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CHAOS EXPANSION FOR MULTIFRACTIONAL LÉVY PROCESSES

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Abstract: In this paper, we study the chaos expansion for multifractional Lévy processes. By using the white noise analysis, we give the chaos expansion of multifractional Lévy Processes. Moreover, we derive their Lévy-Hermite transforms and Malliavin derivatives.

Keywords: muiltifractional Lévy processes; chaos expansion

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1 Introduction

The study on fractional processes started from the fractional Brownian motion (FBM) which was first introduced by Kolmogrov in 1940 and popularized by Mandelbrot and Van Ness [1] in 1968. For a constant $\beta \in (-\frac{1}{2}, \frac{1}{2})$, a FBM W^β is defined by

$$W^\beta = \int_R (I_-^\beta 1_{[0,t]})(s) dW_s = \frac{1}{\Gamma(\beta+1)} \int_{-\infty}^t ((t-s)^\beta - (-s)_+^\beta) dW_s,$$

where W is a standard Brownian motion, I_-^β is the Riemann-Liouville fractional integration operator, $H = \beta + \frac{1}{2}$ is called the Hurst parameter of W^β. FBM exhibits self-similarity and long-range dependence when $0 < \beta < \frac{1}{2}$ while remaining Gaussian, therefore, suits to model driving noise in different applications such as mathematical finance. However, the Hurst parameter $H = \beta + \frac{1}{2}$ is a constant, this property make it unsuitable when some one use it to model some phenomena which do not admit a constant Hölder exponent. To this purpose, [2, 3] independently substitute β by a Hölder continuous function $\beta : [0, \infty) \mapsto (-\frac{1}{2}, \frac{1}{2})$ and define the multifractional Brownian motion (MFBM) by

$$\begin{aligned} \widetilde{W}^\beta(t) = W^{\beta(t)}(t) &= \int_R (I_-^{\beta(t)} 1_{[0,t]})(s) dW_s \\ &= \frac{1}{\Gamma(\beta(t)+1)} \int_{-\infty}^t ((t-s)^{\beta(t)} - (-s)_+^{\beta(t)}) dW_s. \end{aligned}$$

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On the other hand, the light tails of FBM make it unsuitable to model the high volatility phenomenon. [4] and [5] defined fractional Lévy processes and noises on a Gel′fand triple. The β-fractional Lévy process on a Gel′fand $\{X_t^\beta, t \geq 0\}$ $(0 < \beta < \frac{1}{2})$ is defined by

$$X_t^\beta = \int_R (I_-^\beta 1_{[0,t]})(s) dX_s = \frac{1}{\Gamma(\beta+1)} \int_{-\infty}^t ((t-s)^\beta - (-s)_+^\beta) dX_s,$$

where X is a Lévy process on a Gel′fand with zero mean, continuous covariance operator. Lü et al. [5] showed that fractional Lévy process has stationary increments, long-range dependence. Moreover, Lü et al. [7] substitute the β parameter of fractional Lévy processes by a Hölder continuous function with respect to time to define multifractional Lévy processes on Gel′fand triple.

In this paper, based on the white noise analysis of 0 mean Lévy process with finite moments of any orders given by [6], we give the chaos expansion of multifractional Lévy processes. Moreover, we derive their Lévy-Hermite transforms and Malliavin derivatives. The paper is organized as follows: in Section 2, we recall the basic results about white noise analysis of Lévy processes; in Section 3, we we derive the chaos expansion of multifractional Lévy processes.

2 White Noise Aanlysis for Lévy Processes

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions, let $\mathcal{F} = B(\mathcal{S}'(\mathbb{R}))$ be the Borel σ-algebra. By Bochner-Minlos theorem, there exists a probability P on $\mathcal{S}'(\mathbb{R})$ such that

$$\int_\Omega e^{iz\langle\omega,f\rangle} P(d\omega) = \exp\{\int_{-\infty}^{+\infty} \psi(zf(s))ds\},\ z \in \mathbb{R}, f \in \mathcal{S}(\mathbb{R}), \tag{2.1}$$

where

$$\psi(z) = \int_{\mathbb{R}} [e^{izx} - 1 - izx] d\nu(x), u \in \mathbb{R},$$

ν is the Lévy measure satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (|x|^2 \wedge 1) d\nu(x) < \infty,$$

where $a \wedge b = \min\{a, b\}$. Moreover, we assume that

$$\int_{|x|>1} |x|^2 d\nu(x) < \infty.$$

For $f \in \mathcal{S}(\mathbb{R})$, let $\dot{X}(f)(\omega) := \langle\omega, f\rangle$, then by (2.1), we have

$$\mathbb{E}[\dot{X}(f)] = 0,$$
$$\mathbb{E}[\dot{X}(f)]^2 = \int_{\mathbb{R}} f^2(y)dy \int_{\mathbb{R}} x^2 d\nu(x).$$

We can extend the definition of $\dot{X}(f)(\omega)$ for $f \in \mathcal{S}(\mathbb{R})$ to any $f \in L^2(\mathbb{R})$ by choosing $f_n \in \mathcal{S}(\mathbb{R})$ such that $f_n \to f$ in $L^2(\mathbb{R})$ and defining $\dot{X}(f)(\omega) := \lim_{n\to\infty} \dot{X}(f_n)(\omega)$ (in $L^2(P)$). Now define $\eta(t) := \dot{X}(\chi_{[0,t]}(s))$, where

$$\chi_{[0,t]}(s) = \begin{cases} 1, & 0 < s < t, \\ -1, & t < s < 0, \\ 0, & \text{else.} \end{cases}$$

The stochastic process $\{\eta(t), t \in \mathbb{R}\}$ has a càdlàg version, denoted by X. This process $\{X(t), t \in \mathbb{R}\}$ is a pure jump Lévy process with Lévy measure ν, X admits the stochastic integral representation

$$X(t) = \int_0^t \int_{\mathbb{R}} x\widetilde{N}(ds, dx), t \geq 0,$$

where $\widetilde{N}((0,t] \times A) = N((0,t] \times A) - t\nu(A)$ is its compensation Poisson measure. In this case, X is a martingale and we call it pure-jump Lévy process.

From now on we assume that the Levy measure ν satisfies for all $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}_0 \setminus (-\epsilon,\epsilon)} \exp(\lambda|x|) d\nu(x) < \infty, \tag{2.2}$$

where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.This condition means that X has finite moment of any orders. Let $\{p_j(z), j \geq 1\}$ be the orthonormal basis for $L^2(\nu)$ and $\{e_i(t), i \geq 1\}$ be the Hermite functions, where $p_1(z) = m_2^{-1} z$, $m_2 = \int_{\mathbb{R}} x^2 \nu(dx)$. Define $\kappa(i,j) = j + \frac{(i+j-2)(i+j-1)}{2}$, $\delta_{\kappa(i,j)}(t,z) = e_i(t)p_j(z)$. For $\alpha \in \mathcal{J}$ (where $\mathcal{J} = \mathbb{N}^{\mathbb{N}}$), (index)$\alpha = j$, $|\alpha| = m$, that is $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_j, 0, 0, \cdots)$, $\sum_{i=1}^{j} \alpha_i = m, \alpha_j \in \mathbb{N}$, we define

$$\begin{aligned} &\delta^{\otimes\alpha}(t_1, z_1, \cdots, t_m, z_m) \\ =&\delta_1^{\otimes\alpha_1} \otimes \cdots \otimes \delta_j^{\otimes\alpha_j}(t_1, z_1, \ldots, t_m, z_m) \\ =&\delta_1(t_1, z_1) \cdots \delta_1(t_{\alpha_1}, z_{\alpha_1}) \cdots \delta_j(t_{m-\alpha_j+1}, z_{m-\alpha_j+1}) \cdots \delta_j(t_m, z_m). \end{aligned}$$

We set $\delta_j^{\otimes 0} = 1$ and $\delta^{\widehat{\otimes}\alpha}(t_1, z_1, \cdots, t_m, z_m) := \delta_1^{\widehat{\otimes}\alpha_1} \otimes \cdots \otimes \delta_j^{\widehat{\otimes}\alpha_j}(t_1, z_1, \ldots, t_m, z_m)$, define

$$K_\alpha = I_{|\alpha|}(\delta^{\widehat{\otimes}\alpha}),$$

where $I_n(f)$ is the n-fold iterated integral of f with respect to X, $\otimes$ means tensor product, $\widehat{\otimes}$ means the symmetric tensor product (for more details, see [6]).

Theorem 2.1 (see [6]) Any $F \in L^2(P)$ has a unique expansion of the form

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha K_\alpha \tag{2.3}$$

with $c_\alpha \in \mathbb{R}$. Moreover,

$$\| F \|^2_{L^2(\Omega)} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2.$$

If F has the chaos expansion (2.3), its Lévy-Hermite transform of F is defined by

$$\mathcal{H}F(u) := \sum_{\alpha\in\mathcal{J}} c_\alpha u^\alpha = \sum_{\alpha\in\mathcal{J}} c_\alpha \prod_k u_k^{\alpha_k}, \tag{2.4}$$

where $u = (u_1, u_2, \ldots) \in \mathbb{C}^N$.

3 Chaos Expansion for Multifractional Lévy Processes

In this section, we give the chaos expansion of multifractional Lévy processes. Moreover, we derive their Lévy-Hermite transforms and Malliavin derivatives.

By the definition of multi-fractional Lévy processes on Gel′fand triple given by [7], we can easily define the real-valued multi-fractional Lévy process.

Definition 3.1 Let $\beta : \mathbb{R}^+ \to (0, \frac{1}{2})$ be a measurable function, X be a two-side Lévy process satisfying all the assumptions in Section 2. Define

$$\begin{aligned} X_t^{\beta(t)} &:= \int_R (I_-^{\beta(t)} 1_{[0,t]})(s) dX_s \\ &= \frac{1}{\Gamma(\beta(t)+1)} \int_{-\infty}^t ((t-s)^{\beta(t)} - (-s)_+^{\beta(t)}) dX_s, \end{aligned} \tag{3.1}$$

where $x_+ = \max\{x, 0\}$, 1_B is the indicator function on the set B, and I_-^β is the Riemann-Liouville fractional integral operator defined by

$$(I_-^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_t^{+\infty} f(s)(s-t)^{\beta-1} ds,$$

if the integral exist for almost all $x \in \mathbb{R}$ (see [8]), and $\Gamma(\cdot)$ is the Gamma function.

By Theorem 3.2 of [7], we can obtain the one-dimensional distribution of the multi-fractional Lévy process.

Theorem 3.2 Let $X^\beta = \{X_t^{\beta(t)}, t \geq 0\}$ be a β-multifractional Lévy process, then for any $t \geq 0$, $X_t^{\beta(t)}$ is a 0 mean infinitely divisible random variable with characteristic function

$$\mathbb{E}[\exp(izX_t^{\beta(t)})] = \exp\{\int_{\mathbb{R}} [e^{izx} - 1 - izx] d\nu_{\beta(t)}(x)\}, \tag{3.2}$$

where

$$\nu_{\beta(t)}(B) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} 1_B\{(I_-^{\beta(t)} 1_{[0,t]})(s)x\} \nu(dx)ds, \forall B \in \mathcal{B}(\mathbb{R}). \tag{3.3}$$

$\forall s, t \geq 0$,

$$\begin{aligned} &\mathbb{E}[X_t^{\beta(t)} X_s^{\beta(s)}] \\ =&m_2 C(\beta(t), \beta(s))(|t|^{\beta(t)+\beta(s)+1} + |s|^{\beta(t)+\beta(s)+1} - |t-s|^{\beta(t)+\beta(s)+1}) \end{aligned} \tag{3.4}$$

and

$$C(\beta(t), \beta(s)) = \frac{1}{2\Gamma(\beta(t)+\beta(s)+2)\sin\frac{(\beta(t)+\beta(s)+1)\pi}{2}}.$$

Theorem 3.3 Let $X^\beta = \{X_t^{\beta(t)}, t \geq 0\}$ be β-multi-fractional Lévy process, then for any $t \geq 0$, the chaos expansion of $X_t^{\beta(t)}$ is

$$X_t^{\beta(t)} = \sum_{i\geq 1} m_2 \int_{\mathbb{R}} (I_-^{\beta(t)} 1_{[0,t]})(s) e_i(s) ds K_{\varepsilon(i,1)}, \tag{3.5}$$

where

$$K_{\varepsilon(i,1)} = I_1(e_i(t)p_1(z)),$$

$\varepsilon(i,1) = (0,0,\cdots,1,0,\cdots)$ with the 1 on the $\kappa(i,1)$th place.

Proof Since $I_-^{\beta(t)} 1_{[0,t]} \in L^2(\mathbb{R})$, for any $t \geq 0$, $X_t^{\beta(t)} \in L^2(P)$, then by Theorem 2.1,

$$\begin{aligned}
X_t^\beta &= \int_R (I_-^{\beta(t)} 1_{[0,t]})(s) dX_s \\
&= I_1(I_-^{\beta(t)} 1_{[0,t]}(s)z) \\
&= I_1(\sum_i \langle I_-^{\beta(t)} 1_{[0,t]}(\cdot), e_i(\cdot)\rangle_{L^2(\mathbb{R})} e_i(s)z) \\
&= \sum_i \langle (I_-^{\beta(t)} 1_{[0,t]})(\cdot), e_i(\cdot)\rangle_{L^2(\mathbb{R})} I_1(e_i(s)z) \\
&= \sum_{i\geq 1} m_2 \langle (I_-^{\beta(t)} \chi_{[0,t]})(\cdot), e_i(\cdot)\rangle_{L^2(\mathbb{R})} K_{\varepsilon(i,1)} \\
&= \sum_{i\geq 1} m_2 \int_{\mathbb{R}} (I_-^{\beta(t)} \chi_{[0,t]})(s) e_i(s) ds K_{\varepsilon(i,1)}.
\end{aligned}$$

Thus we get the desired.

By the following fractional integral by parts formula of operator $I_\pm^\beta$:

$$\int_{-\infty}^{+\infty} f(s)(I_+^\beta g)(s) ds = \int_{-\infty}^{+\infty} g(s)(I_-^\beta f)(s) ds, \ \ f, g \in \mathcal{S}(\mathbb{R}),$$

which can be extended to $f \in L^p(\mathbb{R})$, $g \in L^r(\mathbb{R})$ with $p > 1$, $r > 1$ and $\frac{1}{p} + \frac{1}{r} = 1 + \beta$, where

$$(I_+^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^t (t-s)^{\beta-1} f(s) ds$$

(see [8]), (3.5) can be written as

$$X_t^{\beta(t)} = m_2 \sum_{i\geq 1} \int_0^t I_+^{\beta(t)} e_i(s) ds K_{\varepsilon(i,1)}.$$

By the chaos expansion of X^β, we get

Corollary 3.4 Let $X^\beta = \{X_t^{\beta(t)}, t \geq 0\}$ be β-multifractional Lévy process, then for any $t \geq 0$, the Lévy-Hermite transform of $X_t^{\beta(t)}$

$$\mathcal{H} X_t^{\beta(t)}(u) = \sum_{i\geq 1} m_2 \int_{\mathbb{R}} (I_-^{\beta(t)} 1_{[0,t]})(s) e_i(s) ds u_{\varepsilon(i,1)}, \tag{3.6}$$

where $u = (u_1, u_2, \cdots) \in \mathbb{C}^N$.

By equality (12.4) of [6], we can easily get

Proposition 3.5 Let $X^\beta = \{X_t^{\beta(t)}, t \geq 0\}$ be β-multifractional Lévy process, then for any $t \geq 0$, the Malliavin derivative of $X_t^{\beta(t)}$

$$D_{s,z} X_t^{\beta(t)} = (I_-^{\beta(t)} 1_{[0,t]})(s) z. \tag{3.7}$$

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多分数Lévy过程的混沌展开

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摘要: 本文研究了多分数Lévy过程的混沌展开. 利用白噪声分析方法, 给出了多分数Lévy过程的混沌展开. 进一步地, 给出其Lévy-Hermite变换和Malliavin导数.

关键词: 多分数Lévy过程; 混沌展开

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