

SUFFICIENT CONDITIONS FOR FINITE-TIME STABILITY OF CONTINUOUS NON-AUTONOMOUS SYSTEMS

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Abstract: In this paper, we study finite-time stability of continuous non-autonomous systems. Through the finite-time stability analysis of continuous time-varying scalar systems and using the comparison principle, some sufficient conditions are presented for finite-time stability of general n -dimensional continuous non-autonomous systems, which improves the existing finite-time stability results for continuous non-autonomous systems.

Keywords: finite-time stability; continuous non-autonomous systems; sufficient conditions

2010 MR Subject Classification: 37C75

Document code: A

Article ID: 0255-7797(2016)04-0737-10

1 Introduction

In recent years, the concept of finite-time stability was used in different dynamical systems to deal with various issues in the applied mathematics field, such as finite-time stabilization control [1], finite-time synchronization [2], finite-time consensus [3], and so on. It shows that developing the theory of finite-time stability is a significant work. So far, some researchers made a contribution to developing and improving the theory of finite-time stability. In [4], a rigorous foundation for the theory of finite-time stability of continuous autonomous systems was provided. In [5] and [6], some Lyapunov results for finite-time stability of continuous non-autonomous systems were proposed.

However, all the finite-time stability results for continuous non-autonomous systems are not perfect. For one thing, the results inevitably lead to continuous settling-time functions and thus, they are not suitable for the finite-time stability analysis of the continuous non-autonomous systems whose settling-time functions may be discontinuous. For another, the conditions of the results can be further relaxed through the study of finite-time stability of general continuous time-varying scalar systems and the use of the comparison principle. Inspired by the two things, the paper will define finite-time stability for continuous non-autonomous systems and attempt to obtain more general results for finite-time stability of continuous non-autonomous systems.

* **Received date:** 2015-07-25

Accepted date: 2015-11-27

Foundation item: Supported by National Natural Science Foundation of China (61273215).

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This paper is organized as follows. In Section 2, some necessary definitions and lemmas are given for later use. In Section 3, finite-time stability of continuous time-varying scalar systems is considered, and in particular, a class of continuous time-varying scalar systems with separated variables is studied in detail. In Section 4, the main results for finite-time stability of continuous non-autonomous systems is presented and some conditions are given to make sure that settling-time functions are continuous or bounded. In Section 5, some applications for the results of this paper are provided.

2 Preliminaries

Let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$. The notions of openness, convergence, continuity, boundary and compactness that we use refer to the topology generated on $\mathbb{R}^n$ by the norm $\|\cdot\|$. We use I to denote the whole nonnegative real numbers. Also, let $\mathring{A}$, $\bar{A}$ and $\text{bd}A$ denote the interior, the closure and the boundary of the set A , respectively.

Consider the system of differential equations

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad (2.1)$$

where $f : I \times D \rightarrow \mathbb{R}^n$ is continuous on $I \times D$ with an open neighborhood $D \subseteq \mathbb{R}^n$ of the origin and $f(t, 0) \equiv 0$. Then according to existence theorem of Peano [7], there exists a sufficiently small positive number τ and a solution $x : (t_0 - \tau, t_0 + \tau) \rightarrow D$ of (2.1) such that $x(t_0) = x_0$ for any initial state $(t_0, x_0) \in I \times D$. Moreover, extension theorem of solution [7] indicates that every solution of (2.1) has an extension that is right maximally defined.

We assume that system (2.1) possesses unique solutions in forward time for all initial states $(t_0, x_0) \in I \times D$ except possibly $(t_0, 0) \in I \times \{0\}$ in the following sense: for every $(t_0, x_0) \in I \times (D \setminus \{0\})$, there exists $\tau_{t_0, x_0} > t_0$ such that for any two right maximally defined solutions $x_1 : [t_0, \tau_1) \rightarrow D$ and $x_2 : [t_0, \tau_2) \rightarrow D$ of (2.1) passing through (t_0, x_0) , we have that $\tau_{t_0, x_0} \leq \min\{\tau_1, \tau_2\}$ and $x_1(t) = x_2(t)$ for all $t \in [t_0, \tau_{t_0, x_0})$. Without any loss of generality, for each (t_0, x_0) , we may assume that τ_{t_0, x_0} is chosen to be the largest such number in I , or $+\infty$. In this case, the unique solution of (2.1) passing through (t_0, x_0) on $[t_0, \tau_{t_0, x_0})$ is denoted by $\phi(\cdot, t_0, x_0)$. Under the above assumption, the definition of finite-time stability of (2.1) is given as follows.

Definition 2.1 The origin is said to be a finite-time-stable equilibrium of (2.1) if there exists a set $\Omega \subseteq I \times D$ satisfying both $I \times \{0\} \subseteq \Omega$ and for any $t \in I$, $\Omega_t = \{x \in \mathbb{R}^n | (t, x) \in \Omega\} \subseteq D$ is an open neighborhood of the origin, and a function $T : \Omega \setminus (I \times \{0\}) \rightarrow (0, +\infty)$ such that the following statements hold.

(i) Finite-time convergence: for every $(t_0, x_0) \in \Omega \setminus (I \times \{0\})$, $\phi(\cdot, t_0, x_0)$ is defined on $[t_0, t_0 + T(t_0, x_0))$, $(t, \phi(t, t_0, x_0)) \in \Omega \setminus (I \times \{0\})$ on $[t_0, t_0 + T(t_0, x_0))$ and

$$\lim_{t \rightarrow (t_0 + T(t_0, x_0))^-} \phi(t, t_0, x_0) = 0.$$

(ii) Lyapunov stability: for any $t_0 \in I$ and any open neighborhood U_ε of the origin,

there exists an open neighborhood $U_\delta \subseteq \Omega_{t_0}$ related with t_0 and U_ε of the origin, such that for every $x_0 \in U_\delta \setminus \{0\}$, $\phi(t, t_0, x_0) \in U_\varepsilon$ on $[t_0, t_0 + T(t_0, x_0))$.

The origin is said to be a globally finite-time-stable equilibrium of (2.1) if it is a finite-time-stable equilibrium of (2.1) with $\Omega = I \times D = I \times \mathbb{R}^n$.

It is easy to prove that if the origin is a finite-time-stable equilibrium of (2.1) with Ω and T as in Definition 2.1 and let $T(t_0, 0) \equiv 0$, then for every $(t_0, x_0) \in \Omega$, $\phi(\cdot, t_0, x_0)$ is defined on $[t_0, +\infty)$, $(t, \phi(t, t_0, x_0)) \in \Omega \setminus (I \times \{0\})$ on $[t_0, t_0 + T(t_0, x_0))$ and $\phi(t, t_0, x_0) = 0$ on $[t_0 + T(t_0, x_0), +\infty)$, what's more,

$$T(t_0, x_0) = \inf\{t - t_0 : t \geq t_0 \text{ and } \phi(t, t_0, x_0) = 0\}.$$

At this time, T is said to be the settling-time function corresponding to Ω of (2.1), which may be continuous or discontinuous, bounded or unbounded.

For later use, we introduce the following definitions and lemmas.

Suppose $V : I \times D \rightarrow \mathbb{R}$ is a continuous function. If $(t, x) \in I \times D$ and $\phi(\cdot, t, x)$ is defined, then we define the derivative $\dot{V}|_{(2.1)}(t, x)$ of V along the solutions of (2.1) as

$$\dot{V}|_{(2.1)}(t, x) = \limsup_{h \rightarrow 0^+} \frac{V(t+h, \phi(t+h, t, x)) - V(t, x)}{h}.$$

Definition 2.2 If there exists an open neighborhood $\mathcal{U} \subseteq D$ of the origin and two continuous functions $W : \mathcal{U} \rightarrow \mathbb{R}$ and $V : I \times \mathcal{U} \rightarrow \mathbb{R}$ such that

- (i) $W(0) = 0$, $W(x) > 0$, $x \in \mathcal{U} \setminus \{0\}$,
- (ii) $V(t, 0) \equiv 0$, $V(t, x) \geq W(x)$, $(t, x) \in I \times \mathcal{U}$,
- (iii) $\dot{V}|_{(2.1)}(t, x) \leq 0$, $(t, x) \in I \times (\mathcal{U} \setminus \{0\})$,

then it is said that system (2.1) satisfies condition $\mathbf{A}_1$.

Definition 2.3 Suppose $r : (a, b) \rightarrow \mathbb{R}$ is a function, where $a < 0 < b$. If $r(0) = 0$, and for every $x \in (a, 0) \cup (0, b)$, $xr(x) < 0$ and $\int_x^0 \frac{1}{r(u)} du < +\infty$, then it is said that the function r satisfies condition $\mathbf{A}_2$.

Lemma 2.1 If system (2.1) satisfies condition $\mathbf{A}_1$, then the origin is a stable equilibrium of (2.1).

Condition $\mathbf{A}_1$ is just the so-called Lyapunov stability condition. According to the Lyapunov stability theory, we can obtain Lemma 2.1.

Lemma 2.2 Assume that the origin is a finite-time-stable equilibrium of (2.1) and T is the settling-time function corresponding to Ω of (2.1), then T is continuous on Ω if and only if for every $t \in I$, $T(t, x) \rightarrow 0$ as $x \rightarrow 0$.

Similarly to [5, Prop. 3.2], we can prove Lemma 2.2.

3 Some Results for Finite-Time Stability of Continuous Time-Varying Scalar Systems

In this section, we will consider system (2.1) with $n = 1$ and $D = \mathbb{R}$. Suppose $f(t, x) = g(t, x)$, where $g : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $g(t, 0) \equiv 0$. Then system (2.1) is

written as

$$\frac{dx(t)}{dt} = g(t, x(t)). \quad (3.1)$$

As in Section 2, system (3.1) has unique solutions in forward time for all initial states $(t_0, x_0) \in I \times (\mathbb{R} \setminus \{0\})$. Distinguished from (2.1), for each (t_0, x_0) , we may use $\psi(\cdot, t_0, x_0)$ to denote the unique solution of (3.1) passing through (t_0, x_0) on $[t_0, \tau_{t_0, x_0})$.

The following conclusion can be directly obtained from the definition of finite-time stability.

Theorem 3.1 If the origin is a stable equilibrium of (3.1), then the following statements hold.

(1) The origin is a finite-time-stable equilibrium of (3.1) if and only if for every $t_0 \in I$, there exist two numbers $x_0 > 0$ and $\hat{x}_0 < 0$ such that the two equations $\psi(t, t_0, x_0) = 0$ and $\psi(t, t_0, \hat{x}_0) = 0$ have a solution with respect to t , respectively.

(2) The origin is a globally finite-time-stable equilibrium of (3.1) if and only if for every $(t_0, x_0) \in I \times \mathbb{R}$, the equation $\psi(t, t_0, x_0) = 0$ has a solution with respect to t .

Next, for simplicity, let $g(t, x) = c(t)r(x)$, where $c : [0, +\infty) \rightarrow \mathbb{R}$ is a continuous function, and $r : \mathbb{R} \rightarrow \mathbb{R}$ is also a continuous function satisfying condition **A**₂. Then system (3.1) is written as

$$\frac{dx(t)}{dt} = c(t)r(x(t)). \quad (3.2)$$

Through verification, system (3.2) has unique solutions in forward time for all initial states $(t_0, x_0) \in I \times (\mathbb{R} \setminus \{0\})$. Furthermore, the following theorem provides a sufficient and necessary condition for finite-time stability of (3.2).

Theorem 3.2 The following two statements hold

(1) The origin is a finite-time-stable equilibrium of (3.2) if and only if $c(t) \geq 0$ for all $t \in I$ and for any $t \in I$, there exists a time $\tau \in [t, +\infty)$ such that $c(\tau) > 0$.

(2) The origin is a globally finite-time-stable equilibrium of (3.2) if and only if $c(t) \geq 0$ and $\int_t^{+\infty} c(\tau)d\tau = +\infty$ for all $t \in I$.

Proof We first prove (1). If there exists a time $t_1 \in I$ such that $c(t_1) < 0$, then the continuity of c implies that there exists a sufficiently small real number $\epsilon > 0$ such that $c(t) < 0$ for all $t \in [t_1, t_1 + \epsilon]$. It is easy to obtain that system (3.2) has a non-zero solution through $(t_1, 0)$ from the equation $\int_{t_1}^t c(\tau)d\tau = \int_0^x \frac{1}{r(u)}du$ for all $t \in [t_1, t_1 + \epsilon]$. This contradicts Lyapunov stability of the origin. Thus, $c(t) \geq 0$ for all $t \in I$.

Also, if there exists a time $t_2 \in I$ such that for every $\tau \in [t_2, +\infty)$, $c(\tau) = 0$, then $\psi(t, t_2, x_0) = x_0$ for all $x_0 \in \mathbb{R}$ and all $t \in [t_2, +\infty)$. This contradicts finite-time stability. Thus, for any $t \in I$, there exists $\tau \in [t, +\infty)$ such that $c(\tau) > 0$. This prove the necessity of (1).

Let $V = x^2$. Then $\dot{V}|_{(3.2)}(t, x) = 2c(t)xr(x) \leq 0$. By Lemma 2.1, the origin is a stable equilibrium of (3.2).

If the origin is not a finite-time-stable equilibrium of (3.2), then by Theorem 3.1, there exists a time $t_3 \in I$ such that for every $x_0 > 0$, $\psi(t, t_3, x_0) \neq 0$ for all $t \in [t_3, \tau_{t_3, x_0})$, or for every $\hat{x}_0 < 0$, $\psi(t, t_3, \hat{x}_0) \neq 0$ for all $t \in [t_3, \tau_{t_3, \hat{x}_0})$. Without any loss of generality, assume for every $x_0 > 0$, $\psi(t, t_3, x_0) \neq 0$ for all $t \in [t_3, \tau_{t_3, x_0})$. Then

$$\begin{aligned} 0 \leq \int_{t_3}^t c(\tau) d\tau &= \int_{t_3}^t \frac{1}{r(\psi(\tau, t_3, x_0))} d\psi(\tau, t_3, x_0) \\ &= \int_{x_0}^{\psi(t, t_3, x_0)} \frac{1}{r(u)} du \leq \int_{x_0}^0 \frac{1}{r(u)} du. \end{aligned}$$

It is easy to see that $0 < \psi(t, t_3, x_0) \leq x_0$ for all $t \in [t_3, \tau_{t_3, x_0})$. By Extension Theorem of Solution, $\psi(\cdot, t_3, x_0)$ is defined on $[t_3, +\infty)$. Moreover,

$$\int_{t_3}^{+\infty} c(\tau) d\tau \leq \int_{x_0}^0 \frac{1}{r(u)} du, \quad x_0 > 0.$$

Since for any $t \in I$, there exists $\tau \in [t, +\infty)$ such that $c(\tau) > 0$, then it can be shown that $\int_{t_3}^{+\infty} c(\tau) d\tau > 0$. Also, by the continuity of r and condition $\mathbf{A}_2$, we have that

$$\lim_{x_0 \rightarrow 0^+} \int_{x_0}^0 \frac{1}{r(u)} du = 0.$$

This leads to a contradiction. Thus, The origin is a finite-time-stable equilibrium of (3.2). This prove the sufficiency of (1).

In the following, we prove the necessity of (2). Similarly to the proof of (1), $c(t) \geq 0$ for all $t \in I$, and for every initial state $(t_0, x_0) \in I \times \mathbb{R}$ satisfying $x_0 \neq 0$,

$$\int_{t_0}^{+\infty} c(\tau) d\tau \geq \int_{t_0}^{t_0 + \tilde{T}(t_0, x_0)} c(\tau) d\tau = \int_{x_0}^0 \frac{1}{r(u)} du > 0.$$

If there exists a time $t_3 \in I$ such that $\int_{t_3}^{+\infty} c(\tau) d\tau < +\infty$, then the infinite integral

$\int_{t_3}^{+\infty} c(\tau) d\tau$ is convergent, and moreover,

$$\lim_{t_0 \rightarrow +\infty} \int_{t_0}^{+\infty} c(\tau) d\tau = 0.$$

This leads to a contradiction. Thus for every $t_0 \in I$, $\int_{t_0}^{+\infty} c(\tau) d\tau = +\infty$.

At last, we can prove the sufficiency of (2) in the same way as the proof of the sufficiency for (1).

Remark If the origin is a finite-time-stable equilibrium of (3.2), and $\tilde{T}$ is the settling-time function corresponding to $\tilde{\Omega}$ of (3.2), then we can prove that $\tilde{T}$ satisfies the following formula:

$$\int_{t_0}^{t_0 + \tilde{T}(t_0, x_0)} c(\tau) d\tau = \int_{x_0}^0 \frac{1}{r(u)} du, \quad (t_0, x_0) \in \tilde{\Omega}. \quad (3.3)$$

Suppose $E = \{t \in I : c(t) = 0\}$. From Lemma 2.2 and formula (3.3), it is easy to see that the following statements hold:

- (1) The settling-time function $\tilde{T}$ of (3.2) is continuous on $\tilde{\Omega}$ if and only if $\mathring{E} = \emptyset$.
- (2) The settling-time function $\tilde{T}$ of (3.2) is bounded on $\tilde{\Omega}$ if and only if there exists a constant $M > 0$ such that $\int_{x_0}^0 \frac{1}{r(u)} du \leq \int_{t_0}^{t_0+M} c(\tau) d\tau$ for all $(t_0, x_0) \in \tilde{\Omega}$.

4 Main Results for Finite-Time Stability of Continuous Non-Autonomous Systems

In this section, we will put forward more general sufficient conditions for finite-time stability of continuous non-autonomous systems. According to the definition of finite-time stability, finite-time stability of the origin is divided into two parts, namely, Lyapunov stability and finite-time convergence. In fact, it is relatively easy to determine Lyapunov stability of the origin since Lyapunov stability theory has been quite mature. Therefore, it will be a good idea to analyse finite-time convergence of the origin under the condition that Lyapunov stability of the origin holds. The results of this section are obtained from this idea.

The comparison principle can be found in Section 2.5 of [8], which will be used in the proof of the following theorem.

Theorem 4.1 If (2.1) satisfies condition $\mathbf{A}_1$ with $\mathcal{U}$, W and V as in Definition 2.2, and there exists an open neighborhood $\mathcal{N} \subseteq D$ of the origin and a continuous function $V_1 : I \times \mathcal{N} \rightarrow \mathbb{R}$ such that

- (i) $V_1(t, 0) \equiv 0$, $V_1(t, x) > 0$, $(t, x) \in I \times (\mathcal{N} \setminus \{0\})$,
- (ii) $\dot{V}_1|_{(2.1)}(t, x) \leq g(t, V_1(t, x))$, $(t, x) \in I \times (\mathcal{N} \setminus \{0\})$,

where let $g : I \times \mathbb{R} \rightarrow \mathbb{R}$ be as in system (3.1) ensuring that the origin is a finite-time-stable equilibrium of (3.1), then the origin is a finite-time-stable equilibrium of (2.1). If in addition the origin is a globally finite-time-stable equilibrium of (3.1), $\mathcal{U} = \mathcal{N} = D = \mathbb{R}^n$ and $W(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$, then the origin is a globally finite-time-stable equilibrium of (2.1).

Proof By Lemma 2.1, Lyapunov stability of the origin holds. It can be shown that the right maximally defined solution of (2.1) starting from the origin at any initial time is unique. Then we have that $\dot{V}|_{(2.1)}(t, 0) \equiv 0$ and $\dot{V}_1|_{(2.1)}(t, 0) \equiv 0$. Thus $\dot{V}|_{(2.1)}(t, x) \leq 0$ for all $(t, x) \in I \times \mathcal{U}$ and $\dot{V}_1|_{(2.1)}(t, x) \leq g(t, V_1(t, x))$ for all $(t, x) \in I \times \mathcal{N}$.

Suppose $\tilde{T}$ is the settling-time function corresponding to $\tilde{\Omega}$ of (3.1). Let $\mathcal{K} \subseteq \mathcal{U} \cap \mathcal{N}$ is a bounded open set containing the origin and $\bar{\mathcal{K}} \subseteq \mathcal{U} \cap \mathcal{N}$. Let $\alpha = \min_{x \in \text{bd}\mathcal{K}} W(x)$, then $\alpha > 0$. Choose β such that $0 < \beta < \alpha$ and let

$$\Omega = \{(t, x) \in I \times \mathcal{K} : V(t, x) < \beta \text{ and } (t, V_1(t, x)) \in \tilde{\Omega}\}.$$

Then $I \times \{0\} \subseteq \Omega$ and for every $t \in I$, $\Omega_t = \{x \in \mathbb{R}^n : (t, x) \in \Omega\} \subseteq \mathcal{K}$ is a bounded open set containing the origin.

Suppose $(t_0, x_0) \in \Omega$. Then it is not difficult to obtain that $\phi(\cdot, t_0, x_0)$ is defined on $[t_0, +\infty)$ and $\phi(t, t_0, x_0) \in \mathcal{K}$ for all $t \in [t_0, +\infty)$ by using condition $\mathbf{A}_1$ and extension theorem of solution. Put $(t, \phi(t, t_0, x_0))$ into $\dot{V}_1|_{(2.1)}(t, x) \leq g(t, V_1(t, x))$. Then by the comparison principle,

$$V_1(t, \phi(t, t_0, x_0)) \leq \psi(t, t_0, V_1(t_0, x_0)), \quad t \geq t_0.$$

Because $(t_0, V_1(t_0, x_0)) \in \tilde{\Omega}$, we have

$$\psi(t, t_0, V_1(t_0, x_0)) = 0, \quad t \geq t_0 + \tilde{T}(t_0, V_1(t_0, x_0)).$$

Thus

$$\phi(t, t_0, x_0) = 0, \quad t \geq t_0 + \tilde{T}(t_0, V_1(t_0, x_0)).$$

This proves finite-time convergence of the origin. So the origin is a finite-time-stable equilibrium.

To prove global finite-time stability of the origin, we only need to prove global finite-time convergence of the origin. In fact, for every $(t_0, x_0) \in I \times \mathbb{R}^n$,

$$V(t, \phi(t, t_0, x_0)) \leq V(t_0, x_0)$$

for all $t \in [t_0, \tau_{t_0, x_0})$ by condition $\mathbf{A}_1$. Also, if $\tau_{t_0, x_0} < +\infty$, then by extension theorem of solution, as $t \rightarrow \tau_{t_0, x_0}^-$, $\|\phi(t, t_0, x_0)\| \rightarrow +\infty$ and moreover,

$$V(t, \phi(t, t_0, x_0)) \geq W(\phi(t, t_0, x_0)) \rightarrow +\infty.$$

This is a contradiction. Thus $\phi(\cdot, t_0, x_0)$ is defined on $[t_0, +\infty)$. Next, similarly to the proof of finite-time stability, the conclusion we need can be proved.

We can seek some practical functions g to make the use of Theorem 4.1 more convenient. The following corollary is just a good example.

Corollary 4.2 If (2.1) satisfies condition $\mathbf{A}_1$ with $\mathcal{U}$, W and V as in Definition 2.2, and there exists an open neighborhood $\mathcal{N} \subseteq D$ of the origin and a continuous function $V_1 : I \times \mathcal{N} \rightarrow \mathbb{R}$ such that

- (i) $V_1(t, 0) \equiv 0$, $V_1(t, x) > 0$, $(t, x) \in I \times (\mathcal{N} \setminus \{0\})$,
- (ii) $\dot{V}_1|_{(2.1)}(t, x) \leq c(t)r(V_1(t, x))$, $(t, x) \in I \times (\mathcal{N} \setminus \{0\})$,

where $c : I \rightarrow \mathbb{R}$ is a continuous function satisfying both $c(t) \geq 0$ and there exists a time $\tau \in [t, +\infty)$ such that $c(\tau) > 0$ for any $t \in I$, and the function $r : \mathbb{R} \rightarrow \mathbb{R}$ satisfies condition $\mathbf{A}_2$, then the origin is a finite-time-stable equilibrium of (2.1). If in addition $\int_t^{+\infty} c(\tau) d\tau = +\infty$ for all $t \in I$, $\mathcal{U} = \mathcal{N} = D = \mathbb{R}^n$ and $W(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$, then the origin is a globally finite-time-stable equilibrium of (2.1).

Proof Here, the function r is not necessarily continuous on its domain. So we can't directly use Theorem 4.1. In this case, let

$$V_2(t, x) = \left[- \int_0^{V_1(t, x)} \frac{1}{r(u)} du \right]^{\frac{1}{1-\alpha}}, \quad (t, x) \in I \times \mathcal{N},$$

where $0 < \alpha < 1$. Then $V_2 : I \times \mathcal{N} \rightarrow \mathbb{R}$ is a continuous function and satisfies the above condition (i). Also,

$$\begin{aligned} \dot{V}_2|_{(2.1)}(t, x) &= -\frac{1}{1-\alpha} \left[-\int_0^{V_1(t,x)} \frac{1}{r(u)} du \right]^{\frac{\alpha}{1-\alpha}} \frac{\dot{V}_1|_{(2.1)}(t, x)}{r(V_1(t, x))} \\ &\leq -\frac{c(t)}{1-\alpha} \left[-\int_0^{V_1(t,x)} \frac{1}{r(u)} du \right]^{\frac{\alpha}{1-\alpha}} \\ &\leq c(t) \left[-\frac{1}{1-\alpha} (V_2(t, x))^\alpha \right] \leq c(t) \tilde{r}(V_2(t, x)) \end{aligned}$$

for all $(t, x) \in I \times (\mathcal{N} \setminus \{0\})$, where $\tilde{r}(x) = -\text{sign}(x) \frac{1}{1-\alpha} |x|^\alpha$ and $\text{sign}(\cdot)$ is the sign function defined by

$$\text{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Obviously, $\tilde{r} : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and satisfies condition $\mathbf{A}_2$. Then Corollary 4.2 can be obtained from Theorem 3.2 and Theorem 4.1.

Remark If the origin is a finite-time-stable equilibrium of (2.1) under the conditions of Corollary 4.2 and T is the settling-time function corresponding to Ω of (2.1), then according to the proofs of Theorem 4.1 and Corollary 4.2, T satisfies the following formula:

$$\int_{t_0}^{t_0+T(t_0, x_0)} c(\tau) d\tau \leq -\int_0^{V_1(t_0, x_0)} \frac{1}{r(u)} du, (t_0, x_0) \in \Omega. \quad (4.1)$$

Moreover, by Lemma 2.2 and formula (4.1), we have that the following statements hold

(1) The settling-time function T of (2.1) is continuous on Ω if $\dot{E} = \emptyset$, where $E = \{t \in I : c(t) = 0\}$.

(2) The settling-time function T of (2.1) is bounded on Ω if there exists a constant $M > 0$ such that $\int_{V_1(t_0, x_0)}^0 \frac{1}{r(u)} du \leq \int_{t_0}^{t_0+M} c(\tau) d\tau$ for all $(t_0, x_0) \in \Omega$.

5 Applications

In this section, we will provide some applications for the results of this paper.

Consider a multi-agent system described by

$$\dot{x}_i = a_i(t) \varphi_i(x_j - x_i), \quad i, j = 1, 2 \text{ and } j \neq i, \quad (5.1)$$

where $x_i \in \mathbb{R}$ is the state variable, $a_i : I \rightarrow \mathbb{R}$ is a continuous function satisfying $a_i(t) \geq 0$ for all $t \in I$, and $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ is also a continuous function and locally Lipschitz outside the origin. Let $y = x_1 - x_2$. Then we have

$$\dot{y} = a_1(t) \varphi_1(-y) - a_2(t) \varphi_2(y). \quad (5.2)$$

Assume that $\varphi = \varphi_1 = \varphi_2$ is an odd function and $-\varphi$ satisfies condition $\mathbf{A}_2$, then system (5.2) can be written as

$$\dot{y} = -(a_1(t) + a_2(t))\varphi(y). \quad (5.3)$$

Obviously, system (5.3) is the same as system (3.2) with $c = a_1 + a_2$ and $r = -\varphi$. By Theorem 3.2, it is easy to see that if for any $t \in I$, there exists a time $\tau \in [t, +\infty)$ such that $a_1(\tau) + a_2(\tau) > 0$, then the origin is a finite-time-stable equilibrium of (5.3), namely, the states of (5.1) can locally reach consensus in finite time; and if

$$\int_t^{+\infty} (a_1(\tau) + a_2(\tau))d\tau = +\infty$$

for all $t \in I$, then the origin is a globally finite-time-stable equilibrium of (5.3), namely, the states of (5.1) can globally reach consensus in finite time, and furthermore, we can make the settling-time function of (5.3) bounded on $\mathbb{R}$, namely, make the states of (5.1) reach consensus in fixed time if there exists a constant $M > 0$ such that

$$\int_0^x \frac{1}{\varphi(u)}du \leq \int_t^{t+M} (a_1(\tau) + a_2(\tau))d\tau$$

for all $(t, x) \in I \times \mathbb{R}$.

Also, assume that $y\varphi_1(-y) \leq 0$ for all $y \in \mathbb{R}$ and there exists a function $r : \mathbb{R} \rightarrow \mathbb{R}$ satisfying condition $\mathbf{A}_2$ such that $-y\varphi_2(y) \leq r(y^2)$, then by Corollary 4.2 with $V = V_1 = y^2$, similarly, we have that the states of (5.1) can locally reach consensus in finite time when for any $t \in I$, there exists a time $\tau \in [t, +\infty)$ such that $a_2(\tau) > 0$, or globally reach consensus in finite time when $\int_t^{+\infty} a_2(\tau)d\tau = +\infty$ for all $t \in I$, or reach consensus in fixed time when there exists a constant $M > 0$ such that

$$\int_x^0 \frac{1}{r(u)}du \leq 2 \int_t^{t+M} a_2(\tau)d\tau$$

for all $(t, x) \in I \times \mathbb{R}$. Of course, if $-y\varphi_2(y) \leq 0$ for all $y \in \mathbb{R}$ and there exists a function $r : \mathbb{R} \rightarrow \mathbb{R}$ satisfying condition $\mathbf{A}_2$ such that $y\varphi_1(-y) \leq r(y^2)$, then we have the same conclusions with a_1 taking the place of a_2 .

From the above analysis, we see that the results of this paper can be applied to the study of finite-time consensus of the multi-agent system (5.1). In fact, it is not difficult to understand that they can be used to analyze finite-time consensus of more general multi-agent systems.

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连续非自治系统的有限时间稳定性及其充分条件

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摘要: 本文研究了连续非自治系统的有限时间稳定性问题. 从一维连续非自治系统的有限时间稳定性分析入手, 本文通过使用比较原理, 获得了一些判定一般 n 维连续非自治系统的有限时间稳定性的充分条件, 这些条件改善了已有的连续非自治系统有限时间稳定性的判定条件.

关键词: 连续非自治系统; 有限时间稳定性; 充分条件

MR(2010)主题分类号: 37C75 中图分类号: O175.13