absolutely convergent series of real numbers with $a=\sum\limits_{i=-\infty}^{\infty}a_i$ and $k\geq 1$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=-\infty}^{\infty}\Big|\sum_{j=i+1}^{i+n}a_j\Big|^k=a^k.$$

Lemma 2.2 Suppose $\{\xi_i,-\infty<i<\infty\}$ is a sequence of identically distributed and φ-mixing random variables with $\mathbb{E}\xi_1=0$, $\mathbb{E}\xi_1^2<\infty$ and $\sum\limits_{m=1}^{\infty}\varphi^{1/2}(m)<\infty$, and suppose $0<\sigma^2=\mathbb{E}\xi_1^2+2\sum\limits_{k=2}^{\infty}\mathbb{E}\xi_1\xi_k<\infty$. For the moving average processes $\{X_k,k\geq 1\}$ defined in (1.1) with $\sum\limits_{i=-\infty}^{\infty}|a_i|<\infty$, we set $S_n=\sum\limits_{k=1}^{n}X_k$. Then we have $\frac{S_n}{\tau\sqrt{n}}\xrightarrow{d}N$, where $\tau=\sigma\cdot\sum\limits_{i=-\infty}^{\infty}a_i$.

Proof The proof is similar to that of Theorem 1 in [3], so we omit it.

Lemma 2.3 (see [4]) Let $\{\xi_i;i\geq 1\}$ be a φ-mixing sequence. $Y_n=\sum\limits_{i=1}^{n}\xi_i$, $n\geq 1$. Suppose that there exists a sequence $\{C_n\}$ of positive numbers such that $\max_{1\leq i\leq n}\mathbb{E}Y_i^2\leq C_n$. Then for any $q\geq 2$, there exists some constant $C=C(q,\varphi(\cdot))$ such that

$$\mathbb{E}(\max_{1\leq i\leq n}|Y_i|^q)\leq C\Big(C_n^{q/2}+\mathbb{E}\max_{1\leq i\leq n}|\xi_i|^q\Big).$$

3 Proof of Theorem 1.1

In this section, for $M > 1$ and $0 < \varepsilon < 1$, we define

$$b_M(\varepsilon) = \lfloor g^{-1}(M\varepsilon^{-1/s}) \rfloor. \tag{3.1}$$

Without loss of generality, we assume $\tau = 1$. Next we calculate the left hand side of (1.3) and (1.4) by approximation of partial sums about the tail probability of standard normal random variable N.

Proposition 3.1 For $b, s > 0$, we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \sum_{n=n_0}^{\infty} \phi(n)\mathbb{P}\{|N| \geq (\varepsilon + a_n)g^s(n)\} = \frac{\mathbb{E}|N|^{b/s}}{b}. \tag{3.2}$$

Proof Since $\lim\limits_{n\to\infty} a_n g^s(n) = 0$, then for any $\tilde{\delta} > 0$ there exists a positive integer $N_0 > n_0$ such that for any $n \geq N_0$, we have $-\tilde{\delta} \leq a_n g^s(n) \leq \tilde{\delta}$. If $\phi(x) = g'(x)g^{b-1}(x)$ is monotone nonincreasing, we have

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \int_{N_0+1}^{\infty} \phi(x)\mathbb{P}\{|N| \geq \varepsilon g^s(x) + \tilde{\delta}\}dx &\leq \lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \sum_{n=N_0+1}^{\infty} \phi(n)\mathbb{P}\{|N| \geq (\varepsilon + a_n)g^s(n)\} \\ &\leq \lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \int_{N_0}^{\infty} \phi(x)\mathbb{P}\{|N| \geq \varepsilon g^s(x) - \tilde{\delta}\}dx.\end{aligned}$$

Let $\tilde{\delta} \downarrow 0$, we obtain

$$\lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \sum_{n=n_0}^{\infty} \phi(n)\mathbb{P}\{|N| \geq (\varepsilon + a_n)g^s(n)\} = \lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \int_{n_0}^{\infty} \phi(x)\mathbb{P}\{|N| \geq \varepsilon g^s(x)\}dx. \tag{3.3}$$

If $\phi(x)$ is monotone nondecreasing, by $\lim\limits_{x\to\infty} \phi(x+1)/\phi(x) = 1$, for the $\tilde{\delta}$ mentioned above, there exists a positive integer N_1 such that for any $x \geq N_1$, we have $(1-\tilde{\delta})\phi(x+1) \leq \phi(x) \leq (1+\tilde{\delta})\phi(x-1)$. Let $N_2 = \max\{N_0, N_1\}$, thus it holds that

$$\begin{aligned}&\lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \sum_{n=n_0}^{\infty} \phi(n)\mathbb{P}\{|N| \geq (\varepsilon + a_n)g^s(n)\} \\ =&\lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \sum_{n=N_2+1}^{\infty} \phi(n)\mathbb{P}\{|N| \geq (\varepsilon + a_n)g^s(n)\} \\ \leq&(1+\tilde{\delta}) \lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \sum_{n=N_2+1}^{\infty} \phi(n-1)\mathbb{P}\{|N| \geq \varepsilon g^s(n) - \tilde{\delta}\} \\ \leq&(1+\tilde{\delta}) \lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \int_{N_2}^{\infty} \phi(x)\mathbb{P}\{|N| \geq \varepsilon g^s(x) - \tilde{\delta}\}dx,\end{aligned}$$

similarly, it holds that

$$\lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \sum_{n=n_0}^{\infty} \phi(n)\mathbb{P}\{|N| \geq (\varepsilon + a_n)g^s(n)\} \geq (1-\tilde{\delta}) \lim_{\varepsilon \searrow 0} \varepsilon^{b/s} \int_{N_2}^{\infty} \phi(x)\mathbb{P}\{|N| \geq \varepsilon g^s(x) + \tilde{\delta}\}dx.$$

Let $\tilde{\delta} \downarrow 0$, we obtain (3.3).

Let $y = \varepsilon g^s(x)$, we have

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0} \varepsilon^{b/s} \int_{n_0}^{\infty} g'(x) g^{b-1}(x) \mathbb{P}\{|N| \geq \varepsilon g^s(x)\} dx \\
=&\frac{1}{b} \lim_{\varepsilon\searrow 0} \int_{\varepsilon g^s(n_0)}^{\infty} \frac{b}{s} y^{\frac{b}{s}-1} \mathbb{P}\{|N| \geq y\} dy \\
=&\frac{1}{b} \int_0^{\infty} \frac{b}{s} y^{\frac{b}{s}-1} \mathbb{P}\{|N| \geq y\} dy = \frac{\mathbb{E}|N|^{b/s}}{b}.
\end{aligned}$$

Thus proof of this proposition is completed.

Proposition 3.2 For $b, s > 0$, we have

$$\lim_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n=n_0}^{b_M(\varepsilon)} \phi(n) |\mathbb{P}\{|S_n| \geq (\varepsilon + a_n)\sqrt{n} g^s(n)\} - \mathbb{P}\{|N| \geq (\varepsilon + a_n) g^s(n)\}| = 0. \tag{3.4}$$

Proof Let

$$\Delta_n = \sup_{x\in\mathbb{R}} |\mathbb{P}(|S_n| \geq \sqrt{n} x) - \mathbb{P}(|N| \geq x)|, \tag{3.5}$$

then $\Delta_n \to 0$ as $n \to \infty$ from Lemma 2.2. Using the Toeplitz lemma [5], we have

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n=n_0}^{b_M(\varepsilon)} \phi(n) |\mathbb{P}\{|S_n| \geq (\varepsilon + a_n)\sqrt{n} g^s(n)\} - \mathbb{P}\{|N| \geq (\varepsilon + a_n) g^s(n)\}| \\
\leq&\lim_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n=n_0}^{b_M(\varepsilon)} g'(n) g^{b-1}(n) \Delta_n \leq \lim_{\varepsilon\searrow 0} \frac{CM^b}{g(b_M(\varepsilon))^b} \sum_{n=n_0}^{b_M(\varepsilon)} g'(n) g^{b-1}(n) \Delta_n = 0.
\end{aligned}$$

Proposition 3.3 For $b, s > 0$, we have

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n > b_M(\varepsilon)} \phi(n) \mathbb{P}\{|N| \geq (\varepsilon + a_n) g^s(n)\} = 0. \tag{3.6}$$

Proof Since $a_n \to 0$ as $n > b_M(\varepsilon) \to \infty$, it is enough to show

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n > b_M(\varepsilon)} \phi(n) \mathbb{P}\Big\{|N| \geq \frac{\varepsilon}{2} g^s(n)\Big\} = 0.$$

Let $y = \varepsilon g^s(x)/2$, we have

$$\begin{aligned}
&\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n > b_M(\varepsilon)} g'(n) g^{b-1}(n) \mathbb{P}\Big\{|N| \geq \frac{\varepsilon}{2} g^s(n)\Big\} \\
\leq&C \lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \int_{b_M(\varepsilon)}^{\infty} g'(x) g^{b-1}(x) \mathbb{P}\Big\{|N| \geq \frac{\varepsilon}{2} g^s(x)\Big\} dx \\
\leq&C \lim_{M\to\infty} \int_{M^s/2}^{\infty} \frac{b}{s} y^{\frac{b}{s}-1} \mathbb{P}\{|N| \geq y\} dy = 0.
\end{aligned}$$

Therefore this proposition is proved.

Proposition 3.4 If $\mathbb{E}|\xi_1|^{2+\delta} < \infty$ for some $\delta \geq 0$, then for $0 < b < (2+\delta)s$, we have

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n>b_M(\varepsilon)} \phi(n)\mathbb{P}\{|S_n| \geq (\varepsilon + a_n)\sqrt{n}g^s(n)\} = 0. \tag{3.7}$$

Proof It suffices to show that

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n>b_M(\varepsilon)} g'(n)g^{b-1}(n)\mathbb{P}\Big\{|S_n| \geq \frac{\varepsilon}{2}\sqrt{n}g^s(n)\Big\} = 0. \tag{3.8}$$

Note that $S_n = \sum\limits_{i=-\infty}^{\infty}\sum\limits_{k=1}^{n} a_{k+i}\xi_i = \sum\limits_{i=-\infty}^{\infty} a_{ni}\xi_i$, where $a_{ni} = \sum\limits_{k=1}^{n} a_{k+i}$. From Lemma 2.1, we can suppose that

$$\sum_{i=-\infty}^{\infty} |a_{ni}|^p \leq n, \quad p \geq 1, \quad \tilde{a} = \sum_{i=-\infty}^{\infty} |a_i| \leq 1. \tag{3.9}$$

Next, for $x \geq 0$, we set

$$S'_n(x) = \sum_{i=-\infty}^{\infty} a_{ni}\xi_i I\{|a_{ni}\xi_i| \leq (\varepsilon + x)\sqrt{n}g^s(n)\}. \tag{3.10}$$

Since $\mathbb{E}\xi_i = 0$, then by (3.9) and Markov's inequality, we have

$$\begin{aligned}
|\mathbb{E}S'_n(x)| =& \Big|\sum_{i=-\infty}^{\infty} \mathbb{E}a_{ni}\xi_i I\{|a_{ni}\xi_i| > (\varepsilon + x)\sqrt{n}g^s(n)\}\Big| \\
\leq& \sum_{i=-\infty}^{\infty} |a_{ni}|\mathbb{E}|\xi_i| I\{|a_{ni}\xi_i| > (\varepsilon + x)\sqrt{n}g^s(n)\} \\
\leq& \sum_{i=-\infty}^{\infty} |a_{ni}|(\mathbb{E}\xi_i^2)^{1/2}\sqrt{\mathbb{P}\{|a_{ni}\xi_i| > (\varepsilon + x)\sqrt{n}g^s(n)\}} \\
\leq& Cn\sqrt{\mathbb{P}\{|\xi_1| > (\varepsilon + x)\sqrt{n}g^s(n)\}} \leq \frac{C\sqrt{n}}{(\varepsilon + x)g^s(n)}.
\end{aligned} \tag{3.11}$$

Since

$$|S_n| \leq |\mathbb{E}S'_n(x)| + |S'_n(x) - \mathbb{E}S'_n(x)| + \Big|\sum_{i=-\infty}^{\infty} a_{ni}\xi_i I\{|a_{ni}\xi_i| > (\varepsilon + x)\sqrt{n}g^s(n)\}\Big|, \tag{3.12}$$

and for M large enough, it holds that

$$\frac{|\mathbb{E}S'_n(0)|}{\varepsilon\sqrt{n}g^s(n)} \leq \frac{C}{\varepsilon^2 g^{2s}(n)} \leq CM^{-2s} < \varepsilon, \quad n > b_M(\varepsilon),$$

then we obtain

$$\mathbb{P}\Big\{|S_n| \geq \frac{\varepsilon}{2}\sqrt{n}g^s(n)\Big\} \leq \mathbb{P}\{\sup_i |a_{ni}\xi_i| > \varepsilon\sqrt{n}g^s(n)\} + \mathbb{P}\Big\{|S'_n(0) - \mathbb{E}S'_n(0)| \geq \frac{\varepsilon}{4}\sqrt{n}g^s(n)\Big\}. \tag{3.13}$$

Set

$$I_{nj} = \{j \in \mathbb{Z}, 1/(j+1) < |a_{ni}| \le 1/j, j = 1, 2, \cdots\},$$

then $\cup_{j\ge1} I_{nj} = \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. Note that (referred by [6])

$$\sum_{j=1}^{k} \#I_{nj} \le n(k+1). \tag{3.14}$$

On the one hand, we have

$$\begin{aligned}
&\mathbb{P}\{\sup_i |a_{ni}\xi_i| > (\varepsilon + x)\sqrt{n} g^s(n)\} \le \sum_{i=-\infty}^{\infty} \mathbb{P}\{|a_{ni}\xi_i| \ge (\varepsilon + x)\sqrt{n} g^s(n)\}\\
\le& \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} \mathbb{P}\{|\xi_1| \ge j(\varepsilon + x)\sqrt{n} g^s(n)\} \le \sum_{j=1}^{\infty} (\#I_{nj})\mathbb{P}\{|\xi_1| \ge j(\varepsilon + x)\sqrt{n} g^s(n)\}\\
\le& \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} (\#I_{nj})\mathbb{P}\Big\{k \le \frac{|\xi_1|}{(\varepsilon + x)\sqrt{n} g^s(n)} < k+1\Big\}\\
=& \sum_{k=1}^{\infty} \sum_{j=1}^{k} (\#I_{nj})\mathbb{P}\Big\{k \le \frac{|\xi_1|}{(\varepsilon + x)\sqrt{n} g^s(n)} < k+1\Big\}\\
\le& \sum_{k=1}^{\infty} n(k+1)\mathbb{P}\Big\{k \le \frac{|\xi_1|}{(\varepsilon + x)\sqrt{n} g^s(n)} < k+1\Big\}\\
\le& \frac{2\sqrt{n}\mathbb{E}|\xi_1| I\{|\xi_1| \ge (\varepsilon + x)\sqrt{n} g^s(n)\}}{(\varepsilon + x) g^s(n)}\\
\le& \frac{2\mathbb{E}|\xi_1|^{2+\delta} I\{|\xi_1| \ge (\varepsilon + x)\sqrt{n} g^s(n)\}}{n^{\delta/2}(\varepsilon + x)^{2+\delta} g^{(2+\delta)s}(n)}\\
\le& \frac{C}{(\varepsilon + x)^{2+\delta} g^{(2+\delta)s}(n)}.
\end{aligned} \tag{3.15}$$

On the other hand, since $\sum\limits_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$, then we have

$$\begin{aligned}
\mathbb{E}(S'_n(x) - \mathbb{E}S'_n(x))^2 \le& C \sum_{i=-\infty}^{\infty} \mathbb{E}(a_{ni}\xi_i)^2 I\{|a_{ni}\xi_i| \le (\varepsilon + x)\sqrt{n} g^s(n)\}\\
\le& C \sum_{i=-\infty}^{\infty} (a_{ni})^2 \mathbb{E}\xi_1^2 \le Cn.
\end{aligned}$$

Thus using Markov's inequality and Lemma 2.3, we have

$$\begin{aligned}
&\mathbb{P}\Big\{|S'_n(x) - \mathbb{E}S'_n(x)| \ge \frac{(x+\varepsilon)}{4}\sqrt{n} g^s(n)\Big\}\\
\le& C\mathbb{E}\{|S'_n(x) - \mathbb{E}S'_n(x)|^q\} n^{-q/2}(x+\varepsilon)^{-q} g^{-qs}(n)\\
\le& \frac{Cn^{-q/2}}{(x+\varepsilon)^q g^{qs}(n)} \sum_{i=-\infty}^{\infty} \mathbb{E}|a_{ni}\xi_1|^q I\{|a_{ni}\xi_1| \le (x+\varepsilon)\sqrt{n} g^s(n)\}\\
&+ C(x+\varepsilon)^{-q} g^{-qs}(n) =: H_1(x) + H_2(x),
\end{aligned} \tag{3.16}$$

where we take $q = 2+\delta$ (actually, the above inequality holds for any $q \geq 2$).

However, from (3.14), it holds that

$$\begin{aligned}
&\sum_{i=-\infty}^{\infty} \mathbb{E}|a_{ni}\xi_1|^{2+\delta} I\{|a_{ni}\xi_1| \leq (x+\varepsilon)\sqrt{n}g^s(n)\} \\
\leq &\sum_{j=1}^{\infty}\sum_{i\in I_{nj}} j^{-(2+\delta)}\mathbb{E}|\xi_1|^{2+\delta} I\{|\xi_1| < (j+1)(x+\varepsilon)\sqrt{n}g^s(n)\} \\
= &\sum_{j=1}^{\infty}(\#I_{nj})j^{-(2+\delta)}\mathbb{E}|\xi_1|^{2+\delta} I\{|\xi_1| < (j+1)(x+\varepsilon)\sqrt{n}g^s(n)\} \\
= &\sum_{j=1}^{\infty}\sum_{k=0}^{j}(\#I_{nj})j^{-(2+\delta)}\mathbb{E}|\xi_1|^{2+\delta} I\Big\{k \leq \frac{|\xi_1|}{(x+\varepsilon)\sqrt{n}g^s(n)} < k+1\Big\} \\
= &\sum_{j=1}^{\infty}(\#I_{nj})j^{-(2+\delta)}\mathbb{E}|\xi_1|^{2+\delta} I\Big\{0 \leq \frac{|\xi_1|}{(x+\varepsilon)\sqrt{n}g^s(n)} < 1\Big\} \\
&+\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}(\#I_{nj})j^{-(2+\delta)}\mathbb{E}|\xi_1|^{2+\delta} I\Big\{k \leq \frac{|\xi_1|}{(x+\varepsilon)\sqrt{n}g^s(n)} < k+1\Big\} \\
=: &L_1(x)+L_2(x).
\end{aligned}$$

Since

$$\sum_{j=k}^{\infty}\frac{\#I_{nj}}{(j+1)^{2+\delta}}(k+1)^{1+\delta} \leq \sum_{j=1}^{\infty}\frac{\#I_{nj}}{j+1} \leq \sum_{i=-\infty}^{\infty}|a_{ni}| = \sum_{j=1}^{\infty}\sum_{i\in I_{nj}}|a_{ni}| \leq n,$$

then we have

$$\sum_{j=k}^{\infty}(\#I_{nj})j^{-(2+\delta)} \leq Cnk^{-(1+\delta)}. \tag{3.17}$$

Therefore, using (3.17), we have

$$\begin{aligned}
L_1(x) \leq &Cn\mathbb{E}|\xi_1|^{2+\delta} I\{|\xi_1| < (x+\varepsilon)\sqrt{n}g^s(n)\} \leq Cn, \\
L_2(x) \leq &Cn\sum_{k=1}^{\infty}k^{-(1+\delta)}\mathbb{E}|\xi_1|^{2+\delta} I\Big\{k \leq \frac{|\xi_1|}{(x+\varepsilon)\sqrt{n}g^s(n)} < k+1\Big\} \\
\leq &Cn\sum_{k=1}^{\infty}\mathbb{E}|\xi_1|^{2+\delta} I\Big\{k \leq \frac{|\xi_1|}{(x+\varepsilon)\sqrt{n}g^s(n)} < k+1\Big\} \\
\leq &Cn\mathbb{E}|\xi_1|^{2+\delta} I\{|\xi_1| \geq (x+\varepsilon)\sqrt{n}g^s(n)\} \leq Cn.
\end{aligned}$$

Thus we have

$$H_1(x) \leq \frac{Cn^{-\delta/2}}{(\varepsilon+x)^{2+\delta}g^{(2+\delta)s}(n)} \leq \frac{C}{(\varepsilon+x)^{2+\delta}g^{(2+\delta)s}(n)}, \tag{3.18}$$

then using (3.13), (3.15), (3.16) and (3.18) with $x=0$, it holds that

$$\begin{aligned}&\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}\varepsilon^{b/s}\sum_{n>b_M(\varepsilon)}g'(n)g^{b-1}(n)\mathbb{P}\Big\{|S_n|\ge\frac{\varepsilon}{2}\sqrt{n}g^s(n)\Big\}\\ \le&\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}C\varepsilon^{\frac{b}{s}-(2+\delta)}\sum_{n>b_M(\varepsilon)}g'(n)[g(n)]^{b-(2+\delta)s-1}\\ \le&\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}C\varepsilon^{\frac{b}{s}-(2+\delta)}[g(b_M(\varepsilon))]^{b-(2+\delta)s}\\ \le&\lim_{M\to\infty}\limsup_{\varepsilon\searrow 0}CM^{b-(2+\delta)s}=0.\end{aligned}$$

Therefore we complete the proof of (3.8).

From Propositions 3.1–3.4, applying the triangle inequality, we complete the proof of (1.3). Next we show (1.4). For simplicity, we let $a_n=0$ and omit the discussion of $\phi(x)$, but the process is similar to that of Proposition 3.1.

Proposition 3.5 For $b,s>0$, one has that

$$\lim_{\varepsilon\searrow 0}\varepsilon^{b/s}\sum_{n=n_0}^{\infty}\phi(n)\mathbb{E}\{|N|-\varepsilon g^s(n)\}_+=\frac{s\mathbb{E}|N|^{b/s+1}}{b(b+s)}. \tag{3.19}$$

Proof We calculate that

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\varepsilon^{b/s}\sum_{n=n_0}^{\infty}g'(n)g^{b-1}(n)\mathbb{E}\{|N|-\varepsilon g^s(n)\}_+\\ =&\lim_{\varepsilon\searrow 0}\varepsilon^{b/s}\int_{n_0}^{\infty}g'(y)g^{b-1}(y)\mathbb{E}\{|N|-\varepsilon g^s(y)\}_+dy\\ =&\lim_{\varepsilon\searrow 0}\varepsilon^{b/s}\int_{n_0}^{\infty}g'(y)g^{b-1}(y)dy\int_{\varepsilon g^s(y)}^{\infty}\mathbb{P}\{|N|\ge x\}dx\quad\text{(double integrals)}\\ =&\frac{1}{s}\lim_{\varepsilon\searrow 0}\int_{\varepsilon g^s(n_0)}^{\infty}t^{b/s-1}dt\int_t^{\infty}\mathbb{P}\{|N|\ge x\}dx\qquad(t=\varepsilon g^s(y))\\ =&\frac{1}{s}\lim_{\varepsilon\searrow 0}\int_{\varepsilon g^s(n_0)}^{\infty}\mathbb{P}\{|N|\ge x\}dx\int_{\varepsilon g^s(n_0)}^{x}t^{b/s-1}dt\\ =&\frac{1}{b}\lim_{\varepsilon\searrow 0}\int_{\varepsilon g^s(n_0)}^{\infty}x^{b/s}\mathbb{P}\{|N|\ge x\}dx=\frac{s\mathbb{E}|N|^{b/s+1}}{b(b+s)}.\end{aligned}$$

Thus proof of this proposition is completed.

Proposition 3.6 For $b,s>0$, one has that

$$\lim_{\varepsilon\searrow 0}\varepsilon^{b/s}\sum_{n=n_0}^{b_M(\varepsilon)}\phi(n)\bigg|\mathbb{E}\{|N|-\varepsilon g^s(n)\}_+-\mathbb{E}\bigg\{\frac{|S_n|}{\sqrt{n}}-\varepsilon g^s(n)\bigg\}_+\bigg|=0. \tag{3.20}$$

Proof It holds that

$$
\begin{aligned}
&\sum_{n=n_0}^{b_M(\varepsilon)}\phi(n)\bigg|\mathbb{E}\{|N|-\varepsilon g^s(n)\}_+-\mathbb{E}\bigg\{\frac{|S_n|}{\sqrt{n}}-\varepsilon g^s(n)\bigg\}_+\bigg|\\
=&\sum_{n=n_0}^{b_M(\varepsilon)}\phi(n)\bigg|\int_0^\infty \mathbb{P}\big(|N|\ge\varepsilon g^s(n)+x\big)-\mathbb{P}\bigg(\frac{|S_n|}{\sqrt{n}}\ge\varepsilon g^s(n)+x\bigg)dx\bigg|\\
\le&\sum_{n=n_0}^{b_M(\varepsilon)}\phi(n)g^s(n)\int_0^\infty\bigg|\mathbb{P}\big(|N|\ge(x+\varepsilon)g^s(n)\big)-\mathbb{P}\bigg(\frac{|S_n|}{\sqrt{n}}\ge(x+\varepsilon)g^s(n)\bigg)\bigg|dx\\
\le&\sum_{n=n_0}^{b_M(\varepsilon)}\phi(n)g^s(n)\bigg[\int_0^{l(n)}\bigg|\mathbb{P}\big(|N|\ge(x+\varepsilon)g^s(n)\big)-\mathbb{P}\bigg(\frac{|S_n|}{\sqrt{n}}\ge(x+\varepsilon)g^s(n)\bigg)\bigg|dx\\
&+\int_{l(n)}^\infty\mathbb{P}\big(|N|\ge(x+\varepsilon)g^s(n)\big)dx+\int_{l(n)}^\infty\mathbb{P}\big(|S_n|\ge(x+\varepsilon)\sqrt{n}g^s(n)\big)dx\bigg]\\
=:&\sum_{n=n_0}^{b_M(\varepsilon)}\phi(n)g^s(n)\big[J_1+J_2+J_3\big],
\end{aligned}
$$

where $l(n)=g^{-s}(n)\Delta_n^{-1/2}$ and Δ_n is defined in (3.5). It is easy to see that

$$
\begin{aligned}
&J_1\le\frac{\Delta_n^{1/2}}{g^s(n)},\\
&J_2\le\int_{l(n)}^\infty\frac{\mathbb{E}N^2}{(x+\varepsilon)^2g^{2s}(n)}dx\le\frac{C}{l(n)g^{2s}(n)}\le\frac{C\Delta_n^{1/2}}{g^s(n)}.
\end{aligned}\tag{3.21}
$$

Next for J_3, from (3.10)–(3.12) and the fact that for $x\ge l(n)$ and n large enough, we have

$$\frac{|\mathbb{E}S_n'(x)|}{(x+\varepsilon)\sqrt{n}g^s(n)}\le\frac{C}{(x+\varepsilon)^2g^{2s}(n)}\le\frac{C}{l^2(n)g^{2s}(n)}\le C\Delta_n<\varepsilon,$$

then we obtain

$$
\begin{aligned}
\mathbb{P}\Big\{|S_n|\ge(x+\varepsilon)\sqrt{n}g^s(n)\Big\}\le&\mathbb{P}\{\sup_i|a_{ni}\xi_i|>(x+\varepsilon)\sqrt{n}g^s(n)\}\\
&+\mathbb{P}\Big\{|S_n'(x)-\mathbb{E}S_n'(x)|\ge\frac{(x+\varepsilon)}{2}\sqrt{n}g^s(n)\Big\}.
\end{aligned}\tag{3.22}
$$

Thus using (3.15), (3.16) and (3.18) with $\delta=0$, we have

$$J_3\le\int_{l(n)}^\infty\frac{C}{(x+\varepsilon)^2g^{2s}(n)}dx\le\frac{C}{l(n)g^{2s}(n)}\le\frac{C\Delta_n^{1/2}}{g^s(n)}.$$

Therefore using (3.21) and the Toeplitz lemma, we have

$$\lim_{\varepsilon\searrow0}\varepsilon^{b/s}\sum_{n=n_0}^{b_M(\varepsilon)}\phi(n)g^s(n)\big[J_1+J_2+J_3\big]\le\lim_{\varepsilon\searrow0}C\varepsilon^{b/s}\sum_{n=n_0}^{b_M(\varepsilon)}g'(n)g^{b-1}(n)\Delta_n^{1/2}=0.$$

Proposition 3.7 For $b, s > 0$, one has that

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n>b_M(\varepsilon)} \phi(n)\mathbb{E}\{|N| - \varepsilon g^s(n)\}_+ = 0. \tag{3.23}$$

Proof It is easy from the proof of Proposition 3.5.

Proposition 3.8 For $0 < b < (1+\delta)s$, one has that

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n>b_M(\varepsilon)} \frac{\phi(n)}{\sqrt{n}}\mathbb{E}\{|S_n| - \varepsilon\sqrt{n} g^s(n)\}_+ = 0. \tag{3.24}$$

Proof For M large enough and any $x \ge 0$, it holds that

$$\frac{|\mathbb{E}S_n'(x)|}{(\varepsilon+x)\sqrt{n}g^s(n)} \le \frac{C}{(\varepsilon+x)^2 g^{2s}(n)} \le \frac{C}{\varepsilon^2 g^{2s}(n)} \le CM^{-2s} < \varepsilon.$$

Then using (3.22), (3.15), (3.16) and (3.18), we have

$$\mathbb{P}\Big\{|S_n| \ge (x+\varepsilon)\sqrt{n}g^s(n)\Big\} \le \frac{C}{(\varepsilon+x)^{2+\delta} g^{(2+\delta)s}(n)}.$$

Hence it holds that

$$\begin{aligned}
&\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n>b_M(\varepsilon)} \frac{g'(n)g^{b-1}(n)}{\sqrt{n}}\mathbb{E}\{|S_n| - \varepsilon\sqrt{n} g^s(n)\}_+ \\
&= \lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n>b_M(\varepsilon)} g'(n)g^{b+s-1}(n) \int_0^\infty \mathbb{P}\{|S_n| \ge (\varepsilon+x)\sqrt{n}g^s(n)\}dx \\
&\le \lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon^{b/s} \sum_{n>b_M(\varepsilon)} g'(n)g^{b+s-1}(n) \int_0^\infty \frac{C}{(x+\varepsilon)^{2+\delta} g^{(2+\delta)s}(n)}dx \\
&\le \lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} C\varepsilon^{\frac{b}{s}-1-\delta} \sum_{n>b_M(\varepsilon)} g'(n)[g(n)]^{b-(1+\delta)s-1} \\
&\le \lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} C\varepsilon^{\frac{b}{s}-1-\delta} [g(b_M(\varepsilon))]^{b-(1+\delta)s} \\
&\le \lim_{M\to\infty} CM^{b-(1+\delta)s} = 0,
\end{aligned}$$

where $b < (1+\delta)s$.

Finally, the proof of (1.4) is completed by combining Propositions 3.5–3.8 together and using the triangle inequality. We omit the proof of Theorem 1.2 since the idea is similar, and we only need to replace $\varepsilon^{b/s}$ by $1/(-\log\varepsilon)$ with $b = 0$.

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相依条件下滑动平均过程精确渐近性的一般规律

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摘要: 本文研究了相依条件下滑动平均过程完全收敛的精确渐近性问题. 利用正态分布逼近的方法及相关不等式, 获得了精确渐近性的一般规律, 推广了对数率和重对数率精确渐近性的已有结果.

关键词: 精确渐近性; 一般规律; 滑动平均过程; φ -混合

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