Vol. 36 (2016)
No. 4

数 学 杂 志
J. of Math. (PRC)

ROBUST SYNCHRONIZATION OF COMPLEX DYNAMICAL NETWORKS

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Abstract: In this paper, we study the synchronization of complex networks with nonlinear perturbations. Using the input-to-state stability method, we obtain the result of robust synchronization of networks. Some sufficient conditions that ensure the robust synchronization of the weighted complex dynamical networks are provided for both systems without time delay and time-delay systems. Numerical simulations are given to validate our theoretical analysis.

Keywords: complex network; robust synchronization; time-delay; perturbation

2010 MR Subject Classification: 34D06; 05C82

Document code: A **Article ID:** 0255-7797(2016)04-0727-10

1 Introduction

Complex networks are ubiquitous in the world nowadays. Examples include transportation and phone networks, internet, wireless networks and the World Wide Web. Significant progress was made in studying complex networks since the discovery of their small-world [1] and scale-free [2] characteristics.

A complex dynamical network can be described by a graph in a mathematical way. In such a graph, each node represents a basic element with certain dynamics, and edges represent interactive topology of the network. Synchronization of complex dynamical networks received a great deal of attention from the systems science community [3–15]. Pecora and Carroll [3] studied the master stability functions for synchronized coupled systems. Using the theory of inhomogeneous Markov chains, Wu [4] proposed a synchronization criterion for nonautonomous discrete-time linear system in random directed network. Belykh et al. [5, 6] proposed the synchronization of coupled chaotic systems via connection graph stability method. Starting with a nearest-neighbor coupled dynamical network, Wang and Chen [7] constructed a small-world dynamical network, and investigated chaos synchronization in the

* **Received date:** 2015-10-21 **Accepted date:** 2016-02-26

Foundation item: Supported by National Natural Science Foundation of China (61203159; 61273232; 61573011).

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network. Lü and Chen [8] introduced a general time-varying complex network model and derived some synchronization criteria for time-varying complex networks.

However, most of the above works focused on the exact synchronization, that is, the synchronization error asymptotically goes to zero. In this paper, we consider networks subject to nonlinear perturbations, where the perturbations could be modeling errors, disturbances or some other types of external inputs. In this setting, the exact synchronization in general cannot be expected. Motivated by the introduction of the input-to-state stability in the systems theory, we propose some notions that characterize the robust behavior of the synchronization error in the presence of the perturbations. Basically, the network is said to achieve robust synchronization if (1) the synchronization error remains small whenever it is initially small and the perturbation is also small and (2) the synchronization error eventually depends on the magnitude of the perturbations. The latter requirement implies that the network achieves exact synchronization without perturbations. By using Lyapunov's direct method, we give sufficient conditions for the network to achieve the robust synchronization. Furthermore, we extend our robust synchronization results to the networks with time-delay individual systems.

2 Preliminaries

In this section, we give some mathematical definitions used in this work.

Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm for vectors and the Frobenius norm for matrices. I_n is the identity matrix of order n. In addition, we use $B \succ 0$ (resp. $B \succeq 0$, $B \prec 0$, $B \preceq 0$) to denote positive definiteness (resp. positive semidefiniteness, negative definiteness, negative semidefiniteness) of matrix B; $\mathbf{1}_q$ represents the vector $(1, \cdots, 1)^T \in R^q$. $\mathcal{W}$ is the set of n by n real matrices with zero row sums and non-positive off-diagonal elements. $\mathcal{W}_s$ is the set of irreducible symmetric matrices in $\mathcal{W}$ [9].

For a matrix $G \in R^{n\times n}$ with non-positive off-diagonal elements, $\mu_A(G)$ is defined as the supremum of the set of real numbers μ such that $A(G-\mu I_n) \succeq 0$ for some $A \in \mathcal{W}_s$. It was proved that if $G \in \mathcal{W}_s$, then $\mu_A(G) > 0$ (see Corollary 4.11 in [9]).

3 Synchronization Analysis of Complex Dynamical Network

Consider the following coupled network of identical dynamical systems

$$\dot{x} = (f(t,x_1), \cdots, f(t,x_n))^T + (G(t)\otimes D(t))x + (g(t,x_1)u_1, \cdots, g(t,x_n)u_n)^T, \tag{3.1}$$

where $x_i \in R^m$ is the state of node i, and $x = (x_1, \cdots, x_n)^T$; $f: R^+ \times R^m \to R^m$ is a continuous vector function; $D(t) \in R^{m\times m}$ is the inner-coupling matrix of each node; $G(t)$ is called as the outer-coupling matrix of the network, and for any t it satisfies zero row sums and non-positive off-diagonal elements; $g(t,x_i)u_i$, $i = 1, \cdots, n$, are the nonlinear perturbation terms, where $g(t,x_i) \in R^{m\times q}$ and $u_i \in R^q$. We will refer to the f system as the individual system.

In this section, we investigate the robust synchronization for the dynamical network (3.1). First we give some definitions.

Definiton 1 Network (3.1) is said to achieve synchronization if, for all i, j, $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$.

Definiton 2 Give an n by n matrix $A \in \mathcal{W}_s$. Define

$$X_A(t) = (\cdots, x_i(t) - x_j(t), \cdots), \quad U_A(t) = (\cdots, u_i(t) - u_j(t), \cdots),$$

where i, j satisfy $i < j$ and $A_{ij} \neq 0$. The dynamical network (3.1) is said to achieve robust synchronization if there exist a matrix $A \in \mathcal{W}_s$, a class $\mathcal{KL}$ function β, and a class $\mathcal{K}$ function γ such that for any $X_A(t_0)$ and bounded U_A,

$$\|X_A(t)\| \leq \beta(\|X_A(t_0)\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|U_A(\tau)\|)$$

for all $t \geq t_0$.

Definiton 3 The dynamical network (3.1) is said to achieve robust exponential synchronization if there exist a matrix $A \in \mathcal{W}_s$, $\lambda > 0$ and class $\mathcal{K}$ functions γ_0, γ_1 such that for any $X_A(t_0)$ and bounded U_A,

$$\|X_A(t)\| \leq \gamma_0(\|X_A(t_0)\|)e^{-\lambda(t-t_0)} + \gamma_1(\sup_{t_0 \leq \tau \leq t} \|U_A(\tau)\|)$$

for all $t \geq t_0$.

Remark 1 The robust synchronization is similar to the conventional input-to-state stability (ISS) notion in systems theory [16, 17].

Now, we are in a position to show our main result on the robust synchronization of dynamical network (3.1).

Theorem 1 The dynamical network (3.1) achieves robust synchronization if the following three conditions are satisfied:

(i) There exists a time-varying matrix $Y(t) \in R^{m\times m}$, a symmetric positive definite matrix $P \in R^{m\times m}$, and a class $\mathcal{K}_\infty$ function $\alpha_3(\cdot)$ such that for any $\xi, \eta \in R^m$ and any t,

$$(\xi - \eta)^T P(f(t,\xi) + Y(t)\xi - f(t,\eta) - Y(t)\eta) \leq -\alpha_3(\|\xi - \eta\|^2) \tag{3.2}$$

for all i, j.

(ii) There exist class $\mathcal{K}_\infty$ functions α_i, $i = 4, 5, 6$, such that, for any $\xi_1, \xi_2 \in R^m$, $\eta_1, \eta_2 \in R^q$ and any t,

$$(\xi_1 - \xi_2)^T P(g(t,\xi_1)\eta_1 - g(t,\xi_2)\eta_2) \leq \alpha_4(\|\xi_1 - \xi_2\|^2) + \alpha_5(\|\eta_1 - \eta_2\|^2); \tag{3.3}$$

and for any $c \geq 0$,

$$-\alpha_3(c) + \alpha_4(c) \leq -\alpha_6(c). \tag{3.4}$$

(iii) There exists a matrix $A \in \mathcal{W}_s$ such that

$$(A \otimes P)(G(t) \otimes D(t) - I \otimes Y(t)) \preceq 0 \tag{3.5}$$

for all t.

Proof Construct the function $E(X_A) = \frac{1}{2} \sum\limits_{i<j} [-A_{ij}(x_i - x_j)^T P(x_i - x_j)] = \frac{1}{2} x^T (A \otimes P) x$, where $A \in \mathcal{W}_s$ satisfying point (iii). It is positive definite and radially unbounded with respect to X_A. By Lemma 3.5 in [18], there exist $\mathcal{K}_\infty$ functions α_1, α_2, such that

$$\alpha_1(\|X_A\|) \le E(X_A) \le \alpha_2(\|X_A\|). \tag{3.6}$$

The derivative of E along trajectories of the dynamical network (3.1) is given by

$$\begin{aligned}
\dot{E}(X_A)|_{(3.1)} &= x^T(A \otimes P)\dot{x} \\
&= x^T(A \otimes P) \begin{pmatrix} f(t, x_1) + Y(t)x_1 + g(t, x_1)u_1(t) \\ \vdots \\ f(t, x_n) + Y(t)x_n + g(t, x_n)u_n(t) \end{pmatrix} \\
&\quad + x^T(A \otimes P)(G \otimes D(t) - I \otimes Y(t))x \\
&\le \sum_{i<j} [-A_{ij}(x_i - x_j)^T P(f(x_i, t) + Y(t)x_i - f(x_j, t) \\
&\quad - Y(t)x_j + g(t, x_i)u_i(t) - g(t, x_j)u_j(t))] \\
&\le \sum_{i<j} [-A_{ij}(-\alpha_3(\|x_i - x_j\|^2) + \alpha_4(\|x_i - x_j\|^2) + \alpha_5(\|u_i - u_j\|^2))] \\
&\le -\sum_{i<j} (-A_{ij}\alpha_6(\|x_i - x_j\|^2)) + \sum_{i<j} (-A_{ij}\alpha_5(\|u_i - u_j\|^2)).
\end{aligned}$$

Note that $\sum\limits_{i<j}(-A_{ij}\alpha_6(\|x_i - x_j\|^2))$ and $\sum\limits_{i<j}(-A_{ij}\alpha_5(\|u_i - u_j\|^2))$ are positive definite and radially unbounded with respect to X_A and U_A, respectively. Thus using Lemma 3.5 in [18], there are $\mathcal{K}_\infty$ function α_7, α_8, such that

$$\begin{aligned}
&\sum_{i<j} (-A_{ij}\alpha_6(\|x_i - x_j\|^2)) \ge \alpha_7(\|X_A\|), \\
&\sum_{i<j} (-A_{ij}\alpha_5(\|u_i - u_j\|^2)) \le \alpha_8(\|U_A\|).
\end{aligned}$$

Hence $\dot{E}(X_A)|_{(3.1)} \le -\alpha_7(\|X_A\|) + \alpha_8(\|U_A\|)$. Since

$$E(X_A) \le \alpha_2(\|X_A\|) \Rightarrow \alpha_2^{-1}(E(X_A)) \le \|X_A\| \Rightarrow \alpha_7(\alpha_2^{-1}(E(X_A))) \le \alpha_7(\|X_A\|).$$

We obtain that E satisfies $\dot{E}(X_A)|_{(3.1)} \le -\alpha_9(E(X_A)) + \alpha_8(\|U_A\|)$, where $\alpha_9(\cdot) := \alpha_7(\alpha_2^{-1}(\cdot))$ is a class $\mathcal{K}_\infty$ function. Further, we have

$$\dot{E}(X_A)|_{(3.1)} \le -(1-\theta)\alpha_9(E(X_A)) - \theta\alpha_9(E(X_A)) + \alpha_8(\|U_A\|),$$

where $0 < \theta < 1$. Therefore $\dot{E}(X_A)|_{(3.1)} \leq -(1-\theta)\alpha_9(E(X_A))$, whenever $\|X_A\| \geq \rho(\|U_A\|)$, where ρ is a class $\mathcal{K}_\infty$ function defined as $\rho(\cdot) = \alpha_1^{-1} \circ \alpha_9^{-1}(\frac{\alpha_8(\cdot)}{\theta})$. By a similar reasoning as in the proofs of Theorem 4.18 and 4.19 in [18], we conclude that there exist class $\mathcal{KL}$ function β and class $\mathcal{K}$ function γ, such that, for any t,

$$\|X_A(t)\| \leq \beta(\|X_A(t_0)\|, t-t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|U_A(\tau)\|), \tag{3.7}$$

which means the dynamical network (3.1) achieves robust synchronization.

Remark 2 Condition (i) can be fulfilled if the system satisfies the QUAD assumption introduced in [10] and [5].

4 Synchronization Analysis of Complex Network with Time-Delay Individual Systems

In this section, we give sufficient conditions for the robust synchronization of complex networks with time-delay individual systems. Similar to Definition , the robust synchronization here is defined in the spirit of input-to-state stability for functional differential equations with disturbances [19].

Let us start by introducing some notations, which are largely inherited from [19]. Given a function $\xi(\cdot)$ that is defined on $[t-\tau, t]$ and takes values in R^l, with $t, \tau \in R^+$, we use $\xi^d(t)(\cdot)$ to represent a function from $[0,\tau]$ to R^l defined by $\xi^d(t)(s) := \xi(t-s)$, and define $|\xi^d(t)| := \max_{0 \leq s \leq \tau} \|\xi^d(t)(s)\| = \max_{t-\tau \leq s \leq t} \|\xi(s)\|$. In addition, for a function $\zeta : [t_0 - \tau, \infty) \to R^l$, define $\|\zeta\|_{t_0} := \sup_{t \geq t_0} |\zeta^d(t)|$.

In this section, we consider the following complex network with time-delay individual systems:

$$\begin{aligned} \dot{x}(t) &= (f(t, x_1^d(t)), \cdots, f(t, x_n^d(t))^T + (G(t) \otimes D(t))x(t) \\ &\quad + (g(t, x_1(t))u_1(t), \cdots, g(t, x_n(t))u_n(t))^T, \end{aligned} \tag{4.1}$$

where f is a function from $R^+ \times C([0,\tau], R^m)$ to R^m, with $C([0,\tau], R^m)$ denoting the set of continuous functions defined on $[0,\tau]$ and taking values in R^m. The other notations hold the same meanings as in (3.1).

Definiton 4 The dynamical network (4.1) is said to achieve robust synchronization if there exists a matrix $A \in \mathcal{W}_s$, a class $\mathcal{KL}$ function β, and a class $\mathcal{K}$ function γ such that, for any bounded $X_A^d(t_0)$ and uniformly bounded U_A, it holds that

$$\|X_A\|_{t_0} \leq \beta(|X_A^d(t_0)|, t-t_0) + \gamma(\|U_A\|_{t_0}), \tag{4.2}$$

where X_A and U_A are given in Definition .

Theorem 2 The dynamical network (4.1) achieves robust synchronization if the following three conditions are satisfied:

(i) There exists a time-varying matrix $Y(t) \in R^{m\times m}$, a symmetric positive definite matrix $P \in R^{m\times m}$, and positive constants $k_1,\ k_2$ such that, for any $\xi, \eta \in C([0,\tau], R^m)$,

$$\begin{aligned}
&(\xi(0)-\eta(0))^T P(f(t,\xi)+Y(t)\xi(0)-f(t,\eta)-Y(t)\eta(0)) \\
\leq\ & -k_1\|\xi(0)-\eta(0)\|^2+k_2\|\xi(\tau)-\eta(\tau)\|^2
\end{aligned}\tag{4.3}$$

for all i, j.

(ii) There exist positive constants $k_i,\ i=3,4$ and a scalar $a>0$ such that for any $\xi_1, \xi_2 \in R^m$, $\eta_1, \eta_2 \in R^q$, and any t,

$$(\xi_1-\xi_2)^T P(g(t,\xi_1)\eta_1-g(t,\xi_2)\eta_2) \leq k_3\|\xi_1-\xi_2\|^2+k_4\|\eta_1-\eta_2\|^2 \tag{4.4}$$

and

$$(k_3-k_1)I_m+aP \prec 0, \quad k_2 I_m - aP \prec 0. \tag{4.5}$$

(iii) There exists a matrix $A \in \mathcal{W}_s$ such that

$$(A\otimes P)(G(t)\otimes D(t)-I\otimes Y(t)) \preceq 0 \tag{4.6}$$

for all t.

Proof Consider the Lyapunov function $E(X_A)=\frac{1}{2}\sum\limits_{i<j}[-A_{ij}(x_i-x_j)^T P(x_i-x_j)] = \frac{1}{2}x^T(A\otimes P)x$. We use the shorthand $E(X_A(t))=E(t)$. And there exist constants $c_1, c_2>0$, such that

$$c_1\|X_A\|^2 \leq E(X_A) \leq c_2\|X_A\|^2. \tag{4.7}$$

The derivative of E along trajectories of the dynamical network (4.1) is given by

$$\begin{aligned}
&\dot{E}(t)|_{(4.1)} = x(t)^T(A\otimes P)\dot{x}(t) \\
=\ & x(t)^T(A\otimes P)\begin{pmatrix} f(t,x_1^d(t))+Y(t)x_1(t)+g(t,x_1(t))u_1(t) \\ \vdots \\ f(t,x_n^d(t))+Y(t)x_n(t)+g(t,x_n(t))u_n(t) \end{pmatrix} \\
& +x(t)^T(A\otimes P)(G(t)\otimes D(t)-I\otimes Y(t))x(t) \\
\leq\ & \sum_{i<j}\{-A_{ij}[(x_i(t)-x_j(t))^T P(f(t,x_i^d(t))+Y(t)x_i(t) \\
& -f(t,x_j^d(t))-Y(t)x_j(t)+g(t,x_i(t))u_i(t)-g(t,x_j(t))u_j(t))]\} \\
\leq\ & \sum_{i<j}\{-A_{ij}[(-k_1+k_3)\|x_i(t)-x_j(t)\|^2+k_2\|x_i(t-\tau)-x_j(t-\tau)\|^2+k_4\|u_i(t)-u_j(t)\|^2]\},
\end{aligned}$$

where we have used conditions (i) and (iii) (note that $x_i(t)=x_i^d(t)(0)$ and $x_i(t-\tau)=x_i^d(t)(\tau)$).

By (4.5), there exists some $\delta > 0$ such that $(-k_1 + k_3)I_m + a(1+2\delta)P \prec 0$. Now, given $p = 1+\delta$, if $E(t)$ satisfies $E(s) < pE(t)$ for all $s \in [t-\tau, t]$, i.e.,

$$x^T(t+\theta)(A \otimes P)x(t+\theta) < px(t)^T(A \otimes P)x(t) \tag{4.8}$$

for all $-\tau \le \theta \le 0$, we can conclude that for any $a > 0$,

$$\begin{aligned}\dot{E}(t)|_{(4.1)} \le & \sum_{i<j}\{-A_{ij}[(-k_1+k_3)\|x_i(t)-x_j(t)\|^2 + k_2\|x_i(t-\tau)-x_j(t-\tau)\|^2 \\ & +k_4\|u_i(t)-u_j(t)\|^2]\} + a[px(t)^T(A\otimes P)x(t) - x^T(t-\tau)(A\otimes P)x(t-\tau)] \\ = & \sum_{i<j}\left\{-A_{ij}\left[\phi^T\begin{pmatrix}(-k_1+k_3)I_m + apP & 0 \\ 0 & k_2 I_m - aP\end{pmatrix}\phi + k_4\|u_i(t)-u_j(t)\|^2\right]\right\},\end{aligned}$$

where $\phi = [(x_i(t)-x_j(t))^T, (x_i(t-\tau)-x_j(t-\tau))^T]^T$. This implies that

$$\begin{aligned}\dot{E}(t)|_{(4.1)} \le & \sum_{i<j}\left\{-A_{ij}\left[-a\delta(x_i(t)-x_j(t))^T P(x_i(t)-x_j(t)) + k_4\|u_i(t)-u_j(t)\|^2\right]\right\} \\ \le & -2a\delta E(t) + b\|U_A(t)\|^2,\end{aligned}$$

where $b = k_4 \max_{i,j}(-A_{ij})$. Therefore, whenever $E(t) \ge \frac{b}{a\delta}\|U_A(t)\|^2$, then $\dot{E}(t)|_{(4.1)} \le -a\delta E(t) \le -a\delta c_1\|X_A(t)\|^2$, where the last inequality is obtained from (4.7).

Define a class $\mathcal{K}$ function γ as $\gamma(y) = \sqrt{\frac{b}{a\delta c_1}}y$, and assume that $\|U_A\|_{t_0} \le \eta$ for some positive real number η. Using the same reasoning as in the proof of Theorem 1 in [19], we conclude that

(i) for each $\epsilon > 0$ there existss $\delta > 0$ such that $|X_A^d(t_0)| \le \delta$ and $\|U_A\|_{t_0} \le \delta$ imply $\|X_A\|_{t_0} \le \epsilon$;

(ii) for any $r, \epsilon > 0$, there is a $T > 0$ so that $\|X_A\|_{t_0+T} \le \epsilon + \gamma(\|U_A\|_{t_0})$ whenever $|X_A^d(t_0)| \le r$.

Lastly, the proof of the theorem is completed by almost repeating the proof of Lemma 2.7 in [17] with some necessary changes of norms.

5 Numerical Simulations

In this section, we show some illustrative examples that validate our results in Section 3 and 4.

Consider a scale-free network with $n = 500$ nodes whose coupling topology is generated by the well-known BA model [2]. If there is a connection between i and j ($j \ne i$) in the network, $G_{ij} = -1$; otherwise $G_{ij} = 0$. It can be checked that the outer-coupling matrix $G \in \mathcal{W}_s$. Furthermore, given a matrix

$$A = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} \in \mathcal{W}_s, \tag{5.1}$$

we can calculate $\mu_A(G) = 2.1$ (see Section 2 for the definition of μ_A). By setting the inner-coupling matrix $D(t) = -cI_3$, we can choose $Y(t) = \mu_A(G)D(t)$ and $P = I_3$ such that (3.5), (4.6) hold for any $c > 0$. Indeed, because $PD(t) = D^T(t)P \preceq 0$ and $A(G - \mu_A(G)I) \succeq 0$, we have

$$(A \otimes P)(G \otimes D(t) - I \otimes \mu_A(G)D(t)) = (A(G - \mu_A(G)I)) \otimes (PD(t)) \preceq 0.$$

Example 1 Each node represents a 3-D neural system, and the individual system function f is defined as

$$f(t, y) = -y + Th(y), \tag{5.2}$$

where $y = (y_1, y_2, y_3)^T \in R^3$,

$$T = \begin{bmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{bmatrix}, \quad h(y) = \begin{bmatrix} (|y_1 + 1| - |y_1 - 1|)/2 \\ (|y_2 + 1| - |y_2 - 1|)/2 \\ (|y_3 + 1| - |y_3 - 1|)/2 \end{bmatrix}. \tag{5.3}$$

As indicated in Ref. [20], the system $\dot{y} = f(t, y)$ has a double-scrolling chaotic attractor. In addition, in (3.1), put

$$g(t, y) = 3\mathrm{diag}\{\sin(y_1), \sin(y_2), \sin(y_3)\}, \quad u_i = \sin(2i)\mathbf{1}_3.$$

Note that, for any $\xi, \eta \in R^3$,

$$\begin{aligned} & (\xi - \eta)^T(f(t, \xi) + Y(t)\xi - f(t, \eta) - Y(t)\eta) \\ \leq\ & -\|\xi - \eta\|^2 + (\xi - \eta)^T T(h(\xi) - h(\eta)) - c\mu_A(G)\|\xi - \eta\|^2 \\ \leq\ & (8.1 - 2.1c)\|\xi - \eta\|^2, \end{aligned}$$

and for any $\xi_1, \xi_2 \in R^3$ and $\eta_1, \eta_2 \in R^3$,

$$(\xi_1 - \xi_2)^T P(g(t, \xi_1)\eta_1 - g(t, \xi_2)\eta_2) \leq \frac{9}{2}\|\xi_1 - \xi_2\|^2 + \frac{3}{2}\|\eta_1 - \eta_2\|^2.$$

Hence the conditions in Theorem 1 are satisfied whenever $c > 6$.

In the simulation, we choose $c = 6.1$. It is shown in Figure 1 (a) (b) that the network (3.1) achieves the robust exponential synchronization with perturbations and synchronization without perturbations, where the synchronization error is measured by $W(t) = \frac{1}{3(n-1)} \sum_{k=1}^{3} \sqrt{\sum_{i=1}^{n-1} (x_i^k - x_{i+1}^k)^2}$.

Example 2 Consider a 2-D time-delay system as the individual system of the above scale-free network, whose system function f is defined as

$$f(t, y) = \begin{pmatrix} -y_1(0) + 3\tanh(y_1(0)) + 5\tanh(y_2(0)) - 2.5\tanh(y_1(\tau)) + 0.2\tanh(y_2(\tau)) \\ -y_2(0) + 0.1\tanh(y_1(0)) + 2\tanh(y_2(0)) + 0.1\tanh(y_1(\tau)) - 1.5\tanh(y_2(\tau)) \end{pmatrix}, \tag{5.4}$$

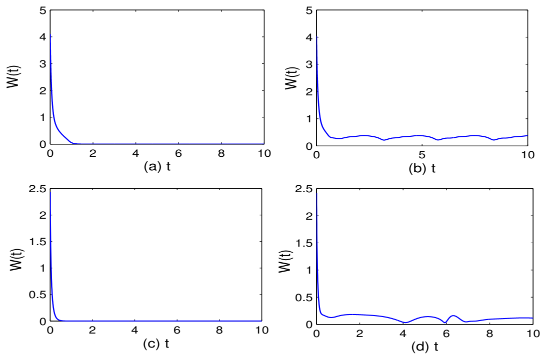

Figure 1: (a) The synchronization errors of network (3.1) without perturbation; (b) The synchronization errors of network (3.1) with perturbation; (c) The synchronization errors of network (4.1) without perturbation; (d) The synchronization errors of network (4.1) with perturbation.

where $y = (y_1, y_2) \in C([0,\tau], R^2)$. Note that the functional differential equation $\dot{x}(t) = f(t, x^d(t))$ describes a neural network model [21].

Let the function g and inputs u_i's are the same as in Example 1. By noting $|\tanh(\theta_1) - \tanh(\theta_2)| \leq |\theta_1 - \theta_2|$, one easily verifies, using similar procedures as shown in Example 1, that conditions (4.3), (4.4), and (4.5) hold if c is large enough. In Figure 1(c)(d), numerical results show that when $c = 5$ the network (4.1) achieves robust synchronized with perturbations and synchronization without perturbations.

6 Conclusions

In this work, we have studied the robust synchronization of the weighted complex dynamical networks with linear coupling and nonlinear perturbations. The novel robust synchronization notion proposed here is a translation of the well-known input-to-state stability in systems theory. For both systems without time delay and time-delay systems, sufficient conditions are given to guarantee the network achieve the robust synchronization. Further discussions will focus on the robust synchronization of the complex dynamical networks with nonlinear or/and time-delay coupling.

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复杂动力网络的鲁棒性同步

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摘要: 本文研究了扰动下复杂动力网络的同步问题. 利用输入状态稳定性分析的方法, 给出了鲁棒同步的概念, 分析了非时间延迟的和含有时间延迟动力网络的同步, 数值仿真也验证了结果的有效性.

关键词: 复杂网络; 鲁棒同步; 时间延迟; 扰动

MR(2010)**主题分类号**: 34D06; 05C82 **中图分类号**: O175.13; O231.5