CENTRAL INVARIANTS OF GENERALIZED HOM-LIE ALGEBRAS

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Abstract: Let \(L\) be a generalized Hom-Lie algebra, \(V\) a \(H\)-Hom-Lie ideal of \([L, L]\). In this paper, we mainly discuss the central invariant of \(L\). Using the method of Hopf algebras, we obtain that \(H\)-invariant of \(V\) is contained in \(H\)-invariant of the center of \(L\). It generalizes the main results by Cohen and Westreich (1994).

Keywords: monoidal Hom-algebra; generalized Hom-Lie algebra; Yetter-Drinfeld category

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1 Introduction

Hom-algebras were firstly studied by Hartwig, Larsson and Silvestrov in [4], where they introduced the structure of Hom-Lie algebras in the context of the deformations of Witt and Virasoro algebras. Determination of derivation algebras is an important task in Lie algebra, see [11]. Later, Larsson and Silvestrov extended the notion of Hom-Lie algebras to quasi-Hom Lie algebras and quasi-Lie algebras, see [5] and [6]. Wang et al. (see [10]) studied the structure of the generalized Hom-Lie algebras (i.e., the Hom-Lie algebras in Yetter-Drinfeld category \(\mathcal{H}YD\)).

Let \(H\) be a Hopf algebra, and \(A\) be an \(H\)-module algebra. Cohen and Westreich [2] showed that if \(H\) is quasitriangular and \(A\) is quantum commutative with respect to \(H\), then \(A_0 \subseteq Z(A)\). It is now a naive but natural question to ask whether we can obtain same results for the generalized Hom-Lie algebras that are analogous to [2]. This becomes our main motivation of the paper.

To give a positive answer to the question above, we organize this paper as follows.

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Biography: Dong Lihong (1980–), female, born at Xinxiang, Henan, doctor, major in Hopf algebra and quantum group.
In Section 2, we recall some basic definitions about Yetter-Drinfeld modules, (monoidal) Hom-Lie algebras and generalized Hom-Lie algebras. In Section 3, we discuss the central invariant of generalized Hom-Lie algebras (see Theorem 3.8).

2 Preliminaries

In this section we recall some basic definitions and results related to our paper. Throughout the paper, all algebraic systems are supposed to be over a field $k$. The reader is referred to [1] as general references about monoidal Hom-algebras and monoidal Hom-Lie algebras, to [7] and [9] about Hopf algebras, and [3] about Yetter-Drinfeld categories. If $C$ is a coalgebra, we use the Sweedler-type notation for the comultiplication: $\Delta(c) = c_1 \otimes c_2$ for all $c \in C$.

From now on we always assume that $H$ is a Hopf algebra with a bijective antipode $S$. The Yetter-Drinfeld category $H^H\mathcal{YD}$ is a braided monoidal category whose objects $M$ are both left $H$-modules and left $H$-comodules, morphisms are both left $H$-linear and $H$-colinear maps and satisfy the compatibility condition

$$h_1m_{(-1)} \otimes h_2 \cdot m_0 = (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_0$$

or equivalently $\rho(h \cdot m) = h_1 m_{(-1)} S(h_3) \otimes (h_2 \cdot m_0)$, where the $H$-module action is denoted $h \cdot m$ and the $H$-comodule structure map is denoted by $\rho_M : M \rightarrow H \otimes M, \rho(m) = m_{(-1)} \otimes m_0$ for all $h \in H, m \in M$. The braiding $\tau$ is given by $\tau(m \otimes n) = m_{(-1)} \cdot n \otimes m_0$ for all $m \in M, n \in N$, $M, N$ are objects in $H^H\mathcal{YD}$.

Let $A$ be an object in $H^H\mathcal{YD}$, the braiding $\tau$ is called symmetric on $A$ if the following condition holds, for any $a, b \in A$, $(a_{(-1)} \cdot b)_{(-1)} \cdot a_0 \otimes (a_{(-1)} \cdot b)_0 = a \otimes b$, which is equivalent to the following condition

$$a_{(-1)} \cdot b \otimes a_0 = b_0 \otimes S^{-1}(b_{(-1)}) \cdot a. \quad (2.1)$$

2.1 Monoidal Hom-Algebra

Recall from [1] that a monoidal Hom-algebra is a triple $(A, \mu, \alpha)$ consisting of a linear space $A$, a $k$-linear map $\mu : A \otimes A \rightarrow A$ and a homomorphism $\alpha : A \rightarrow A$ for all $a, b, c \in A$, such that

$$\alpha(ab) = \alpha(a)\alpha(b), \quad \alpha(1_A) = 1_A, \quad \alpha(a)(bc) = (ab)\alpha(c), \quad 1_Aa = a1_A = \alpha(a).$$

2.2 Monoidal Hom-Lie Algebra

Recall from [1] that a monoidal Hom-Lie algebra is a triple $(L, [,], \alpha)$ consisting of a linear space $L$, a $k$-linear map $[,] : L \otimes L \rightarrow L$ and a homomorphism $\alpha : L \rightarrow L$ satisfying

$$\alpha[l, l'] = [\alpha(l), \alpha(l')]$$

$$[l, l'] = -[l', l] \quad \text{(Skew symmetry)}$$

$$\odot_{l, l', l''} [\alpha(l), [l', l'']] = 0 \quad \text{(Hom – Jacobi identity)}$$

The braiding $\tau$ is given by $\tau(m \otimes n) = m_{(-1)} \cdot n \otimes m_0$ for all $m \in M, n \in N$, $M, N$ are objects in $H^H\mathcal{YD}$. The braiding $\tau$ is called symmetric on $A$ if the following condition holds, for any $a, b \in A$, $(a_{(-1)} \cdot b)_{(-1)} \cdot a_0 \otimes (a_{(-1)} \cdot b)_0 = a \otimes b$, which is equivalent to the following condition

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for all $l, l', l'' \in L$, where $\circ$ denotes the summation over the cyclic permutation on $l, l', l''$.

### 2.2 Generalized Hom-Lie Algebra

Let $H$ be a Hopf algebra. Recall from [10] that a generalized Hom-Lie algebra is a triple $(L, [\cdot, \cdot], \alpha)$, which is a monoidal Hom-Lie algebra in a Yetter-Drinfeld category $\mathcal{YD}_H$, where $L$ is an object in $\mathcal{YD}_H$, $\alpha : L \to L$ is a homomorphism in $\mathcal{YD}_H \mathcal{YD}$ and $[\cdot] : L \otimes L \to L$ is a morphism in $\mathcal{YD}_H \mathcal{YD}$ satisfying

1. $H$-Hom-skew-symmetry
\[
[l, l'] = -[l_{(-1)} \cdot l', l_0], \quad l, l' \in L. \quad (2.2)
\]

2. $H$-Hom-Jacobi identity
\[
\{l \otimes l' \otimes l'' \} + \{(l \otimes 1)(1 \otimes \tau)(l \otimes l' \otimes l'')\} + \{(1 \otimes l)(l \otimes l' \otimes l'')\} = 0 \quad (2.3)
\]

for all $l, l', l'' \in L$, where $\{l \otimes l' \otimes l''\}$ denotes $[\alpha(l), [l', l'']]$ and $\tau$ the braiding for $L$.

### 3 Main Results

In this section we always assume that the braiding $\tau$ is symmetric. We consider some $H$-analogous of classical concepts of ring theory and of Lie theory as follows.

Let $A$ be a monoidal Hom-algebra in $\mathcal{YD}_H$. An $H$-Hom-ideal $U$ of $A$ is not only $H$-stable but also $H$-costable such that $\alpha(U) \subseteq U$ and $(AU)A = A(UA) \subseteq U$.

Let $L$ be a generalized Hom-Lie algebra. An $H$-Hom-Lie ideal $U$ of $L$ is not only $H$-stable but also $H$-costable such that $\alpha(U) \subseteq U$ and $[U, L] \subseteq U$.

Define the center of $L$ to be $Z_H(L) = \{l \in L|[l, L]_H = 0\}$. It is easy to see that $Z_H(L)$ is not only $H$-stable but also $H$-costable.

$L$ is called $H$-prime if the product of any two non-zero $H$-Hom-ideals of $L$ is non-zero. It is called $H$-semiprime if it has no non-zero nilpotent $H$-Hom-ideals, and is called $H$-simple if it has no nontrivial $H$-Hom-ideals.

**Definition 3.1** If $A$ is a monoidal Hom-algebra in $\mathcal{YD}_H$, the monoidal Hom-subalgebra of $H$-invariant is the set
\[
A_0 = \{a \in A|\h \cdot a = \varepsilon(h)a, \alpha(a) = a\}.
\]

**Example 3.2** Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional linear space $A$. The following multiplication $m$ and linear map $\alpha$ on $A$ define a monoidal Hom-algebra (see [8]):

$m(x_1, x_1) = x_1, m(x_1, x_2) = x_2, m(x_1, x_3) = bx_3,\\m(x_2, x_1) = x_2, m(x_2, x_2) = x_2, m(x_2, x_3) = bx_3,\\m(x_3, x_1) = bx_3, m(x_3, x_2) = 0, m(x_3, x_3) = 0,\\\alpha(x_1) = x_1, \alpha(x_2) = x_2, \alpha(x_3) = bx_3,$
where \( b \) is a parameter in \( k \). Let \( G \) be the cyclic group of order 2 generated by \( g \). The group algebra \( H = kG \) is a Hopf algebra in the usual way,
\[
\rho(x_1) = e \otimes x_1, \rho(x_2) = e \otimes x_2, \rho(x_3) = g \otimes x_3,
\]
\[
e \cdot x_i = x_i, g \cdot x_1 = x_1, g \cdot x_2 = x_2, g \cdot x_3 = -x_3, i = 1, 2, 3,
\]
where \( e \) is the unit of the group \( G \). It is not hard to check that \( A \) is a monoidal Hom-algebra in \( H \).

We assume that \( A_0 \) is a linear space spanned by \( \{x_1, x_2\} \), then \( A_0 \) is the monoidal Hom-subalgebra of \( H \)-invariant.

From a monoidal Hom-algebra \( (L, \alpha) \) in \( H \), Wang et al. [10] gave a derived monoidal Hom-Lie algebra \( (L, [,], \alpha) \) in \( H \) (that is a generalized Hom-Lie algebra) as follows for all \( a, b, c \in L \),
\[
[\cdot] : L \otimes L \to L, \quad [a, b] = ab - (a_{(-1)} \cdot b)a_0. \tag{3.1}
\]

In what follows, we always assume that the generalized Hom-Lie algebra means the above generalized Hom-Lie algebra.

The following lemma is referred to [10].

**Lemma 3.3** Let \( (L, [,], \alpha) \) be a generalized Hom-Lie algebra. Then
(1) \([\alpha(a), bc] = [a, b]\alpha(c) + (a_{(-1)} \cdot \alpha(b))[a_0, c];\)
(2) \([ab, \alpha(c)] = \alpha(a)[b, c] + [a, b_{(-1)} \cdot c] \alpha(b_0);\)
(3) \([ab, \alpha(c)] = [\alpha(a), bc] + [a_{(-1)} \cdot \alpha(b), (a_0_{(-1)} \cdot c)a_0] \) for all \( a, b, c \in L \).

Define \( ad_x(l) = [x, l] \) for all \( x, l \in L \). By Lemma 3.3 (1), we have
\[
ad_{\alpha(x)}(lm) = \alpha(l)ad_x(m).
\]

**Lemma 3.4** Let \( (L, [,], \alpha) \) be a generalized Hom-Lie algebra, and let \( x \in L_0 \). Then
(1) \( \tau_{L,L}(x \otimes y) = y \otimes x, \tau_{L,L}(y \otimes x) = x \otimes y;\)
(2) \( ad_x(y) = xy - yx;\)
(3) \( \alpha(xz) = \alpha(z)x + \alpha(y)ad_x(z);\)
(4) \( ad_x^2(z) = \alpha(1)ad_x(z) + 2\alpha(ad_x(y)ad_x(z)) + \alpha^2(y)ad_x^2(z), \) for all \( y, z \in L \).

**Lemma 3.5** Let \( (L, [,], \alpha) \) be a generalized Hom-Lie algebra. Assume that \( L \) is \( H \)-simple. Then \( Z_H(L)_0 \) is a field.

**Proof** Note that \( Z_H(L)_0 = Z_H(L) \cap L_0 = Z(L) \cap L_0 = Z(L)_0 \), where \( Z(L) \) is the usual center of \( L \). Taking \( 0 \neq x \in Z_H(L)_0 \), we have that \( Lx = I \neq 0 \) is an \( H \)-Hom-ideal, thus \( I = L \). That is to say that for some \( y \in L \), we obtain \( xy = yx = 1 \). Since
\[
\alpha^2(h \cdot y) = \alpha(h \cdot y)1 = \alpha(h \cdot y)(xy) = \alpha(h_1 \cdot y)(\varepsilon(h_2)xy)
\]
\[
= \alpha(h_1 \cdot y)((h_2 \cdot x)y) = (h \cdot (yx))\alpha(y)
\]
\[
= (h \cdot 1)\alpha(y) = \varepsilon(h)1\alpha(y) = \varepsilon(h)\alpha^2(y).
\]

We can get \( h \cdot y = \varepsilon(h)y \), that is, \( y \in L_0 \).
We need to show $y \in Z_H(L)$. For any $z \in L$, by Lemma 3.4 (1), $[z,x] = zx - xz = 0$. Thus $yz - zy = 0$, i.e., $[y,z] = yz - zy = 0$ by Lemma 3.4 (2). This shows that $y \in Z_H(L)$.

**Lemma 3.6** Let $(L, [\cdot, \cdot], \alpha)$ be a generalized Hom-Lie algebra, and let $x \in L_0$, $l, m \in L$.

Then

1. $\text{ad}_x^2(xl) = \alpha^2(x)\text{ad}_x^2(l)$;
2. If $\text{ad}_x^2(L) = 0$ and $\text{char}(k) \neq 2$, then $\text{ad}_x(L)(\text{ad}_x(m)) = 0$.

**Proof**

(1) It is easy to show that (1) holds by Lemma 3.4 (4).

(2) For all $l, m \in L$, we have

$$0 = \text{ad}_x^2(lm) = \text{ad}_x^2(l)\alpha^2(m) + 2\alpha(\text{ad}_x(l)\text{ad}_x(m)) + \alpha^2(l)\text{ad}_x^2(m)$$

$$= 2\alpha(\text{ad}_x(l)\text{ad}_x(\alpha(m)))$$

and so $\text{ad}_x(L)\text{ad}_x(m) = 0$ since $\text{char}(k) \neq 0$. Thus by Lemma 3.4 (3), one gets

$$\text{ad}_x(l)(\text{ad}_x(m)) = 0$$

for all $l, m \in L$.

**Lemma 3.7** Let $(L, [\cdot, \cdot], \alpha)$ be a generalized Hom-Lie algebra. If $L$ is $H$-simple with $\text{char}(k) \neq 2$, assume that $I$ is an $H$-Hom-Lie ideal of $[L,L]$. Let $x \in I_0$ satisfying

1. $\text{ad}_x(I) = 0$;
2. $\text{ad}_x^2([L,L]) = 0$.

Then $x \in Z_H(L)$.

**Proof** Let $x \in I_0$. For any $m \in L$, $l \in [L,L]$ and $y \in I$. By Lemma 3.3 (1),

$$0 = \text{ad}_x^2([\alpha(l),my]) = \text{ad}_x^2([l,m]\alpha(y)) + \text{ad}_x^2((l_{-1}) \cdot \alpha(m))[l_0,y]).$$

(3.2)

First, we have

$$\text{ad}_x^2([l,m]\alpha(y))$$

$$\begin{aligned}
&= \text{ad}_x^2([l,m]\alpha^3(y) + 2\alpha(\text{ad}_x([l,m])\text{ad}_x(\alpha(y))) + \alpha^2([l,m])\text{ad}_x^2(\alpha(y)) \\
&\quad \overset{(i)}{=} \text{ad}_x^2([l,m]\alpha^3(y) \overset{(ii)}{=} 0.
\end{aligned}$$

Hence

$$\text{ad}_x^2([l,m]\alpha(y)) = 0.$$  

(3.3)

Similarly,

$$\text{ad}_x^2((l_{-1}) \cdot \alpha(m))[l_0,y])$$

$$\begin{aligned}
&= \text{ad}_x^2((l_{-1}) \cdot \alpha(m))\alpha^2([l_0,y]) + 2\alpha(\text{ad}_x((l_{-1}) \cdot \alpha(m))\text{ad}_x([l_0,y])) \\
&\quad + \alpha^2((l_{-1}) \cdot \alpha(m))\text{ad}_x^2([l_0,y])
\end{aligned}$$

Since $l \in [L,L]$ and $[,]$ is $H$-colinear, $l_0 \in [L,L]$, $\text{ad}_x([l_0,y]) \overset{(i)}{=} 0$ and $\text{ad}_x^2([l_0,y]) \overset{(ii)}{=} 0$. Hence

$$\text{ad}_x^2((l_{-1}) \cdot \alpha(m))[l_0,y]) = \text{ad}_x^2((l_{-1}) \cdot \alpha(m))\alpha^2([l_0,y]).$$  

(3.4)
Substituting (3.3) and (3.4) into (3.2), we obtain
\[
\text{ad}^2_x((l_{-1}) \cdot \alpha(m))\alpha^2([l_0, y]) = 0
\]  \hspace{1cm} (3.5)

for all \(y \in I, l \in [L, L], m \in L\). We now consider two cases.

(1) If \([I, [L, L]] = 0\), then we have \(\text{ad}^2_x(L) = 0\). By Lemma 3.6 (2), \(\text{ad}_x(l)(\text{ad}_x(m)) = 0\).

Since \(L, L, L\) is a costable left ideal of \(L\), we get \(\text{ad}_x(L) = 0, \forall l \in L\), and hence \(x \in Z_H(L)\).

(2) Now assume \(U = [I, [L, L]] \neq 0\). It is easy to see that \(U\) is an \(H\)-Hom-Lie ideal of \([L, L]\). By (3.5) we have \(\text{ad}^2_x(L)U = 0\). Let \(Q = \{y \in L | yU = 0\}\), then \(Q\) is an \(H\)-stable \(H\)-costable left ideal of \(L\), we claim \(Q = 0\). If not, then \(L = QL\) since \(L\) is \(H\)-simple. By (2.1) we have
\[
QL \subseteq [Q, L] + LQ \subseteq [Q, L] + Q.
\]

Thus \(L = Q + [Q, L]\). Let \(y \in Q, l \in [L, L]\) and \(u \in U\). Then
\[
[y, l]u = ylu = y[l, u].
\]

Since \([l, u] \in U, y[l, u] = 0\), and thus \([Q, [L, L]] \subseteq Q\) and \([Q, L] \subseteq Q\). Hence
\[
L = QL = Q(Q + [Q, L]) \subseteq Q.
\]

This implies \(LU = 0\), which contradicts the assumption \(U \neq 0\). Hence, \(Q = 0\), and so \(\text{ad}^2_x(L) = 0\). Similarly to case (1), one gets \(x \in Z_H(L)\).

**Theorem 3.8** Let \((L, [\cdot, \cdot], \alpha)\) be a generalized Hom-Lie algebra. Let \(L\) be \(H\)-simple with \(\text{char}(k) \neq 2\), and assume that \(V\) is an \(H\)-Hom-Lie ideal of \([L, L]\) such that \([V_0, V] \subseteq Z_H(L)_0\). Then \(V_0 \subseteq Z_H(L)_0\).

**Proof** Let \(V\) be an \(H\)-Hom-Lie ideal of \([L, L]\) such that \([V_0, V] \subseteq Z_H(L)_0\). Let \(x \in V_0\). We consider the following two cases:

(1) \(\text{ad}_x(V) = 0\), which implies that \(x \in Z_H(L)_0\) by Lemma 3.7.

(2) \(\text{ad}_x(V) \neq 0\). For \(v \in V\), we have
\[
[[x, [x, L]], \alpha(v)] \equiv [[x, [x, L]]_{(-1)}(\cdot \alpha(v)), [x, [x, L]]] = 0
\]
\[
-\alpha(v_0), [x, [x, S^{-1}(v_{-1}) \cdot L]]
\]
\[
[[x, v_0(-1), S^{-1}(v_{-1}) \cdot \alpha(L)], [\alpha(v_0), x]] + [x, [[x, L], v]]
\]
\[
[[x, \alpha(L)], [\alpha(v), x]] + [x, [[x, L], v]]
\]
\[
\subseteq 0 + [x, [[L, L], v]] \subseteq [x, V] \subseteq Z_H(L)_0.
\]

We obtain \([\text{ad}^2_x(L), V] \subseteq Z_H(L)_0\). By Lemma 3.6 (1), we have \(\text{ad}^2_x(vl) = \alpha^2(x)\text{ad}^2_x(l)\). If \(\text{ad}^2_x(l) \neq 0\) for some \(l \in L\), then \((\text{ad}^2_x(l))^{-1} \in Z_H(L)_0\) by Lemma 3.5. In this case, it is easy to see that \(x \in Z_H(L)_0\). Now we assume \(\text{ad}^2_x(L) \subseteq Z_H(L)_0\). Let \(y \in L\) with \(\text{ad}^2_x(y) \notin Z_H(L)_0\).
Then we choose \( z \in V \) such that \( 0 \neq \text{ad}_z(x) = u \in Z_H(L)_0 \). Thus there exist \( v_1, v_2, v_3 \in Z_H(L)_0 \) such that 
\[
[z, \text{ad}_z^2(y)] = v_1, \quad [\alpha(z), \text{ad}_z^2(xy)] = v_2 \quad \text{and} \quad [\alpha^2(z), \text{ad}_z^2(x^2y)] = v_3.
\]
Now we have
\[
v_2 = [\alpha(z), \text{ad}_z^2(xy)] = [\alpha(z), x\text{ad}_z^2(y)] = [z, x][\alpha(\text{ad}_z^2(y))] + x[z, \text{ad}_z^2(y)] = u\alpha(\text{ad}_z^2(y)) + xv_1.
\]
By Lemma 3.5, \( u \) is invertible. Thus \( \text{ad}_z^2(y) = \alpha^{-1}(u^{-1}v_2 - u^{-1}(v_1x)) \). However, by \( v_1 \in Z_H(L), x \in V_0 \) and Lemma 3.4 (1), we have \( xv_1 = v_1x \), and so \( \text{ad}_z^2(y) = \alpha^{-1}(u^{-1}v_2 - u^{-1}(v_1x)) \). Similarly, we have
\[
v_3 = [\alpha^2(z), \text{ad}_z^2(x^2y)] = [\alpha(z), x\text{ad}_z^2(xy)] = [\alpha(z), x[\alpha(\text{ad}_z^2(y))] + x[z, \text{ad}_z^2(y)] = \alpha(u)\alpha(\text{ad}_z^2(y)) + xv_2 = u\alpha(\text{ad}_z^2(y)) + xv_2,
\]
and thus \( \text{ad}_z^2(xy) = \alpha^{-1}(u^{-1}v_3 - u^{-1}(v_2x)) \). Using Lemma 3.6 (1), we have
\[
\text{ad}_z^2(xy) = x\text{ad}_z^2(y) = \alpha^{-1}(\alpha(x)(u^{-1}v_2) - \alpha(x)(u^{-1}(v_1x))) = \alpha^{-1}((xu^{-1})\alpha(v_2) - (xu^{-1})\alpha(v_1x)) = \alpha^{-1}(u^{-1}v_2x) = u^{-1}(v_1x^2).
\]
Hence \( v_1x^2 - 2v_2x + v_3 = 0 \), that is, \( x^2 + \theta^1x + \theta^0 = 0 \), where \( \theta^1 = -2v_2/v_1, \theta^0 = v_3/v_1 \), and \( \theta^1, \theta^0 \in Z_H(L) \). And so by Lemma 3.3 (2) and Lemma 3.4 (1) we have
\[
0 = [-\theta^0, \alpha(z)] = [x^2, \alpha(z)] + [\theta^1x, \alpha(z)] = \alpha([x^2, z]) + \alpha(\theta^1)[x, z] + [\theta^1, x(-1) \cdot z]\alpha(x_0) = \alpha([x^2, z]) + \alpha(\theta^1)[x, z].
\]
By Lemma 3.4 (1), one has \( \alpha([x^2, z]) = -\alpha(\theta^1)[x, z] = \alpha(\theta^1)u \), and similarly we have
\[
\alpha([x^2, z]) = \alpha(x[x, z] + [x, z]x) = 2\alpha([x, z]x) = -2\alpha(ux) = -2ux.
\]
Since \( u \in Z_H(L)_0 \), \( \alpha(\theta^1) = -2x \). Since \( \text{char}(k) \neq 2 \), \( x = -(1/2)\theta^1 \in Z_H(L) \).

References


广义Hom-李代数的中心不变量

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